Chapter 7: The Exact Form and General Integrating Factors

7.1 a. By definition $\phi = 3xy$ is potential function for

$$M + N\frac{dy}{dx} = 0$$

if

$$M = \frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}[3xy] = 3y \qquad \text{and} \qquad N = \frac{\partial\phi}{\partial y} = \frac{\partial}{\partial y}[3xy] = 3x \quad .$$

So, $\phi = 3xy$ is potential function for

$$3y + 3x\frac{dy}{dx} = 0 \quad .$$

An implicit solution is given by $\phi = c$; that is,

$$3xy = c \quad .$$

Solving this for y gives the explicit solution $\quad y = \frac{c}{3x} = \frac{C}{x} \quad .$

7.1 c. By definition $\phi = x^2y - xy^3$ is potential function for

$$M + N\frac{dy}{dx} = 0$$

if

$$M = \frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}\left[x^2y - xy^3\right] = 2xy - y^3$$

and

$$N = \frac{\partial\phi}{\partial y} = \frac{\partial}{\partial y}\left[x^2y - xy^3\right] = x^2 - 3xy^2 \quad .$$

So, $\phi = x^2y - xy^3$ is potential function for

$$2xy - y^3 + \left[x^2 - 3xy^2\right]\frac{dy}{dx} = 0 \quad .$$

An implicit solution is given by $\phi = c$; that is,

$$x^2y - xy^3 = c \quad .$$

Solving this for y to obtain an explicit solution is, alas, not practical.

7.2 a. By subtracting the left side of the equation from both sides, and then multiplying both sides by $2xy$,

$$\frac{dy}{dx} = \frac{1}{y} - \frac{y}{2x} \quad \rightarrowtail \quad \frac{y}{2x} - \frac{1}{y} + \frac{dy}{dx} = 0$$

$$\hookrightarrow \qquad \underbrace{\left[y^2 - 2x\right]}_{M} + \underbrace{2xy}_{N}\frac{dy}{dx} = 0 \quad .$$

To show this equation is exact with potential function $\phi = xy^2 - x^2$, we only need to show that ${}^{\partial\phi}/_{\partial x} = M$ and ${}^{\partial\phi}/_{\partial y} = N$ where M and N are as indicated above. Checking this, we see that

$$\frac{\partial\phi}{\partial x} = \frac{\partial}{\partial x}\left[xy^2 - x^2\right] = y^2 - 2x = M$$

and

$$\frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}\left[xy^2 - x^2\right] = 2xy = N \quad ,$$

confirming the claim.

7.2 b. Starting with the implicit solution $\phi = c$,

$$xy^2 - x^2 = c \quad \rightarrowtail \quad y^2 = \frac{x^2 + c}{x} = x + \frac{c}{x} \quad \rightarrowtail \quad y = \pm\sqrt{x + \frac{c}{x}} \quad .$$

7.2 c. By definition, the equation $M + N^{dy}/_{dx} = 0$ ψ is in exact form and has potential function ψ if

$$\frac{\partial \psi}{\partial x} = e^{xy^2 - x^2}\left[y^2 - 2x\right] \qquad \text{and} \qquad \frac{\partial \psi}{\partial y} = e^{xy^2 - x^2} 2xy \quad .$$

Computing the derivatives, we get that, in fact,

$$\frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x} e^{xy^2 - x^2} = e^{xy^2 - x^2}\left[y^2 - 2x\right]$$

and

$$\frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y} e^{xy^2 - x^2} = e^{xy^2 - x^2} 2xy \quad ,$$

confirming the claim.

7.4 a. Here, the potential function $\phi(x, y)$ must satisfy the pair of equations

$$\frac{\partial \phi}{\partial x} = M(x, y) = 2xy + y^2 \qquad \text{and} \qquad \frac{\partial \phi}{\partial y} = N(x, y) = 2xy + x^2 \quad .$$

Integrating the first equation with respect to x yields

$$\phi(x, y) = \int \frac{\partial \phi}{\partial x}\, dx = \int \left[2xy + y^2\right] dx = x^2 y + xy^2 + h(y) \qquad (\star)$$

where $h(y)$ is a yet undetermined function of y. To determine this function, plug the last formula for ϕ into the second of the initial pair of equations:

$$\frac{\partial \phi}{\partial y} = 2xy + x^2$$

$$\hookrightarrow \qquad \frac{\partial}{\partial y}\left[x^2 y + xy^2 + h(y)\right] = 2xy + x^2$$

$$\hookrightarrow \qquad x^2 + 2xy + h'(y) = 2xy + x^2$$

$$\hookrightarrow \qquad h'(y) = 0 \quad .$$

Hence, since $h(y)$ is a function of y only,

$$h(y) = \int h'(y)\, dy = \int 0\, dy = c_1 \quad ,$$

and formula ($\star$) for ϕ becomes

$$\phi(x,y) = x^2y + xy^2 + h(y) = x^2y + xy^2 + c_1 \quad .$$

An implicit solution to the differential equation is then given by $\phi = c_2$,

$$x^2y + xy^2 + c_1 = c_2 \quad ,$$

which can be rewritten as

$$xy^2 + x^2y + C = 0 \qquad (\text{with } c = c_1 - c_2) \quad .$$

Solving for y via the quadratic formula then yields the explicit solution

$$y(x) = \frac{-x^2 \pm \sqrt{\left(x^2\right)^2 - 4xc}}{2x} = -\frac{x}{2} \pm \frac{1}{2}\sqrt{x^2 + \frac{C}{x}} \quad .$$

7.4 c. Here, the potential function $\phi(x,y)$ must satisfy

$$\frac{\partial\phi}{\partial x} = M(x,y) = 2 - 2x \qquad \text{and} \qquad \frac{\partial\phi}{\partial y} = N(x,y) = 3y^2 \quad .$$

Integrating the first equation with respect to x yields

$$\phi(x,y) = \int \frac{\partial\phi}{\partial x}\,dx = \int [2-2x]\,dx = 2x - x^2 + h(y) \quad . \qquad (\star)$$

Combining this formula for ϕ with the second of the initial pair of equations:

$$3y^2 = \frac{\partial\phi}{\partial y} = \frac{\partial}{\partial y}\left[2x - x^2 + h(y)\right] = h'(y) \quad .$$

Hence,

$$h(y) = \int h'(y)\,dy = \int 3y^2\,dy = y^3 + c_1 \quad ,$$

and formula ($\star$) for ϕ becomes

$$\phi(x,y) = 2x - x^2 + h(y) = 2x - x^2 + y^3 + c_1 \quad .$$

An implicit solution to the differential equation is then given by $\phi = c_2$,

$$2x - x^2 + y^3 + c_1 = c_2 \quad .$$

Solving for y then yields the explicit solution

$$y(x) = \left(x^2 - 2x + C\right)^{1/3} \quad .$$

7.4 e. Here, the potential function $\phi(x,y)$ must satisfy

$$\frac{\partial\phi}{\partial x} = M(x,y) = 4x^3y \qquad \text{and} \qquad \frac{\partial\phi}{\partial y} = N(x,y) = x^4 - y^4 \quad .$$

Integrating the first equation with respect to x yields

$$\phi(x,y) = \int \frac{\partial\phi}{\partial x}\,dx = \int 4x^3y\,dx = x^4y + h(y) \quad . \qquad (\star)$$

Combining this formula for ϕ with the second of the initial pair of equations:

$$x^4 - y^4 = \frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}\left[x^4 y + h(y)\right] = x^4 + h'(y) \quad .$$

Hence, $h'(y) = -y^4$,

$$h(y) = \int h'(y)\,dy = -\int y^4\,dy = -\tfrac{1}{5}y^5 + c_1 \quad ,$$

and formula ($\star$) for ϕ becomes

$$\phi(x,y) = x^4 y + h(y) = x^4 y - \tfrac{1}{5}y^5 + c_1 \quad .$$

An implicit solution to the differential equation is then given by $\phi = c_2$, which, after combining arbitrary constants, is

$$x^4 y - \tfrac{1}{5}y^5 = c \quad .$$

Solving this for an explicit solution y is not practical.

7.4 g. Here, the potential function $\phi(x,y)$ must satisfy

$$\frac{\partial \phi}{\partial x} = M(x,y) = 1 + e^y \qquad \text{and} \qquad \frac{\partial \phi}{\partial y} = N(x,y) = xe^y \quad .$$

Integrating the first equation with respect to x yields

$$\phi(x,y) = \int \frac{\partial \phi}{\partial x}\,dx = \int \left[1 + e^y\right]dx = x + xe^y + h(y) \quad . \qquad (\star)$$

Combining this formula for ϕ with the second of the initial pair of equations:

$$xe^y = \frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}\left[x + xe^y + h(y)\right] = xe^y + h'(y) \quad .$$

Hence, $h'(y) = 0$,

$$h(y) = \int h'(y)\,dy = \int 0\,dy = c_1 \quad ,$$

and formula ($\star$) for ϕ becomes

$$\phi(x,y) = x + xe^y + h(y) = x + xe^y + c_1 \quad .$$

An implicit solution to the differential equation is then given by $\phi = c_2$, which, after combining arbitrary constants, is

$$x + xe^y = c \quad .$$

Solving for y then yields the explicit solution $\quad y(x) = \ln\left|\frac{c-x}{x}\right| \quad .$

7.5 a. *For part i:* For $M + N^{dy}/_{dx} = 0$ to be in exact form, we must have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad .$$

Here
$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[1 + y^4\right] = 4y^3 \qquad \text{and} \qquad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[xy^3\right] = y^3 \quad .$$

So ${}^{\partial M}/_{\partial y} \neq {}^{\partial N}/_{\partial x}$, telling us the equation is not in exact form.

For part ii: We need to find a function $\mu = \mu(x, y)$ such that
$$\frac{\partial}{\partial y}[\mu M] = \frac{\partial}{\partial x}[\mu N] \quad .$$

For this problem, this equation is
$$\frac{\partial}{\partial y}\left[\mu\left[1 + y^4\right]\right] = \frac{\partial}{\partial x}\left[\mu\left[xy^3\right]\right] \quad .$$

Let us see if the integrating factor μ can be a function of x only, $\mu = \mu(x)$:
$$\frac{\partial}{\partial y}\left[\mu(x)\left[1 + y^4\right]\right] = \frac{\partial}{\partial x}\left[\mu(x)\left[xy^3\right]\right]$$

$$\hookrightarrow \qquad 4\mu(x)y^3 = \frac{d\mu}{dx}xy^3 + \mu(x)y^3$$

$$\hookrightarrow \qquad \frac{d\mu}{dx}xy^3 = 3\mu(x)y^3$$

$$\hookrightarrow \qquad \frac{d\mu}{dx} = \frac{3\mu(x)}{x} \quad .$$

The last equation is a differential equation for $\mu(x)$ not involving the variable y. A solution (easily found by treating this as a separable or linear differential equation) is
$$\mu(x) = x^3 \quad .$$

This is what we can use as an integrating factor.

For part iii: Multiplying the given differential equation by the integrating factor μ just found:
$$x^3\left[1 + y^4 + xy^3\frac{dy}{dx}\right] = x^3 \cdot 0$$

$$\hookrightarrow \qquad x^3\left[1 + y^4\right] + x^4y^3\frac{dy}{dx} = 0 \quad .$$

Any potential function ϕ of this equation must satisfy
$$\frac{\partial \phi}{\partial x} = x^3\left[1 + y^4\right] \qquad \text{and} \qquad \frac{\partial \phi}{\partial y} = x^4y^3 \quad . \tag{$\star$}$$

Integrating the first equation with respect to x:
$$\phi(x, y) = \int \frac{\partial \phi}{\partial x}\,dx = \int x^3\left[1 + y^4\right]dx = \frac{1}{4}x^4\left[1 + y^4\right] + h(y) \quad .$$

Combining this result with the second equation in ($\star$):
$$x^4y^3 = \frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}\left[\frac{1}{4}x^4\left[1 + y^4\right] + h(y)\right] = x^4y^3 + h'(y)$$

$$\hookrightarrow \qquad h'(y) = 0 \quad \rightarrowtail \quad h(y) = \int 0\,dy = c_1 \quad .$$

So the potential function is

$$\phi(x,y) = \tfrac{1}{4}x^4\left[1+y^4\right] + h(y) = \tfrac{1}{4}x^4\left[1+y^4\right] + c_1 \quad ,$$

and an implicit solution to the given differential equation is

$$\tfrac{1}{4}x^4\left[1+y^4\right] + c_1 = c_2 \quad ,$$

which we rewrite as

$$x^4\left[1+y^4\right] = C \quad .$$

Solving this for y then yields $\quad y(x) = \pm\left(Cx^{-4}-1\right)^{1/4} \quad .$

7.5 c. *For part i:* Here

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[2x^{-1}y\right] = 2x^{-1}$$

and

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[4x^2y-3\right] = 8xy \quad .$$

So ${}^{\partial M}/_{\partial y} \neq {}^{\partial N}/_{\partial x}$, telling us the equation is not in exact form.

For part ii: We need to find a function $\mu = \mu(x,y)$ such that

$$\frac{\partial}{\partial y}\left[2\mu x^{-1}y\right] = \frac{\partial}{\partial y}[\mu M] = \frac{\partial}{\partial x}[\mu N] = \frac{\partial}{\partial x}\left[\mu\left[4x^2y-3\right]\right] \quad .$$

Let us see if an integrating factor μ can be a function of x only, $\mu = \mu(x)$:

$$\frac{\partial}{\partial y}\left[\mu(x)\left[2x^{-1}y\right]\right] = \frac{\partial}{\partial x}\left[\mu(x)\left[4x^2y-3\right]\right]$$

$$\hookrightarrow \qquad \mu(x)\left[2x^{-1}\right] = \frac{d\mu}{dx}\left[4x^2y-3\right] + \mu(x)\,[8xy]$$

$$\hookrightarrow \qquad \frac{d\mu}{dx}\left[4x^2y-3\right] = -\mu(x)\left[8xy-2x^{-1}\right]$$

$$\hookrightarrow \qquad \frac{d\mu}{dx}\left[4x^2y-3\right] = -2x^{-1}\mu(x)\left[4x^2y-1\right] \quad .$$

In this case, we clearly cannot cancel out the y variable, and cannot obtain a differential equation for $\mu = \mu(x)$ in terms of x and μ only. So there is not an integrating factor of x alone.

Attempting to find an integrating factor of y only, $\mu = \mu(y)$:

$$\frac{\partial}{\partial y}\left[\mu(y)\left[2x^{-1}y\right]\right] = \frac{\partial}{\partial x}\left[\mu(y)\left[4x^2y-3\right]\right]$$

$$\hookrightarrow \qquad \frac{d\mu}{dy}\left[2x^{-1}y\right] + \mu(y)\left[2(y)x^{-1}\right] = \mu(y)\,[8xy]$$

$$\hookrightarrow \qquad \frac{d\mu}{dy}\left[2x^{-1}y\right] = \mu(y)\left[8xy-2x^{-1}\right]$$

$$\hookrightarrow \qquad \frac{d\mu}{dy} = \mu(y)\left[4x^2-y^{-1}\right] \quad .$$

Since we clearly cannot obtain a differential equation for $\mu = \mu(y)$ in terms of y and μ only, there is not an integrating factor of y alone.

Finally, let us attempt to find an integrating factor of the form $\mu = \mu(x, y) = x^\alpha y^\beta$:

$$\frac{\partial}{\partial y}\left[\mu\left[2x^{-1}y\right]\right] = \frac{\partial}{\partial x}\left[\mu\left[4x^2y - 3\right]\right]$$

$$\hookrightarrow \quad \frac{\partial}{\partial y}\left[x^\alpha y^\beta\left[2x^{-1}y\right]\right] = \frac{\partial}{\partial x}\left[x^\alpha y^\beta\left[4x^2y - 3\right]\right]$$

$$\hookrightarrow \quad \frac{\partial}{\partial y}\left[2x^{\alpha-1}y^{\beta+1}\right] = \frac{\partial}{\partial x}\left[4x^{\alpha+2}y^{\beta+1} - 3x^\alpha y^\beta\right]$$

$$\hookrightarrow \quad 2(\beta+1)x^{\alpha-1}y^\beta = 4(\alpha+2)x^{\alpha+1}y^{\beta+1} - 3\alpha x^{\alpha-1}y^\beta \quad .$$

Combining like terms yields

$$[2(\beta+1) + 3\alpha]\,x^{\alpha-1}y^\beta - 4(\alpha+2)x^{\alpha+1}y^{\beta+1} = 0 \quad ,$$

which requires that

$$2(\beta+1) + 3\alpha = 0 \qquad \text{and} \qquad -4(\alpha+2) = 0 \quad ,$$

and from which it follows that

$$\alpha = -2 \qquad \text{and} \qquad \beta = -1 - \tfrac{3}{2}\alpha = -1 - \tfrac{3}{2}(-2) = 2 \quad .$$

So,

$$\mu = \mu(x, y) = x^\alpha y^\beta = x^{-2}y^2 \quad .$$

For part iii: Multiplying the given differential equation by the integrating factor μ just found:

$$x^{-2}y^2\left[2x^{-1}y + \left[4x^2y - 3\right]\frac{dy}{dx}\right] = x^{-2}y^2 \cdot 0$$

$$\hookrightarrow \quad 2x^{-3}y^3 + \left[4y^3 - 3x^{-2}y^2\right]\frac{dy}{dx} = 0 \quad .$$

Any potential function ϕ of this equation must satisfy

$$\frac{\partial\phi}{\partial x} = 2x^{-3}y^3 \qquad \text{and} \qquad \frac{\partial\phi}{\partial y} = 4y^3 - 3x^{-2}y^2 \quad . \tag{$\star$}$$

Integrating the first with respect to x :

$$\phi(x, y) = \int \frac{\partial\phi}{\partial x}\,dx = \int 2x^{-3}y^3\,dx = -x^{-2}y^3 + h(y) \quad .$$

Combining this result with the second equation in ($\star$):

$$4y^3 - 3x^{-2}y^2 = \frac{\partial\phi}{\partial y} = \frac{\partial}{\partial y}\left[-x^{-2}y^3 + h(y)\right]$$

$$\hookrightarrow \quad 4y^3 - 3x^{-2}y^2 = -3x^{-2}y^2 + h'(y)$$

$$\hookrightarrow \quad h'(y) = 4y^3 \quad \rightarrowtail \quad h(y) = \int 4y^3\,dy = y^4 + c_1 \quad .$$

So the potential function is

$$\phi(x,y) = -x^{-2}y^3 + h(y) = -x^{-2}y^3 + h(y) = -x^{-2}y^3 + y^4 + c_1 \quad ,$$

and an implicit solution to the given differential equation is

$$-x^{-2}y^3 + y^4 + c_1 = c_2 \quad ,$$

which we rewrite as

$$y^4 - x^{-2}y^3 = c \quad ,$$

but will not attempt to solve for y.

7.5 e. *For part i:* Here

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[3y + 3y^2\right] = 3 + 6y$$

and

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}[2x + 4xy] = 2 + 4y \quad .$$

So ${}^{\partial M}/_{\partial y} \neq {}^{\partial N}/_{\partial x}$, telling us the equation is not in exact form.

For part ii: We need to find a function $\mu = \mu(x,y)$ such that

$$\frac{\partial}{\partial y}\left[\mu\left[3y + 3y^2\right]\right] = \frac{\partial}{\partial y}[\mu M] = \frac{\partial}{\partial x}[\mu N] = \frac{\partial}{\partial x}\left[\mu\,[2x + 4xy]\right] \quad .$$

Let us see if an integrating factor μ can be a function of x only, $\mu = \mu(x)$:

$$\frac{\partial}{\partial y}\left[\mu(x)\left[3y + 3y^2\right]\right] = \frac{\partial}{\partial x}\left[\mu(x)\,[2x + 4xy]\right]$$

$$\hookrightarrow \qquad \mu(x)\,[3 + 6y] = \frac{d\mu}{dx}[2x + 4xy] + \mu(x)\,[2 + 4y]$$

$$\hookrightarrow \qquad \frac{d\mu}{dx}[2x + 4xy] = \mu(x)\,[3 + 6y - 2 - 4y]$$

$$\hookrightarrow \qquad \frac{d\mu}{dx}2x\,[1 + 2y] = \mu(x)\,[1 + 2y]$$

$$\hookrightarrow \qquad \frac{d\mu}{dx} = \frac{\mu(x)}{2x} \quad .$$

The last equation is a differential equation for $\mu(x)$ not involving the variable y. A solution (easily found by treating this as a separable or linear differential equation) is

$$\mu(x) = x^{1/2} \quad .$$

This is what we can use as an integrating factor.

For part iii: Multiplying the given differential equation by the integrating factor μ just found:

$$x^{1/2}\left[3y + 3y^2 + [2x + 4xy]\frac{dy}{dx}\right] = x^{1/2}\cdot 0$$

$$\hookrightarrow \qquad 3x^{1/2}\left[y + y^2\right] + 2x^{3/2}[1 + 2y]\frac{dy}{dx} = 0 \quad .$$

Any potential function ϕ of this equation must satisfy

$$\frac{\partial\phi}{\partial x} = 3x^{1/2}\left[y+y^2\right] \qquad \text{and} \qquad \frac{\partial\phi}{\partial y} = 2x^{3/2}\left[1+2y\right] \quad . \tag{$\star$}$$

Integrating the first equation with respect to x :

$$\phi(x,y) = \int \frac{\partial\phi}{\partial x}\,dx = \int 3x^{1/2}\left[y+y^2\right]dx = 2x^{3/2}\left[y+y^2\right] + h(y) \quad .$$

Combining this result with the second equation in ($\star$):

$$2x^{3/2}\left[1+2y\right] = \frac{\partial\phi}{\partial y} = \frac{\partial}{\partial y}\left[2x^{3/2}\left[y+y^2\right]+h(y)\right]$$

$$\hookrightarrow \qquad 2x^{3/2}\left[1+2y\right] = 2x^{3/2}\left[1+2y\right] + h'(y)$$

$$\hookrightarrow \qquad h'(y) = 0 \quad \rightarrowtail \quad h(y) = \int 0\,dy = c_1 \quad .$$

So the potential function is

$$\phi(x,y) = 2x^{3/2}\left[y+y^2\right] + h(y) = 2x^{3/2}\left[y+y^2\right] + c_1 \quad ,$$

and an implicit solution to the given differential equation is

$$-2x^{3/2}\left[y+y^2\right] + c_1 = c_2 \quad ,$$

which we rewrite as

$$2y^2 + 2y + cx^{-3/2} = 0$$

so that we can solve for y using the quadratic formula,

$$y(x) = \frac{-2\pm\sqrt{2^2-4\cdot 2cx^{-3/2}}}{2\cdot 2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1+Cx^{-3/2}} \quad .$$

7.5 g. *For part i:*

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[2y^3\right] = 6y^2$$

and

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[4x^3y^3 - 3xy^2\right] = 12x^2y^3 - 3y^2 \quad .$$

So $\partial M/\partial y \neq \partial N/\partial x$, telling us the equation is not in exact form.

For part ii: We need to find a function $\mu = \mu(x,y)$ such that

$$\frac{\partial}{\partial y}\left[2\mu y^3\right] = \frac{\partial}{\partial y}\left[\mu M\right] = \frac{\partial}{\partial x}\left[\mu N\right] = \frac{\partial}{\partial x}\left[\mu\left[4x^3y^3-3xy^2\right]\right] \quad .$$

Let us see if an integrating factor μ can be a function of x only, $\mu = \mu(x)$:

$$\frac{\partial}{\partial y}\left[2\mu(x)y^3\right] = \frac{\partial}{\partial x}\left[\mu(x)\left[4x^3y^3-3xy^2\right]\right]$$

$$\hookrightarrow \qquad 6\mu(x)y^2 = \frac{d\mu}{dx}\left[4x^3y^3-3xy^2\right] + \mu(x)\left[12x^2y^3-3y^2\right]$$

$$\hookrightarrow \qquad \frac{d\mu}{dx}\left[4x^3y^3 - 3xy^2\right] = -\mu(x)\left[4x^2y^3 - 3y^2 - 6y^2\right]$$

$$\hookrightarrow \qquad \frac{d\mu}{dx}\left[4x^3y^3 - 3xy^2\right] = -3\mu(x)\left[4x^2y^3 - 3y^2\right]$$

$$\hookrightarrow \qquad \frac{d\mu}{dx} = -3\mu(x) \quad .$$

The last equation does not contain the variable y, and is a simple differential equation for $\mu(x)$. A solution (easily found by treating this as a separable or linear differential equation) is

$$\mu(x) = x^{-3} \quad .$$

This is what we can use as an integrating factor.

For part iii: Multiplying the given differential equation by the integrating factor μ just found:

$$x^{-3}\left[2y^3 + \left[4x^3y^3 - 3xy^2\right]\frac{dy}{dx}\right] = x^{-3} \cdot 0$$

$$\hookrightarrow \qquad 2x^{-3}y^3 + \left[4y^3 - 3x^{-2}y^2\right]\frac{dy}{dx} = 0 \quad .$$

Any potential function ϕ of this equation must satisfy

$$\frac{\partial \phi}{\partial x} = 2x^{-3}y^3 \qquad \text{and} \qquad \frac{\partial \phi}{\partial y} = 4y^3 - 3x^{-2}y^2 \quad . \tag{$\star$}$$

Integrating the first equation with respect to x:

$$\phi(x, y) = \int \frac{\partial \phi}{\partial x}\, dx = \int 2x^{-3}y^3\, dx = -x^{-2}y^3 + h(y) \quad .$$

Combining this result with the second equation in ($\star$):

$$4y^3 - 3x^{-2}y^2 = \frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}\left[-x^{-2}y^3 + h(y)\right]$$

$$\hookrightarrow \qquad 4y^3 - 3x^{-2}y^2 = -3x^{-2}y^2 + h'(y)$$

$$\hookrightarrow \qquad h'(y) = 4y^3 \quad \rightarrowtail \quad h(y) = \int 4y^3\, dy = y^4 + c_1 \quad .$$

So the potential function is

$$\phi(x, y) = -x^{-2}y^3 + h(y) = -x^{-2}y^3 + y^4 + c_1 \quad ,$$

and an implicit solution to the given differential equation is

$$-x^{-2}y^3 + y^4 + c_1 = c_2 \quad ,$$

which we rewrite as

$$y^4 - x^{-2}y^3 = c \quad ,$$

but will not attempt to solve for y.

7.5 i. *For part i:* Here

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[6 + 12x^2y^2\right] = 24x^2y$$

and

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[7x^3y + xy^{-1}\right] = 21x^2y + y^{-1} \quad .$$

So ${}^{\partial M}/_{\partial y} \neq {}^{\partial N}/_{\partial x}$, telling us the equation is not in exact form.

For part ii: We need to find a function $\mu = \mu(x, y)$ such that

$$\frac{\partial}{\partial y}\left[\mu\left[6 + 12x^2y^2\right]\right] = \frac{\partial}{\partial y}[\mu M] = \frac{\partial}{\partial x}[\mu N] = \frac{\partial}{\partial x}\left[\mu\left[7x^3y + xy^{-1}\right]\right] \quad .$$

Let us see if an integrating factor μ can be a function of x only, $\mu = \mu(x)$:

$$\frac{\partial}{\partial y}\left[\mu(x)\left[6 + 12x^2y^2\right]\right] = \frac{\partial}{\partial x}\left[\mu(x)\left[7x^3y + xy^{-1}\right]\right]$$

$$\hookrightarrow \qquad \mu(x)\left[24x^2y\right] = \frac{d\mu}{dx}\left[7x^3y + xy^{-1}\right] + \mu(x)\left[21x^2y + y^{-1}\right]$$

$$\hookrightarrow \qquad \frac{d\mu}{dx}\left[7x^3y + xy^{-1}\right] = \mu(x)\left[3x^2y - xy^2\right] \quad .$$

In this case, we clearly cannot cancel out the y variable, and cannot obtain a differential equation for $\mu = \mu(x)$ in terms of x and μ only. So there is not an integrating factor of x alone.

Attempting to find an integrating factor of y only, $\mu = \mu(y)$:

$$\frac{\partial}{\partial y}\left[\mu(y)\left[6 + 12x^2y^2\right]\right] = \frac{\partial}{\partial x}\left[\mu(y)\left[7x^3y + xy^{-1}\right]\right]$$

$$\hookrightarrow \qquad \frac{d\mu}{dy}\left[6 + 12x^2y^2\right] + \mu(y)\left[24x^2y\right] = \mu(y)\left[21x^2y + y^{-1}\right]$$

$$\hookrightarrow \qquad \frac{d\mu}{dy}\left[6 + 12x^2y^2\right] = \mu(y)\left[y^{-1} - 3x^2y\right] \quad .$$

Since we clearly cannot obtain a differential equation for $\mu = \mu(y)$ in terms of y and μ only, there is not an integrating factor of y alone.

Finally, let's attempt to find an integrating factor of the form $\mu = x^\alpha y^\beta$:

$$\frac{\partial}{\partial y}\left[\mu\left[6 + 12x^2y^2\right]\right] = \frac{\partial}{\partial x}\left[\mu\left[7x^3y + xy^{-1}\right]\right]$$

$$\hookrightarrow \qquad \frac{\partial}{\partial y}\left[x^\alpha y^\beta\left[6 + 12x^2y^2\right]\right] = \frac{\partial}{\partial x}\left[x^\alpha y^\beta\left[7x^3y + xy^{-1}\right]\right]$$

$$\hookrightarrow \qquad \frac{\partial}{\partial y}\left[6x^\alpha y^\beta + 12x^{\alpha+2}y^{\beta+2}\right] = \frac{\partial}{\partial x}\left[7x^{\alpha+3}y^{\beta+1} + x^{\alpha+1}y^{\beta-1}\right]$$

$$\hookrightarrow \qquad 6\beta x^\alpha y^{\beta-1} + 12(\beta+2)x^{\alpha+2}y^{\beta+1} = 7(\alpha+3)x^{\alpha+2}y^{\beta+1} + (\alpha+1)x^\alpha y^{\beta-1} \quad .$$

Combining like terms yields

$$[\alpha - 6\beta + 1]\, x^{\alpha} y^{\beta-1} \;+\; [7\alpha - 12\beta - 3]\, x^{\alpha+2} y^{\beta+1} \;=\; 0 \quad ,$$

which requires that

$$\alpha \;-\; 6\beta \;+\; 1 \;=\; 0 \qquad \text{and} \qquad 7\alpha \;-\; 12\beta \;-\; 3 \;=\; 0 \quad .$$

Solving for α and β :

$$\alpha \;-\; 6\beta \;+\; 1 \;=\; 0 \qquad \text{and} \qquad 7\alpha \;-\; 12\beta \;-\; 3 \;=\; 0$$

$$\hookrightarrow \qquad \alpha \;=\; 6\beta \;-\; 1 \qquad \text{and} \qquad 7(6\beta - 1) \;-\; 12\beta \;=\; 3$$

$$\hookrightarrow \qquad \alpha \;=\; 6\beta \;-\; 1 \qquad \text{and} \qquad \beta \;=\; \frac{3+7}{7 \cdot 6 - 12} \;=\; \frac{1}{3}$$

$$\hookrightarrow \qquad \alpha \;=\; 6 \cdot \frac{1}{3} \;-\; 1 \;=\; 1 \qquad \text{and} \qquad \beta \;=\; \frac{1}{3} \quad .$$

So,

$$\mu \;=\; x^{\alpha} y^{\beta} \;=\; x y^{1/3} \quad .$$

For part iii: Multiplying the given differential equation by the integrating factor μ just found:

$$x y^{1/3} \left[6 \;+\; 12x^2 y^2 \;+\; \left[7x^3 y + \frac{x}{y} \right] \frac{dy}{dx} \right] \;=\; x y^{1/3} \cdot 0$$

$$\hookrightarrow \qquad 6x y^{1/3} \;+\; 12x^3 y^{7/3} \;+\; \left[7x^4 y^{4/3} + x^2 y^{-2/3} \right] \frac{dy}{dx} \;=\; 0 \quad .$$

Any potential function ϕ of this equation must satisfy

$$\frac{\partial \phi}{\partial x} \;=\; 6x y^{1/3} \;+\; 12x^3 y^{7/3} \qquad \text{and} \qquad \frac{\partial \phi}{\partial y} \;=\; 7x^4 y^{4/3} \;+\; x^2 y^{-2/3} \quad . \qquad (\star)$$

Integrating the first equation with respect to x :

$$\begin{aligned} \phi(x, y) \;&=\; \int \frac{\partial \phi}{\partial x}\, dx \\ &=\; \int \left[6x y^{1/3} + 12x^3 y^{7/3} \right] dx \;=\; 3x^2 y^{1/3} \;+\; 3x^4 y^{7/3} \;+\; h(y) \quad . \end{aligned}$$

Combining this result with the second equation in $(\star)$:

$$7x^4 y^{4/3} + x^2 y^{-2/3} \;=\; \frac{\partial \phi}{\partial y} \;=\; \frac{\partial}{\partial y} \left[3x^2 y^{1/3} \;+\; 3x^4 y^{7/3} \;+\; h(y) \right]$$

$$\hookrightarrow \qquad 7x^4 y^{4/3} + x^2 y^{-2/3} \;=\; x^2 y^{-2/3} \;+\; 7x^4 y^{4/3} \;+\; h'(y)$$

$$\hookrightarrow \qquad h'(y) \;=\; 0 \quad \rightarrowtail \quad h(y) \;=\; \int 0\, dy \;=\; c_1 \quad .$$

So the potential function is

$$\phi(x, y) \;=\; 3x^2 y^{1/3} \;+\; 3x^4 y^{7/3} \;+\; h(y) \;=\; 3x^2 y^{1/3} \;+\; 3x^4 y^{7/3} \;+\; c_1 \quad ,$$

and an implicit solution to the given differential equation is

$$3x^2 y^{1/3} + 3x^4 y^{7/3} + c_1 = c_2 \quad ,$$

which we rewrite as

$$x^2 y^{1/3} + x^4 y^{7/3} = c \quad ,$$

but will not attempt to solve for y .

7.6 a. If

$$\mu = \mu(x) = e^{\int P(x)\,dx} \qquad \text{with} \quad P = P(x) = \frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} \quad ,$$

then

$$\frac{\partial}{\partial y}[\mu M] = \frac{\partial}{\partial y}\left[e^{\int P(x)\,dx} M(x,y)\right] = e^{\int P(x)\,dx}\frac{\partial M}{\partial y} = \mu\frac{\partial M}{\partial y}$$

and

$$\begin{aligned}
\frac{\partial}{\partial x}[\mu N] &= \frac{\partial}{\partial x}\left[e^{\int P(x)\,dx} N(x,y)\right] \\
&= \left[e^{\int P(x)\,dx} P(x)\right] N(x,y) + e^{\int P(x)\,dx}\frac{\partial N}{\partial x} \\
&= e^{\int P(x)\,dx}\left[PN + \frac{\partial N}{\partial x}\right] \\
&= \mu\left[\left(\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}\right) N + \frac{\partial N}{\partial x}\right] \\
&= \mu\left[\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) + \frac{\partial N}{\partial x}\right] = \mu\frac{\partial M}{\partial y} \quad .
\end{aligned}$$

Thus,

$$\frac{\partial}{\partial y}[\mu M] = \frac{\partial}{\partial x}[\mu N]$$

which, according to the complete test for exactness (Theorem 7.4 on page 137 of the text), ensures that the given μ is an integrating factor for the given equation.