Chapter 6: Simplifying Through Substitution

6.1 a. Let $u = 3x + 3y + 2$. Solving for y and computing y', we then see that

$$3y = u - 3x - 2 \quad\rightarrowtail\quad y = \frac{u}{3} - x - \frac{2}{3}$$

and

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{u}{3} - x - \frac{2}{3}\right] = \frac{1}{3}\frac{du}{dx} - 1 \quad .$$

Substituting this into the differential equation and solving for u, we have

$$\frac{dy}{dx} = \frac{1}{(3x+3y+2)^2} \quad\rightarrowtail\quad \frac{1}{3}\frac{du}{dx} - 1 = \frac{1}{u^2}$$

$$\hookrightarrow \quad \frac{1}{3}\frac{du}{dx} = \frac{1}{u^2} + 1 = \frac{1}{u^2} + \frac{u^2}{u^2} = \frac{1+u^2}{u^2}$$

$$\hookrightarrow \quad \frac{u^2}{1+u^2}\frac{du}{dx} = 3 \quad\rightarrowtail\quad \int \frac{u^2}{1+u^2}\frac{du}{dx}\,dx = \int 3\,dx$$

$$\hookrightarrow \quad \int \frac{u^2}{1+u^2}\,du = 3x + c \quad . \tag{$\star$}$$

Computing the last integral (and ignoring the arbitrary constant):

$$\int \frac{u^2}{1+u^2}\,du = \int \frac{1+u^2-1}{1+u^2}\,du$$

$$= \int \left[1 - \frac{1}{u^2+1}\right] du = u - \arctan(u) \quad .$$

So equation ($\star$) becomes

$$u - \arctan(u) = 3x + c \quad ,$$

which cannot be solved for u. So we now replace each u with its formula in terms of y from the original substitution ($u = 3x + 3y + 2$), and simplify the resulting equation as much as practical:

$$3x + 3y + 2 - \arctan(3x+3y+2) = 3x + c$$

$$\hookrightarrow \quad \arctan(3x+3y+2) = 3y + 2 + c = 3y + C$$

$$\hookrightarrow \quad 3x + 3y + 2 = \tan(3y+C) \quad .$$

6.1 c. Using the substitution $u = 4y - 8x + 3$ we have

$$y = \frac{1}{4}[u + 8x - 3] = \frac{u}{4} + 2x - \frac{3}{4} \quad , \qquad \frac{dy}{dx} = \frac{1}{4}\frac{du}{dx} + 2 \quad ,$$

and

$$\cos(4y-8x+3)\frac{dy}{dx} = 2 + 2\cos(4y-8x+3)$$

$$\hookrightarrow \quad \cos(u)\left[\frac{1}{4}\frac{du}{dx} + 2\right] = 2 + 2\cos(u)$$

$$\hookrightarrow \quad \frac{1}{4}\cos(u)\frac{du}{dx} + 2\cos(u) = 2 + 2\cos(u) \quad \rightarrowtail \quad \cos(u)\frac{du}{dx} = 8$$

$$\hookrightarrow \quad \int \cos(u)\frac{du}{dx}\,dx = \int 8\,dx \quad \rightarrowtail \quad \sin(u) = 8x + c$$

$$\hookrightarrow \quad u = \arcsin(8x+c) \quad .$$

Plugging the last formula for u into the formula for y based on the original substitution yields the final answer,

$$\begin{aligned} y &= \frac{1}{4}[u + 8x - 3] \\ &= \frac{1}{4}[\arcsin(8x+c) + 8x - 3] = \frac{1}{4}\arcsin(8x+c) + 2x - \frac{3}{4} \quad . \end{aligned}$$

6.3 a. Rewriting the differential equation:

$$x^2\frac{dy}{dx} - xy = y^2 \quad \rightarrowtail \quad \frac{dy}{dx} = \frac{y}{x} + \left(\frac{y}{x}\right)^2 \quad .$$

Since the equation is homogeneous, our substitution is based on $u = \frac{y}{x}$, from which we derive

$$y = xu \qquad \text{and} \qquad \frac{dy}{dx} = \frac{d}{dx}[xu] = \frac{dx}{dx}\cdot u + x\frac{du}{dx} = u + x\frac{du}{dx} \quad .$$

Using this with our differential equation:

$$\frac{dy}{dx} = \frac{y}{x} + \left(\frac{y}{x}\right)^2 \quad \rightarrowtail \quad u + x\frac{du}{dx} = u + u^2 \quad \rightarrowtail \quad \frac{du}{dx} = \frac{u^2}{x} \quad .$$

Obviously, $u = 0$ is the only constant solution, which, in turn, also yields

$$y = xu = x \cdot 0 = 0$$

as a constant solution to the original equation. For the other solutions:

$$\frac{du}{dx} = \frac{u^2}{x} \quad \rightarrowtail \quad u^{-2}\frac{du}{dx} = \frac{1}{x} \quad \rightarrowtail \quad \int u^{-2}\frac{du}{dx}\,dx = \int \frac{1}{x}\,dx$$

$$\hookrightarrow \quad -u^{-1} = \ln|x| + c \quad \rightarrowtail \quad u = \frac{-1}{\ln|x| + c} \quad .$$

This, along with our formula for y in terms of u, yields

$$y = xu = x\left[\frac{-1}{\ln|x| + c}\right] = \frac{-x}{\ln|x| + c} \quad ,$$

which, along with $y = 0$, describes all the solutions.

6.3 c. The substitution: $\quad u = \frac{y}{x} \quad \rightarrowtail \quad y = xu$

$$\hookrightarrow \quad \frac{dy}{dx} = \frac{d}{dx}[xu] = u + x\frac{du}{dx} \quad .$$

So,

$$\cos\left(\frac{y}{x}\right)\left[\frac{dy}{dx} - \frac{y}{x}\right] = 1 + \sin\left(\frac{y}{x}\right)$$

$$\hookrightarrow \qquad \cos(u)\left[\left(u + x\frac{du}{dx}\right) - u\right] = 1 + \sin(u)$$

$$\hookrightarrow \qquad \frac{du}{dx} = \frac{1+\sin(u)}{x\cos(u)}$$

The constant solutions of this are given by those values of u where $1+\sin(u) = 0$; that is,

$$u = \arcsin(-1) = 2n\pi - \frac{\pi}{2} \qquad \text{for} \quad n = 0,\ \pm 1,\ \pm 2,\ \dots \quad .$$

For the nonconstant solutions:

$$\frac{du}{dx} = \frac{1+\sin(u)}{x\cos(u)} \quad \rightarrowtail \quad \frac{\cos(u)}{1+\sin(u)}\frac{du}{dx} = \frac{1}{x}$$

$$\hookrightarrow \quad \int \frac{\cos(u)}{1+\sin(u)}\frac{du}{dx}\,dx = \int \frac{1}{x}\,dx \quad \rightarrowtail \quad \ln|1+\sin(u)| = \ln|x| + c$$

$$\hookrightarrow \quad 1 + \sin(u) = \pm e^{\ln|x|+c} = Ax \quad \rightarrowtail \quad u = \arcsin(Ax - 1) \quad .$$

Note that the last formula for u also yields the constant solutions. So all the solutions to our original differential equation are given by

$$y = xu = x\arcsin(Ax - 1) \quad .$$

6.5 a. First, observe that $y = 0$ is clearly a constant solution.
Here, $n = 3$ (from the $3y^3$ term). So the substitution is based on $u = y^{1-3} = y^{-2}$. Hence, we also have

$$y = \pm u^{-1/2} \qquad \text{and} \qquad \frac{dy}{dx} = \frac{d}{dx}\left[\pm u^{-1/2}\right] = \pm\left[-\frac{1}{2}u^{-3/2}\frac{du}{dx}\right] \quad .$$

Using this substitution:

$$\frac{dy}{dx} + 3y = 3y^3 \quad \rightarrowtail \quad -\frac{1}{2}u^{-3/2}\frac{du}{dx} + 3u^{-1/2} = 3\left(u^{-1/2}\right)^3$$

$$\hookrightarrow \qquad \left(-2u^{3/2}\right)\left[-\frac{1}{2}u^{-3/2}\frac{du}{dx} + 3u^{-1/2} = 3u^{-3/2}\right]$$

$$\hookrightarrow \qquad \frac{du}{dx} - 6u = -6 \quad .$$

The integrating factor for the last differential equation is

$$\mu(x) = e^{\int(-6)\,dx} = e^{-6x}$$

Using this with the last differential equation, above:

$$e^{-6x}\left[\frac{du}{dx} - 6u = -6\right] \quad \rightarrowtail \quad e^{-6x}\frac{du}{dx} - 6e^{-6x}u = -6e^{-6x}$$

$$\hookrightarrow \quad \frac{d}{dx}\left[e^{-6x}u\right] = -6e^{-6x} \quad \rightarrowtail \quad e^{-6x}u = -\int 6e^{-6x}\,dx = e^{-6x} + c$$

$$\hookrightarrow \qquad u = 1 + ce^{6x} \quad .$$

This with the original substitution then gives

$$y \;=\; \pm u^{-1/2} \;=\; \pm\left(1 \;+\; ce^{6x}\right)^{-1/2} \quad .$$

So any solution to the original differential equation is given by the above formula for y or $y = 0$.

6.5 c. Again, we note that $y = 0$ is a solution.
The substitution:

$$u \;=\; y^{1-2/3} \;=\; y^{1/3} \quad\rightarrowtail\quad y \;=\; u^3 \quad\rightarrowtail\quad \frac{dy}{dx} \;=\; 3u^2\frac{du}{dx} \quad .$$

Using the substitution to find the linear equation for u:

$$\frac{dy}{dx} \;+\; 3\cot(x)y \;=\; 6\cos(x)y^{2/3}$$

$$\hookrightarrow \qquad 3u^2\frac{du}{dx} \;+\; 3\cot(x)u^3 \;=\; 6\cos(x)\left(u^3\right)^{2/3}$$

$$\hookrightarrow \qquad \frac{du}{dx} \;+\; \cot(x)u \;=\; 2\cos(x) \quad .$$

The integrating factor is given by

$$\mu(x) \;=\; e^{\int \cot(x)\,dx} \;=\; \exp\left(\int \frac{\cos(x)}{\sin(x)}\,dx\right) \;=\; \exp(\ln|\sin(x)|) \;=\; |\sin(x)| \quad .$$

As noted in Chapter 5, we can simply use $\mu = \sin(x)$. Doing so:

$$\sin(x)\left[\frac{du}{dx} \;+\; \cot(x)u \;=\; 2\cos(x)\right]$$

$$\hookrightarrow \qquad \frac{d}{dx}[\sin(x)u] \;=\; 2\sin(x)\cos(x)$$

$$\hookrightarrow \qquad \sin(x)u \;=\; \int 2\sin(x)\cos(x)\,dx \;=\; \sin^2(x) \;+\; c$$

$$\hookrightarrow \qquad u \;=\; \sin(x) \;+\; \frac{c}{\sin(x)}$$

$$\hookrightarrow \qquad y \;=\; u^3 \;=\; \left(\sin(x) \;+\; \frac{c}{\sin(x)}\right)^3 \quad .$$

The last equation, along with $y = 0$, describes all solutions to our original differential equation.

6.7 a. This is obviously in the form

$$\frac{dy}{dx} \;=\; \text{formula of } \frac{y}{x} \quad .$$

So it is a homogeneous equation, and we use the corresponding substitution,

$$u \;=\; \frac{y}{x} \quad , \quad y \;=\; xu \quad \text{and} \quad \frac{dy}{dx} \;=\; \frac{d}{dx}[xu] \;=\; u \;+\; x\frac{du}{dx} \quad .$$

Doing so:

$$\frac{dy}{dx} = \frac{y}{x} + \left(\frac{x}{y}\right)^2 \quad\rightarrowtail\quad u + x\frac{du}{dx} = u + u^{-2}$$

$$\hookrightarrow \qquad u^2\frac{du}{dx} = \frac{1}{x} \quad\rightarrowtail\quad \int u^2\frac{du}{dx}\,dx = \int \frac{1}{x}\,dx$$

$$\hookrightarrow \qquad \frac{1}{3}u^3 = \ln|x| + c \quad\rightarrowtail\quad u = (3\ln|x| + 3c)^{1/3}$$

$$\hookrightarrow \qquad y = xu = x\,(3\ln|x| + C)^{1/3} \quad .$$

6.7 c. Since the equation is

$$\frac{dy}{dx} + \frac{2}{x}y = 4y^{1/2} \quad ,$$

it is a Bernoulli equation with $n = 1/2$. One solution is $y = 0$. The others are obtained using the substitution

$$u = y^{1-1/2} = y^{1/2} \quad .$$

Hence,

$$y = u^2 \quad , \quad \frac{dy}{dx} = 2u\frac{du}{dx} \quad ,$$

and

$$\frac{dy}{dx} + \frac{2}{x}y = 4y^{1/2} \quad\rightarrowtail\quad 2u\frac{du}{dx} + \frac{2}{x}u^2 = 4\left(u^2\right)^{1/2}$$

$$\hookrightarrow \qquad \frac{du}{dx} + \frac{1}{x}u = 2 \quad .$$

The integrating factor is $\mu(x) = e^{\int 1/x\,dx} = \cdots = x$.
So

$$\frac{du}{dx} + \frac{1}{x}u = 2 \quad\rightarrowtail\quad x\left[\frac{du}{dx} + \frac{1}{x}u = 2\right]$$

$$\hookrightarrow \qquad \frac{d}{dx}[xu] = 2x \quad\rightarrowtail\quad xu = \int 2x\,dx = x^2 + c$$

$$\hookrightarrow \qquad u = x + \frac{c}{x} \quad\rightarrowtail\quad y = u^2 = \left(x + \frac{c}{x}\right)^2 \quad .$$

6.7 e. The $y - x$ factor suggests using the linear substitution $u = y - x$. Then

$$y = u + x \quad \text{and} \quad \frac{dy}{dx} = \frac{d}{dx}[u + x] = \frac{du}{dx} + 1 \quad .$$

Using this,

$$(y - x)\frac{dy}{dx} = 1 \quad\rightarrowtail\quad u\left[\frac{du}{dx} + 1\right] = 1$$

$$\hookrightarrow \qquad u\frac{du}{dx} = 1 - u \quad\rightarrowtail\quad \frac{du}{dx} = \frac{1 - u}{u} \quad .$$

Note that the derivative of u is zero if and only if $u = 1$. So $u = 1$ is the only constant solution to this last equation. The corresponding solution to the original equation is then

$$y = u + x = 1 + x \quad .$$

For the other solutions, we'll need

$$\int \frac{u}{1-u}\,du \;=\; -\int \frac{1-u-1}{1-u}\,du$$
$$=\; -\int \left[1 - \frac{1}{1-u}\right] du \;=\; -u \;-\; \ln|1-u| \;+\; C \quad .$$

Back to the last differential equation above:

$$\frac{du}{dx} \;=\; \frac{1-u}{u} \quad\rightarrowtail\quad \frac{u}{1-u}\frac{du}{dx} \;=\; 1$$

$$\hookrightarrow \qquad \int \frac{u}{1-u}\frac{du}{dx}\,dx \;=\; \int 1\,dx \quad\rightarrowtail\quad -u \;-\; \ln|1-u| \;=\; x \;+\; c$$

$$\hookrightarrow \qquad -(y-x) \;-\; \ln|1-(y-x)| \;=\; x \;+\; c$$

$$\hookrightarrow \qquad \ln|1-y+x| \;=\; -c \;-\; y \quad .$$

$$\hookrightarrow \qquad 1 \;-\; y \;+\; x \;=\; \pm e^{-c-y} \;=\; Ae^{-y} \quad\rightarrowtail\quad y \;=\; 1 \;+\; x \;-\; Ae^{-y} \quad .$$

This last equation describes all solutions since it does reduce to the solution $y(x) = 1 + x$ when $A = 0$.

6.7 g.

$$\left(2xy + 2x^2\right)\frac{dy}{dx} \;=\; x^2 \;+\; 2xy \;+\; 2y^2$$

$$\hookrightarrow \qquad \frac{dy}{dx} \;=\; \frac{x^2+2xy+2y^2}{2xy+2x^2} \;=\; \frac{x^2\left[1+2\frac{y}{x}+2\left(\frac{y}{x}\right)^2\right]}{x^2\left[2\frac{y}{x}+2\right]} \;=\; \frac{1+2\frac{y}{x}+2\left(\frac{y}{x}\right)^2}{2\frac{y}{x}+2} \quad .$$

So the equation is homogeneous, and we use the substitution $\quad u \;=\; \frac{y}{x} \quad .$
Hence

$$y \;=\; xu \quad , \quad \frac{dy}{dx} \;=\; \frac{d}{dx}[xu] \;=\; u \;+\; x\frac{du}{dx} \quad ,$$

and

$$\frac{dy}{dx} \;=\; \frac{1+2\frac{y}{x}+2\left(\frac{y}{x}\right)^2}{2\frac{y}{x}+2} \quad\rightarrowtail\quad u \;+\; x\frac{du}{dx} \;=\; \frac{1+2u+2u^2}{2u+2}$$

$$\hookrightarrow \qquad x\frac{du}{dx} \;=\; \frac{1+2u+2u^2}{2u+2} \;-\; u \;=\; \frac{1+2u+2u^2}{2u+2} \;-\; \frac{2u^2+2u}{2u+2} \;=\; \frac{1}{2u+2}$$

$$\hookrightarrow \qquad (2u+2)\frac{du}{dx} \;=\; \frac{1}{x} \quad\rightarrowtail\quad \int (2u+2)\frac{du}{dx}\,dx \;=\; \int \frac{1}{x}\,dx$$

$$\hookrightarrow \qquad u^2 \;+\; 2u \;=\; \ln|x| \;+\; c$$

$$\hookrightarrow \qquad u^2 \;+\; 2u \;-\; (\ln|x| + c) \;=\; 0$$

$$\hookrightarrow \qquad u = \frac{-2 \pm \sqrt{2^2 - 4[-(\ln|x| + c)]}}{2} = -1 \pm \sqrt{\ln|x| + C}$$

$$\hookrightarrow \qquad y = xu = x\left[-1 \pm \sqrt{\ln|x| + C}\right] = -x \pm x\sqrt{\ln|x| + C} \quad .$$

6.7 i. Clearly, the substitution $u = 2x + y - 3$ is in order. With this,

$$y = u - 2x + 3 \quad , \quad \frac{dy}{dx} = \frac{d}{dx}[u - 2x + 3] = \frac{du}{dx} - 2 \quad ,$$

and

$$\frac{dy}{dx} = 2\sqrt{2x + y - 3} - 2 \quad \rightarrowtail \quad \frac{du}{dx} - 2 = 2\sqrt{u} - 2$$

$$\hookrightarrow \qquad \frac{du}{dx} = 2\sqrt{u} \quad .$$

Note that this last equation has one constant solution $u = 0$. Corresponding to this is the solution to the original equation

$$y = u - 2x + 3 = 0 - 2x + 3 = 3 - 2x \quad .$$

For the other solutions, we continue the above computations:

$$\frac{du}{dx} = \sqrt{u} \quad \rightarrowtail \quad \int u^{-1/2}\frac{du}{dx}\,dx = \int 2\,dx$$

$$\hookrightarrow \qquad 2u^{1/2} = 2x + C \quad \rightarrowtail \quad u = (x + c)^2$$

$$\hookrightarrow \qquad y = u - 2x + 3 = (x + c)^2 - 2x + 3 \quad .$$

So each solution is given by either the last formula or by $y = 3 - 2x$.

6.7 k.

$$x\frac{dy}{dx} - y = \sqrt{xy + x^2}$$

$$\hookrightarrow \qquad \frac{dy}{dx} = \frac{y + \sqrt{xy + x^2}}{x} \quad \rightarrowtail \quad \frac{dy}{dx} = \frac{y}{x} + \left(\frac{y}{x} + 1\right)^{1/2} \quad .$$

So the equation is homogeneous, and we use the substitution $u = \dfrac{y}{x}$.
Hence

$$y = xu \quad , \quad \frac{dy}{dx} = \frac{d}{dx}[xu] = u + x\frac{du}{dx} \quad ,$$

and

$$\frac{dy}{dx} = \frac{y}{x} + \left(\frac{y}{x} + 1\right)^{1/2} \quad \rightarrowtail \quad u + x\frac{du}{dx} = u + (u + 1)^{1/2}$$

$$\hookrightarrow \qquad \frac{du}{dx} = \frac{1}{x} \cdot (u + 1)^{1/2} \quad . \qquad (\star)$$

Note that the derivative of u is zero if $u = -1$. So $u = -1$ is a constant solution, and the corresponding solution to our original equation is

$$y = xu = x(-1) = -x \quad .$$

For the rest of the solutions, we divide equation ($\star$) by $(u+1)^{1/2}$ and continue the process:

$$(u+1)^{-1/2}\frac{du}{dx} = \frac{1}{x} \quad\rightarrowtail\quad \int (u+1)^{-1/2}\frac{du}{dx}\,dx = \int \frac{1}{x}\,dx$$

$$\hookrightarrow \qquad 2(u+1)^{1/2} = \ln|x| + C \quad\rightarrowtail\quad u = \left(\tfrac{1}{2}\ln|x| + c\right)^2 - 1$$

$$\hookrightarrow \qquad y = xu = x\left[\left(\tfrac{1}{2}\ln|x| + c\right)^2 - 1\right] = x\left(\tfrac{1}{2}\ln|x| + c\right)^2 - x \quad .$$

So every solution is given by either the last formula for y or by $y = -x$.

6.7 m. Clearly, the linear substitution $u = x - y + 3$ is appropriate. Using this,

$$y = x - u + 3 \qquad , \qquad \frac{dy}{dx} = \frac{d}{dx}[x - u + 3] = 1 - \frac{du}{dx} \quad ,$$

and

$$\frac{dy}{dx} = (x - y + 3)^2 \quad\rightarrowtail\quad 1 - \frac{du}{dx} = u^2$$

$$\hookrightarrow \qquad \frac{du}{dx} = 1 - u^2 \quad .$$

Here, the derivative of u is zero if $u = \pm 1$. So this differential equation has two constant solutions $u = 1$ and $u = -1$. Since $y = x - u + 3$, the corresponding solutions to the original equation are

$$y = x - 1 + 3 = x + 2 \qquad \text{and} \qquad y = x - (-1) + 3 = x + 4 \quad .$$

To find the other solutions, we divide the last differential equation above by $1 - u^2$ and proceed as usual, using partial fractions to rewrite $(1 - u^2)^{-1}$ in more convenient form for integration:

$$\frac{1}{1-u^2}\frac{du}{dx} = 1 \quad\rightarrowtail\quad \int \frac{1}{1-u^2}\frac{du}{dx}\,dx = \int 1\,dx$$

$$\hookrightarrow \qquad \int \frac{1}{(1+u)(1-u)}\,dx = x + c \quad\rightarrowtail\quad \cdots$$

$$\hookrightarrow \qquad \int \frac{1}{2}\left[\frac{1}{1+u} + \frac{1}{1-u}\right]dx = x + c$$

$$\hookrightarrow \qquad \frac{1}{2}\left[\ln|1+u| - \ln|1-u|\right] = x + c$$

$$\hookrightarrow \qquad \ln\left|\frac{1+u}{1-u}\right| = 2x + 2c \quad\rightarrowtail\quad \frac{1+u}{1-u} = \pm e^{2s+2c} = Ae^{2x}$$

$$\hookrightarrow \qquad 1 + u = Ae^{2x} - uAe^{2x} \quad\rightarrowtail\quad u = \frac{Ae^{2x} - 1}{Ae^{2x} + 1}$$

$$\hookrightarrow \qquad y(x) = x - u + 3 = x + 3 + \frac{Ae^{2x} - 1}{Ae^{2x} + 1} \quad .$$

Note that the last formula for y reduces to the constant solution $y = x + 2$ if $A = 0$, but does not reduce to the constant solution $y = x + 4$ for any value of A. So to describe all solutions, we need both

$$y = x + 3 + \frac{Ae^{2x} - 1}{Ae^{2x} + 1} \qquad \text{and} \qquad y = x + 4 \quad .$$

6.7 o. We need a formula of y and maybe x, $f(x, y)$, such that the differential equation reduces to a 'simple' differential equation for u when we let $u = f(x, y)$. Because the given equation involves both $\sin(y)$ and $\cos(y)$, two possible choices for the substitution are obviously

$$u = \cos(y) \qquad \text{and} \qquad u = \sin(y) \quad .$$

Differentiating these yield, respectively,

$$\frac{du}{dx} = -\sin(y)\frac{dy}{dx} \qquad \text{and} \qquad \frac{du}{dx} = \cos(y)\frac{dy}{dx} \quad ,$$

the second of which matches exactly the left side of the given differential equation. So let's use the second choice,

$$u = \sin(y) \qquad \text{with} \qquad \frac{du}{dx} = \cos(y)\frac{dy}{dx}$$

as our substitution. Doing so, we get

$$\cos(y)\,\frac{dy}{dx} = e^{-x} - \sin(y) \quad \rightarrowtail \quad \frac{du}{dx} = e^{-x} - u$$

$$\hookrightarrow \qquad \frac{du}{dx} + u = e^{-x} \quad .$$

This is a simple linear equation with integrating factor $\mu(x) = e^{\int 1\,dx} = e^{x}$.
Multiplying our differential equation for u by this integrating factor and continuing:

$$e^{x}\left[\frac{du}{dx} + u = e^{-x}\right] \quad \rightarrowtail \quad \frac{d}{dx}\left[e^{x}u\right] = 1$$

$$\hookrightarrow \qquad e^{x}u = \int 1\,dx = x + c \quad \rightarrowtail \quad u = (x + c)e^{-x}$$

$$\hookrightarrow \qquad \sin(y) = (x + c)e^{-x} \quad \rightarrowtail \quad y = \arcsin\big((x + c)e^{-x}\big) \quad .$$