Chapter 40: Numerical Methods III: Systems and Higher-Order Equations

40.1 a. $\begin{array}{l} x' = -x + 2y \\ y' = x - 2y \end{array} \longmapsto \begin{array}{l} f(t,x,y) = -x + 2y \\ g(t,x,y) = x - 2y \end{array}$.

From the initial conditions and given parameters

$$t_0 = 0 \quad , \quad x_0 = 4 \quad , \quad y_0 = 1 \quad , \quad N = 6 \quad , \quad \Delta t = \frac{1}{2}$$

and

$$t_{\max} = t_0 + N\Delta t = 3 \quad .$$

Since we are doing these computations "by hand", let us do a little "pre-computing" to save time and space later:

$$t_{k+1} = t_k + \Delta x = t_k + \frac{1}{2} \quad ,$$

$$x_{k+1} = x_k + \Delta t \cdot f(t_k, x_k, y_k)$$

$$= x_k + \frac{1}{2}[-x_k + 2y_k] = \frac{1}{2}x_k + y_k$$

and

$$y_{k+1} = y_k + \Delta t \cdot g(t_k, x_k, y_k)$$

$$= y_k + \frac{1}{2}[x_k - 2y_k] = \frac{1}{2}x_k \quad .$$

Repeatedly using these with $k = 0, 1, \ldots$:

With $k = 0$:

$$t_1 = 0 + \frac{1}{2} = \frac{1}{2} \quad ,$$

$$x_1 = \frac{1}{2}x_0 + y_0 = \frac{1}{2} \cdot 4 + 1 = 3$$

and

$$y_1 = \frac{1}{2}x_k = \frac{1}{2} \cdot 4 = 2 \quad .$$

With $k = 1$:

$$t_2 = t_1 + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1 \quad ,$$

$$x_2 = \frac{1}{2}x_1 + y_1 = \frac{1}{2} \cdot 3 + 2 = \frac{7}{2}$$

and

$$y_2 = \frac{1}{2}x_1 = \frac{1}{2} \cdot 3 = \frac{3}{2} \quad .$$

With $k = 2$:

$$t_3 = t_2 + \frac{1}{2} = +1\frac{1}{2} = \frac{3}{2} \quad ,$$

$$x_3 = \frac{1}{2}x_2 + y_2 = \frac{1}{2} \cdot \frac{7}{2} + \frac{3}{2} = \frac{13}{4}$$

and

$$y_3 = \frac{1}{2}x_2 = \frac{1}{2} \cdot \frac{7}{2} = \frac{7}{4} \quad .$$

With $k = 3$:

$$t_4 = t_3 + \frac{1}{2} = \frac{3}{2} + \frac{1}{2} = 2 \quad ,$$

$$x_4 = \frac{1}{2}x_3 + y_3 = \frac{1}{2} \cdot \frac{13}{4} + \frac{7}{4} = \frac{27}{8}$$

and

$$y_4 = \frac{1}{2}x_3 = \frac{1}{2} \cdot \frac{13}{4} = \frac{13}{8} \quad .$$

With $k = 4$:

$$t_5 = t_4 + \frac{1}{2} = 2 + \frac{1}{2} = \frac{5}{2} \quad ,$$

$$x_5 = \frac{1}{2}x_4 + y_4 = \frac{1}{2} \cdot \frac{27}{8} + \frac{13}{8} = \frac{53}{16}$$

and

$$y_5 = \frac{1}{2}x_k = \frac{1}{2} \cdot \frac{27}{8} = \frac{27}{16} \quad .$$

With $k = 5$:

$$t_6 = t_5 + \frac{1}{2} = \frac{5}{2} + \frac{1}{2} = 3 \quad (= t_{\max}) \quad ,$$

$$x_6 = \frac{1}{2}x_5 + y_5 = \frac{1}{2} \cdot \frac{53}{16} + \frac{27}{16} = \frac{107}{32}$$

and

$$y_6 = \frac{1}{2}x_5 = \frac{1}{2} \cdot \frac{53}{16} = \frac{53}{32} \quad .$$

In summary:

k	t_k	x_k	y_k
0	0	4	1
1	$1/2$	3	2
2	1	$7/2$	$3/2$
3	$3/2$	$13/4$	$7/4$
4	2	$27/8$	$13/8$
5	$5/2$	$53/16$	$27/16$
6	3	$107/32$	$53/32$

40.1 c.

$$\begin{array}{l} tx' = -x + 2y \\ ty' = 2 - 2x \end{array} \quad \rightarrowtail \quad \begin{array}{l} x' = \dfrac{-x+2y}{t} \\ y' = \dfrac{2-2x}{t} \end{array} \quad \rightarrowtail \quad \begin{array}{l} f(t,x,y) = \dfrac{-x+2y}{t} \\ g(t,x,y) = \dfrac{2-2x}{t} \end{array} \quad .$$

From the initial conditions and given parameters

$$t_0 = 1 \quad , \quad x_0 = 2 \quad , \quad y_0 = 0 \quad , \quad N = 6 \quad , \quad \Delta t = \frac{1}{3}$$

and

$$t_{\max} = t_0 + N\Delta t = 3 \quad .$$

Since we are doing these computations “by hand”, let us do a little “pre-computing” to save time and space later:

$$t_{k+1} = t_k + \Delta x = t_k + \frac{1}{3} \quad ,$$

$$x_{k+1} = x_k + \Delta t \cdot f(t_k, x_k, y_k)$$

$$= x_k + \frac{1}{3} \cdot \frac{-x_k + 2y_k}{t_k} = x_k + \frac{1}{3t_k}[-x_k + 2y_k]$$

and

$$y_{k+1} = y_k + \Delta t \cdot g(t_k, x_k, y_k)$$

$$= y_k + \frac{1}{3} \cdot \frac{2-2x}{t} = y_k + \frac{2}{3t_k}[1 - x_k] \quad .$$

Repeatedly using these with $k = 0, 1, \ldots$:

With $k = 0$:

$$t_1 = t_0 + \tfrac{1}{3} = 1 + \tfrac{1}{3} = \tfrac{4}{3} \quad ,$$

$$x_1 = x_0 + \frac{1}{3t_0}[-x_0 + 2y_0] = 2 + \frac{1}{3 \cdot 1}[-2 + 2 \cdot 0] = \frac{4}{3}$$

and

$$y_1 = y_0 + \frac{2}{3t_0}[1 - x_0] = 0 + \frac{2}{3 \cdot 1}[1 - 2] = -\frac{2}{3} \quad .$$

With $k = 1$:

$$t_2 = t_1 + \tfrac{1}{3} = \tfrac{4}{3} + \tfrac{1}{3} = \tfrac{5}{3} \quad ,$$

$$x_2 = x_1 + \frac{1}{3t_1}[-x_1 + 2y_1] = \frac{4}{3} + \frac{1}{3 \cdot {}^4\!/_3}\left[-\frac{4}{3} + 2 \cdot \left(-\frac{2}{3}\right)\right] = \frac{2}{3}$$

and

$$y_2 = y_1 + \frac{2}{3t_1}[1 - x_1] = -\frac{2}{3} + \frac{2}{3 \cdot {}^4\!/_3}\left[1 - \frac{4}{3}\right] = -\frac{5}{6} \quad .$$

With $k = 2$:

$$t_3 = t_2 + \tfrac{1}{3} = \tfrac{5}{3} + \tfrac{1}{3} = 2 \quad ,$$

$$x_3 = x_2 + \frac{1}{3t_2}[-x_2 + 2y_2] = \frac{2}{3} + \frac{1}{3 \cdot {}^5\!/_3}\left[-\frac{2}{3} + 2 \cdot \left(-\frac{5}{6}\right)\right] = \frac{1}{5}$$

and

$$y_3 = y_2 + \frac{2}{3t_2}[1 - x_2] = -\frac{5}{6} + \frac{2}{3 \cdot {}^5\!/_3}\left[1 - \frac{2}{3}\right] = -\frac{7}{10} \quad .$$

With $k = 3$:

$$t_4 = t_3 + \tfrac{1}{3} = 2 + \tfrac{1}{3} = \tfrac{7}{3} \quad ,$$

$$\begin{aligned} x_4 &= x_3 + \frac{1}{3t_3}[-x_3 + 2y_3] \\ &= \frac{1}{5} + \frac{1}{3 \cdot 2}\left[-\frac{1}{5} + 2 \cdot \left(-\frac{7}{10}\right)\right] = -\frac{1}{15} \end{aligned}$$

and

$$y_4 = y_3 + \frac{2}{3t_3}[1 - x_3] = -\frac{7}{10} + \frac{2}{3 \cdot 2}\left[1 - \frac{1}{5}\right] = -\frac{13}{30} \quad .$$

With $k = 4$:

$$t_5 = t_4 + \tfrac{1}{3} = \tfrac{7}{3} + \tfrac{1}{3} = \tfrac{8}{3} \quad ,$$

$$\begin{aligned} x_5 &= x_4 + \frac{1}{3t_4}[-x_4 + 2y_4] \\ &= -\frac{1}{15} + \frac{1}{3 \cdot {}^7\!/_3}\left[-\left(-\frac{1}{15}\right) + 2 \cdot \left(-\frac{13}{30}\right)\right] = -\frac{19}{105} \end{aligned}$$

and

$$y_5 = y_4 + \frac{2}{3t_4}[1 - x_4] = -\frac{13}{30} + \frac{2}{3 \cdot {}^7\!/_3}\left[1 - \left(-\frac{1}{15}\right)\right] = -\frac{9}{70} \quad .$$

With $k = 5$:

$$t_6 = t_5 + \tfrac{1}{3} = \tfrac{8}{3} + \tfrac{1}{3} = 3 \quad ,$$

$$x_6 = x_5 + \frac{1}{3t_5}[-x_5 + 2y_5]$$

$$= -\frac{19}{105} + \frac{1}{3 \cdot {}^8\!/_3}\left[-\left(-\frac{19}{105}\right) + 2\cdot\left(-\frac{9}{70}\right)\right] = -\frac{4}{21}$$

and

$$y_6 = y_5 + \frac{2}{3t_5}[1 - x_5] = -\frac{9}{70} + \frac{2}{3 \cdot {}^8\!/_3}\left[1 - \left(-\frac{19}{105}\right)\right] = \frac{1}{6} \quad .$$

In summary:

k	t_k	x_k	y_k
0	1	2	0
1	$^4/_3$	$^4/_3$	$-^2/_3$
2	$^5/_3$	$^2/_3$	$-^5/_6$
3	2	$^1/_5$	$-^7/_{10}$
4	$^7/_3$	$-^1/_{15}$	$-^{13}/_{30}$
5	$^8/_3$	$-^{19}/_{105}$	$-^9/_{70}$
6	3	$-^4/_{21}$	$^1/_6$

40.5 a. Letting $x = y'$,

$$y'' - 2y' + y = 0 \quad \rightarrowtail \quad y'' = 2y' - y \quad \rightarrowtail \quad x' = 2x - y \quad .$$

So, the system is

$$\begin{aligned} x' &= h(t, y, x) \\ y' &= x \end{aligned} \qquad \text{with} \quad h(t, y, x) = 2x - y \quad .$$

From the initial data and given parameters, we have

$$t_0 = 0 \quad , \quad y_0 = 1 \quad , \quad x_0 = y'(0) = 1 \quad , \quad \Delta t = \tfrac{1}{2} \quad , \quad t_{max} = 2$$

and, from $t_{max} = t_0 + N\Delta t$,

$$N = 4 \quad .$$

Thus, the iterative steps are given by

$$t_{k+1} = t_k + \Delta t = t_k + \tfrac{1}{2} \quad ,$$

$$x_{k+1} = x_k + \Delta t \cdot h(t_k, y_k, x_k) = x_k + \tfrac{1}{2} \cdot [2x_k - y_k]$$

and

$$y_{k+1} = y_k + \Delta t \cdot x_k = y_k + \tfrac{1}{2} \cdot x_k \quad .$$

Repeatedly using these with $k = 0, 1, \ldots$:
With $k = 0$:

$$t_1 = t_0 + \tfrac{1}{2} = 0 + \tfrac{1}{2} = \tfrac{1}{2} \quad ,$$

$$x_1 = x_0 + \tfrac{1}{2} \cdot [2x_0 - y_0]$$

$$= 1 + \tfrac{1}{2} \cdot [2 \cdot 1 - 1] = \tfrac{3}{2}$$

and

$$y_1 = y_0 + \tfrac{1}{2} \cdot x_0$$

$$= 1 + \tfrac{1}{2} \cdot 1 = \tfrac{3}{2} \quad .$$

With $k = 1$:

$$t_2 = t_1 + \tfrac{1}{2} = \tfrac{1}{2} + \tfrac{1}{2} = 1 \quad ,$$

$$x_2 = x_1 + \tfrac{1}{2} \cdot [2x_1 - y_1]$$

$$= \tfrac{3}{2} + \tfrac{1}{2} \cdot \left[2 \cdot \tfrac{3}{2} - \tfrac{3}{2}\right] = \tfrac{9}{4}$$

and

$$y_2 = y_1 + \tfrac{1}{2} \cdot x_1$$

$$= \tfrac{3}{2} + \tfrac{1}{2} \cdot \tfrac{3}{2} = \tfrac{9}{4} \quad .$$

With $k = 2$:

$$t_3 = t_2 + \tfrac{1}{2} = 1 + \tfrac{1}{2} = \tfrac{3}{2} \quad (= t_{max}) \quad ,$$

$$x_3 = x_2 + \tfrac{1}{2} \cdot [2x_2 - y_2]$$

$$= \tfrac{9}{4} + \tfrac{1}{2} \cdot \left[2 \cdot \tfrac{9}{4} - \tfrac{9}{4}\right] = \tfrac{27}{8}$$

and

$$y_3 = y_2 + \tfrac{1}{2} \cdot x_2 = \tfrac{9}{4} + \tfrac{1}{2} \cdot \tfrac{9}{4} = \tfrac{27}{8} \quad .$$

With $k = 3$:

$$t_4 = t_3 + \tfrac{1}{2} = \tfrac{3}{2} + \tfrac{1}{2} = 2 \quad ,$$

$$x_4 = x_3 + \tfrac{1}{2} \cdot [2x_3 - y_3]$$

$$= \tfrac{27}{8} + \tfrac{1}{2} \cdot \left[2 \cdot \tfrac{27}{8} - \tfrac{27}{8}\right] = \tfrac{81}{16}$$

and

$$y_4 = y_3 + \tfrac{1}{2} \cdot x_3$$

$$= \tfrac{27}{8} + \tfrac{1}{2} \cdot \tfrac{27}{8} = \tfrac{81}{16} \quad .$$

In summary:

k	t_k	y_k
0	0	1
1	$1/2$	$3/2$
2	1	$9/4$
3	$3/2$	$27/8$
4	2	$81/16$

40.5 c. Letting $x = y'$,

$$t^2 y'' - 6ty' + 10y = 0 \quad \rightarrowtail \quad y'' = \left[6ty' - 10y\right] t^{-2}$$

$$\hookrightarrow \qquad x' = [6tx - 10y]\, t^{-2} \quad .$$

So, the system is

$$\begin{aligned} x' &= h(t,y,x) \\ y' &= x \end{aligned} \qquad \text{with} \quad h(t,y,x) = [6tx - 10y]\,t^{-2} \quad .$$

From the initial data and given parameters, we have

$$t_0 = 1 \quad , \quad y_0 = 1 \quad , \quad x_0 = y'(1) = 2 \quad , \quad \Delta t = \tfrac{1}{2} \quad , \quad N = 4$$

and, from $t_{\max} = t_0 + N\Delta t$,

$$t_{\max} = 3 \quad .$$

Thus, the iterative steps are given by

$$t_{k+1} = t_k + \Delta t = t_k + \tfrac{1}{2} \quad ,$$

$$\begin{aligned} x_{k+1} &= x_k + \Delta t \cdot h(t_k, y_k, x_k) \\ &= x_k + \tfrac{1}{2} \cdot [6t_k x_k - 10y_k] \cdot [t_k]^{-2} = x_k + [3t_k x_k - 5y_k] \cdot [t_k]^{-2} \end{aligned}$$

and

$$y_{k+1} = y_k + \Delta t \cdot x_k = y_k + \tfrac{1}{2} \cdot x_k \quad .$$

Repeatedly using these with $k = 0, 1, \ldots$:
With $k = 0$:

$$t_1 = 1 + \tfrac{1}{2} = 1 + \tfrac{1}{2} = \tfrac{3}{2} \quad ,$$

$$\begin{aligned} x_1 &= x_0 + [3t_0 x_0 - 5y_0] \cdot [t_k]^{-2} \\ &= 2 + [3 \cdot 1 \cdot 2 - 5 \cdot 1] \cdot [1]^{-2} = 3 \end{aligned}$$

and

$$\begin{aligned} y_1 &= y_0 + \tfrac{1}{2} \cdot x_0 \\ &= 1 + \tfrac{1}{2} \cdot 2 = 2 \quad . \end{aligned}$$

With $k = 1$:

$$t_2 = t_1 + \tfrac{1}{2} = \tfrac{3}{2} + \tfrac{1}{2} = 2 \quad ,$$

$$\begin{aligned} x_2 &= x_1 + [3t_1 x_1 - 5y_1] \cdot [t_1]^{-2} \\ &= 3 + \left[3 \cdot \tfrac{3}{2} \cdot 3 - 5 \cdot 2\right] \cdot \left[\tfrac{3}{2}\right]^{-2} = \frac{41}{9} \end{aligned}$$

and

$$\begin{aligned} y_2 &= y_1 + \tfrac{1}{2} \cdot x_1 \\ &= 2 + \tfrac{1}{2} \cdot 3 = \tfrac{7}{2} \quad . \end{aligned}$$

With $k = 2$:

$$t_3 = t_2 + \tfrac{1}{2} = 2 + \tfrac{1}{2} = \tfrac{5}{2} \quad ,$$

$$\begin{aligned} x_3 &= x_2 + [3t_2 x_2 - 5y_2] \cdot [t_2]^{-2} \\ &= \frac{41}{9} + \left[3 \cdot 2 \cdot \frac{41}{9} - 5 \cdot \tfrac{7}{2}\right] \cdot [2]^{-2} = \frac{505}{72} \end{aligned}$$

and

$$y_3 = y_2 + \tfrac{1}{2} \cdot x_2 = \tfrac{7}{2} + \tfrac{1}{2} \cdot \frac{41}{9} = \frac{52}{9} \quad .$$

With $k = 3$:

$$t_4 = t_3 + \frac{1}{2} = \frac{5}{2} + \frac{1}{2} = 3 \quad (= t_{\max}) \quad ,$$

$$x_4 = x_3 + [3t_3x_3 - 5y_3] \cdot [t_3]^{-2}$$

$$= \frac{505}{72} + \left[3 \cdot \frac{5}{2} \cdot \frac{505}{72} - 5 \cdot \frac{52}{9}\right] \cdot \left[\frac{5}{2}\right]^{-2} = \frac{1297}{120}$$

and

$$y_4 = y_3 + \frac{1}{2} \cdot x_3$$

$$= \frac{52}{9} + \frac{1}{2} \cdot \frac{505}{72} = \frac{1337}{144} \quad .$$

In summary:

k	t_k	y_k
0	1	1
1	$^3/_2$	2
2	2	$^7/_2$
3	$^5/_2$	$^{52}/_9$
4	3	$^{1337}/_{144}$