Chapter 5: Linear First-Order Equations

5.1 a. Dividing through by x^2 we get $\frac{dy}{dx} + 3y = x^{-2}\sin(x)$,

which is the standard form $\frac{dy}{dx} + p(x)y = q(x)$

with $p(x) = 3$ and $q(x) = x^{-2}\sin(x)$.

So the equation is linear.

5.1 c. Because of the y^2, this cannot be put into the standard form

$$\frac{dy}{dx} + p(x)y = q(x) \quad .$$

So the differential equation is not linear.

5.1 e. $\frac{dy}{dx} = 1 + xy + 3y = 1 + (x+3)y \quad \rightarrowtail \quad \frac{dy}{dx} - (x+3)y = 1$.

The last equation is in standard form for a first-order linear differential equation (with $p(x) = -(x+3)$ and $q(x) = 1$). So the equation is linear.

5.1 g. $\frac{dy}{dx} - e^{2x} = 0 \quad \rightarrowtail \quad \frac{dy}{dx} + 0 \cdot y = e^{2x}$.

This equation is linear since it can be rewritten in standard form

$$\frac{dy}{dx} + p(x)y = q(x) \qquad \text{with} \quad p(x) = 0 \quad \text{and} \quad q(x) = e^{2x} \quad .$$

5.1 i. This equation cannot be put into standard form because of the y^3 term. So it is not linear.

5.2 a. The equation is already in standard form with $p(x) = 2$.
The corresponding integrating factor is

$$\mu(x) = e^{\int p(x)\,dx} = e^{\int 2\,dx} = e^{2x} \quad .$$

Multiplying the differential equation by the integrating factor and proceeding with the procedure:

$$\mu\left[\frac{dy}{dx} + py = q\right] \quad \rightarrowtail \quad e^{2x}\left[\frac{dy}{dx} + 2y = 6\right]$$

$$\hookrightarrow \qquad \underbrace{e^{2x}\frac{dy}{dx} + 2e^{2x}y}_{\frac{d}{dx}\left[e^{2x}y\right]} = 6e^{2x} \quad \rightarrowtail \quad \frac{d}{dx}\left[e^{2x}y\right] = 6e^{2x}$$

$$\hookrightarrow \qquad e^{2x}y = \int \frac{d}{dx}\left[e^{2x}y\right] dx = \int 6e^{2x}\,dx = 3e^{2x} + c$$

$$\hookrightarrow \qquad y = e^{-2x}\left[3e^{2x} + c\right] = 3 + ce^{-2x} \quad .$$

5.2 c. Get the equation into standard form:

$$\frac{dy}{dx} = 4y + 16x \quad \rightarrowtail \quad \frac{dy}{dx} - 4y = 16x \quad .$$

Find the corresponding integrating factor:

$$\mu(x) = e^{\int p(x)\,dx} = e^{\int (-4)\,dx} = e^{-4x} \quad .$$

Multiply the differential equation in standard form by μ and proceed:

$$e^{-4x}\left[\frac{dy}{dx} - 4y = 16x\right] \quad \rightarrowtail \quad e^{-4x}\frac{dy}{dx} - 4e^{-4x}y = 16xe^{-4x}$$

$$\hookrightarrow \qquad \frac{d}{dx}\left[e^{-4x}y\right] = 16xe^{-4x}$$

$$\hookrightarrow \qquad e^{-4x}y = \int \frac{d}{dx}\left[e^{-4x}y\right] dx = \int 16xe^{-4x}\,dx \quad . \qquad (\star)$$

For the last integral, we use integration by parts $\left(\int u\,dv = uv - \int v\,du\right.$ with $u = x$ and $\left.dv = 16e^{-4x}\right)$:

$$\int 16xe^{-4x}\,dx = x\left(-4e^{-4x}\right) - \int \left(-4e^{-4x}\right) dx = -4xe^{-4x} - e^{-4x} + c \quad .$$

Plugging this back into equation ($\star$) and solving for y:

$$e^{-4x}y = -4xe^{-4x} - e^{-4x} + c$$

$$\hookrightarrow \qquad y = e^{4x}\left[-4xe^{-4x} - e^{-4x} + c\right] = -4x - 1 + ce^{4x} \quad .$$

5.2 e. Standard form: $$x\frac{dy}{dx} + 3y - 10x^2 = 0$$

$$\hookrightarrow \qquad x\frac{dy}{dx} + 3y = 10x^2 \quad \rightarrowtail \quad \frac{dy}{dx} + \frac{3}{x}y = 10x \quad .$$

Integrating factor: $\mu(x) = e^{\int 3/x\,dx} = e^{3\ln|x|} = |x|^3$.
However, as noted on page 99 of the text, we can simply continue with $\mu(x) = x^3$ as the integrating factor:

$$x^3\left[\frac{dy}{dx} + \frac{3}{x}y = 10x\right]$$

$$\hookrightarrow \qquad x^3\frac{dy}{dx} + 3x^2 = 10x^4 \quad \rightarrowtail \quad \frac{d}{dx}\left[x^3y\right] = 10x^4$$

$$\hookrightarrow \qquad x^3y = \int \frac{d}{dx}\left[x^3y\right] dx = \int 10x^4\,dx = 2x^5 + c$$

$$\hookrightarrow \qquad y = x^{-3}\left[2x^5 + c\right] = 2x^2 + cx^{-3} \quad .$$

5.2 g. Standard form: $x\dfrac{dy}{dx} = \sqrt{x} + 3y \quad \rightarrowtail \quad \dfrac{dy}{dx} - \dfrac{3}{x}y = x^{-1/2}$.

Integrating factor: $\mu(x) \ = \ e^{\int \left(-3/x\right) dx} \ = \ e^{-3\ln|x|} \ = \ |x|^{-3}$.
Continuing with $\mu(x) = x^{-3}$ as the integrating factor:

$$x^{-3}\left[\frac{dy}{dx} - \frac{3}{x}y \ = \ x^{-1/2}\right]$$

$$\hookrightarrow \quad x^{-3}\frac{dy}{dx} - 3x^{-4}y \ = \ x^{-7/2} \ \rightarrowtail \ \frac{d}{dx}\left[x^{-3}y\right] \ = \ x^{-7/2}$$

$$\hookrightarrow \quad x^{-3}y \ = \ \int \frac{d}{dx}\left[x^{-3}y\right] dx \ = \ \int x^{-7/2}\,dx \ = \ -\tfrac{2}{5}x^{-5/2} + c$$

$$\hookrightarrow \quad y \ = \ x^3\left[c - \tfrac{2}{5}x^{-5/2}\right] \ = \ cx^3 - \tfrac{2}{5}x^{1/2} \quad .$$

5.2 i. Standard form: $x\frac{dy}{dx} + (5x+2)y \ = \ \frac{20}{x} \ \rightarrowtail \ \frac{dy}{dx} + \left(5 + \frac{2}{x}\right)y \ = \ \frac{20}{x^2}$.

Integrating factor: $\mu(x) \ = \ e^{\int \left[5 + 2/x\right] dx} \ = \ e^{5x + 2\ln x} \ = \ e^{5x}x^2$.

Continuing with the procedure:

$$e^{5x}x^2\left[\frac{dy}{dx} + \left(5 + \frac{2}{x}\right)y \ = \ \frac{20}{x^2}\right] \ \rightarrowtail \ \frac{d}{dx}\left[e^{5x}x^2y\right] \ = \ 20e^{5x}$$

$$\hookrightarrow \quad e^{5x}x^2y \ = \ \int \frac{d}{dx}\left[e^{5x}x^2y\right] dx \ = \ \int 20e^{5x}\,dx \ = \ 4e^{5x} + c$$

$$\hookrightarrow \quad y \ = \ e^{-5x}x^{-2}\left[4e^{5x} + c\right] \ = \ x^{-2}\left[4 + ce^{-5x}\right] \quad .$$

5.3 a. First, find the general solution:

$$\mu(x) \ = \ e^{\int (-3)\,dx} \ = \ e^{-3x}$$

$$\hookrightarrow \quad e^{-3x}\left[\frac{dy}{dx} - 3y \ = \ 6\right] \ \rightarrowtail \ \frac{d}{dx}\left[e^{-3x}y\right] \ = \ 6e^{-3x}$$

$$\hookrightarrow \quad e^{-3s}y \ = \ \int 6e^{-3x}\,dx \ = \ -2e^{-3x} + c \ \rightarrowtail \ y \ = \ ce^{3x} - 2 \quad .$$

Then apply the initial condition:

$$5 \ = \ y(0) \ = \ ce^{3\cdot 0} - 2 \ = \ c - 2 \ \rightarrowtail \ c \ = \ 5 + 2 \ = \ 7 \quad .$$

So the solution is $y(x) = 7e^{3x} - 2$.

5.3 c. General solution:

$$\mu(x) \ = \ e^{\int 5\,dx} \ = \ e^{5x}$$

$$\hookrightarrow \quad e^{5x}\left[\frac{dy}{dx} + 5y \ = \ e^{-3x}\right] \ \rightarrowtail \ \frac{d}{dx}\left[e^{5x}y\right] \ = \ e^{2x}$$

$$\hookrightarrow \quad e^{5x}y \ = \ \int e^{2x}\,dx \ = \ \tfrac{1}{2}e^{2x} + c \ \rightarrowtail \ y \ = \ \tfrac{1}{2}e^{-3x} + ce^{-5x} \quad .$$

Apply the initial condition:

$$0 = y(0) = \frac{1}{2}e^{-3\cdot 0} + ce^{-5\cdot 0} = \frac{1}{2} + c \quad \rightarrowtail \quad c = -\frac{1}{2} \quad .$$

So the solution is $y(x) = \frac{1}{2}e^{-3x} - \frac{1}{2}e^{-5x}$.

5.3 e. General solution:

$$x\frac{dy}{dx} = y + x^2\cos(x) \quad \rightarrowtail \quad \frac{dy}{dx} - \frac{1}{x}y = x\cos(x)$$

$$\hookrightarrow \quad \mu(x) = e^{\int \left(-1/x\right) dx} = e^{-\ln|x|} = |x|^{-1} \quad \left(\text{use } \mu(x) = x^{-1}\right)$$

$$\hookrightarrow \quad x^{-1}\left[\frac{dy}{dx} - \frac{1}{x}y = x\cos(x)\right] \quad \rightarrowtail \quad \frac{d}{dx}\left[x^{-1}y\right] = \cos(x)$$

$$\hookrightarrow \quad x^{-1}y = \int \cos(x)\, dx = \sin(x) + c \quad \rightarrowtail \quad y = x\sin(x) + cx \quad .$$

Apply the initial condition:

$$0 = y\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) + c\cdot\frac{\pi}{2} = \frac{\pi}{2}\cdot 1 + c\cdot\frac{\pi}{2} \quad \rightarrowtail \quad c = -1 \quad .$$

Hence, $y(x) = x\sin(x) - x$.

5.4 a.

$$\mu(x) = e^{\int 6x\, dx} = e^{3x^2}$$

$$\hookrightarrow \quad e^{3x^2}\left[\frac{dy}{dx} + 6xy = \sin(x)\right] \quad \rightarrowtail \quad \frac{d}{dx}\left[e^{3x^2}y\right] = e^{3x^2}\sin(x)$$

$$\hookrightarrow \quad \int_0^x \frac{d}{ds}\left[e^{3s^2}y\right] ds = \int_0^x e^{3s^2}\sin(s)\, ds$$

$$\hookrightarrow \quad e^{3x^2}y(x) - \underbrace{e^{3\cdot 0^2}y(0)}_{=1\cdot 4} = \int_0^x e^{3s^2}\sin(s)\, ds$$

$$\hookrightarrow \quad e^{3x^2}y(x) = 4 + \int_0^x e^{3s^2}\sin(s)\, ds$$

$$\hookrightarrow \quad y = e^{-3x^2}\left[4 + \int_0^x e^{3s^2}\sin(s)\, ds\right] \quad .$$

5.4 c.

$$x\frac{dy}{dx} - y = x^2e^{-x^2} \quad \rightarrowtail \quad \frac{dy}{dx} - \frac{1}{x}y = xe^{-x^2}$$

$$\hookrightarrow \quad \mu(x) = e^{\int \left(-1/x\right) dx} = \cdots = \frac{1}{x}$$

$$\hookrightarrow \quad \frac{1}{x}\left[\frac{dy}{dx} - \frac{1}{x}y = xe^{-x^2}\right] \quad \rightarrowtail \quad \frac{d}{dx}\left[\frac{y}{x}\right] = e^{-x^2}$$

$$\hookrightarrow \quad \int_3^x \frac{d}{ds}\left[\frac{y}{s}\right] ds = \int_3^x e^{-s^2}\, ds$$

$$\hookrightarrow \quad \frac{y(x)}{x} - \frac{y(3)}{3} = \int_3^x e^{-s^2}\,ds \quad \rightarrowtail \quad \frac{y(x)}{x} - \frac{8}{3} = \int_3^x e^{-s^2}\,ds$$

$$\hookrightarrow \qquad y(x) = x\left[\frac{8}{3} + \int_3^x e^{-s^2}\,ds\right] \quad .$$