

### Chapter 40: Numerical Methods III: Systems and Higher-Order Equations

**40.1 a.**  $\begin{aligned} x' &= -x + 2y & \rightarrow f(t, x, y) &= -x + 2y \\ y' &= x - 2y & g(t, x, y) &= x - 2y \end{aligned}$

From the initial conditions and given parameters

$$t_0 = 0, \quad x_0 = 4, \quad y_0 = 1, \quad N = 6, \quad \Delta t = \frac{1}{2}$$

and

$$t_{\max} = t_0 + N\Delta t = 3.$$

Since we are doing these computations “by hand”, let us do a little “pre-computing” to save time and space later:

$$\begin{aligned} t_{k+1} &= t_k + \Delta x = t_k + \frac{1}{2}, \\ x_{k+1} &= x_k + \Delta t \cdot f(t_k, x_k, y_k) \\ &= x_k + \frac{1}{2}[-x_k + 2y_k] = \frac{1}{2}x_k + y_k \end{aligned}$$

and

$$\begin{aligned} y_{k+1} &= y_k + \Delta t \cdot g(t_k, x_k, y_k) \\ &= y_k + \frac{1}{2}[x_k - 2y_k] = \frac{1}{2}x_k. \end{aligned}$$

Repeatedly using these with  $k = 0, 1, \dots$ :

With  $k = 0$ :

$$\begin{aligned} t_1 &= 0 + \frac{1}{2} = \frac{1}{2}, \\ x_1 &= \frac{1}{2}x_0 + y_0 = \frac{1}{2} \cdot 4 + 1 = 3 \end{aligned}$$

and

$$y_1 = \frac{1}{2}x_0 = \frac{1}{2} \cdot 4 = 2.$$

With  $k = 1$ :

$$\begin{aligned} t_2 &= t_1 + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1, \\ x_2 &= \frac{1}{2}x_1 + y_1 = \frac{1}{2} \cdot 3 + 2 = \frac{7}{2} \end{aligned}$$

and

$$y_2 = \frac{1}{2}x_1 = \frac{1}{2} \cdot 3 = \frac{3}{2}.$$

With  $k = 2$ :

$$\begin{aligned} t_3 &= t_2 + \frac{1}{2} = 1 + \frac{1}{2} = \frac{3}{2}, \\ x_3 &= \frac{1}{2}x_2 + y_2 = \frac{1}{2} \cdot \frac{7}{2} + \frac{3}{2} = \frac{13}{4} \end{aligned}$$

and

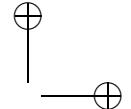
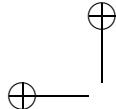
$$y_3 = \frac{1}{2}x_2 = \frac{1}{2} \cdot \frac{7}{2} = \frac{7}{4}.$$

With  $k = 3$ :

$$\begin{aligned} t_4 &= t_3 + \frac{1}{2} = \frac{3}{2} + \frac{1}{2} = 2, \\ x_4 &= \frac{1}{2}x_3 + y_3 = \frac{1}{2} \cdot \frac{13}{4} + \frac{7}{4} = \frac{27}{8} \end{aligned}$$

and

$$y_4 = \frac{1}{2}x_3 = \frac{1}{2} \cdot \frac{13}{4} = \frac{13}{8}.$$



With  $k = 4$ :

$$t_5 = t_4 + \frac{1}{2} = 2 + \frac{1}{2} = \frac{5}{2},$$

$$x_5 = \frac{1}{2}x_4 + y_4 = \frac{1}{2} \cdot \frac{27}{8} + \frac{13}{8} = \frac{53}{16}$$

and

$$y_5 = \frac{1}{2}x_k = \frac{1}{2} \cdot \frac{27}{8} = \frac{27}{16}.$$

With  $k = 5$ :

$$t_6 = t_5 + \frac{1}{2} = \frac{5}{2} + \frac{1}{2} = 3 (= t_{\max}),$$

$$x_6 = \frac{1}{2}x_5 + y_5 = \frac{1}{2} \cdot \frac{53}{16} + \frac{27}{16} = \frac{107}{32}$$

and

$$y_6 = \frac{1}{2}x_5 = \frac{1}{2} \cdot \frac{53}{16} = \frac{53}{32}.$$

In summary:

$k$	$t_k$	$x_k$	$y_k$
0	0	4	1
1	$\frac{1}{2}$	3	2
2	1	$\frac{7}{2}$	$\frac{3}{2}$
3	$\frac{3}{2}$	$\frac{13}{4}$	$\frac{7}{4}$
4	2	$\frac{27}{8}$	$\frac{13}{8}$
5	$\frac{5}{2}$	$\frac{53}{16}$	$\frac{27}{16}$
6	3	$\frac{107}{32}$	$\frac{53}{32}$

**40.1 c.** 
$$\begin{aligned} tx' &= -x + 2y & \rightarrow & x' = \frac{-x + 2y}{t} \\ ty' &= 2 - 2x & \rightarrow & y' = \frac{2 - 2x}{t} \end{aligned} \quad \begin{aligned} f(t, x, y) &= \frac{-x + 2y}{t} \\ g(t, x, y) &= \frac{2 - 2x}{t} \end{aligned}.$$

From the initial conditions and given parameters

$$t_0 = 1, \quad x_0 = 2, \quad y_0 = 0, \quad N = 6, \quad \Delta t = \frac{1}{3}$$

and

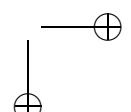
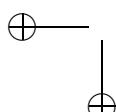
$$t_{\max} = t_0 + N\Delta t = 3.$$

Since we are doing these computations “by hand”, let us do a little “pre-computing” to save time and space later:

$$\begin{aligned} t_{k+1} &= t_k + \Delta x = t_k + \frac{1}{3}, \\ x_{k+1} &= x_k + \Delta t \cdot f(t_k, x_k, y_k) \\ &= x_k + \frac{1}{3} \cdot \frac{-x_k + 2y_k}{t_k} = x_k + \frac{1}{3t_k}[-x_k + 2y_k] \end{aligned}$$

and

$$\begin{aligned} y_{k+1} &= y_k + \Delta t \cdot g(t_k, x_k, y_k) \\ &= y_k + \frac{1}{3} \cdot \frac{2 - 2x}{t} = y_k + \frac{2}{3t_k}[1 - x_k]. \end{aligned}$$



Repeatedly using these with  $k = 0, 1, \dots$ :

With  $k = 0$ :

$$t_1 = t_0 + \frac{1}{3} = 1 + \frac{1}{3} = \frac{4}{3},$$

$$x_1 = x_0 + \frac{1}{3t_0}[-x_0 + 2y_0] = 2 + \frac{1}{3 \cdot 1}[-2 + 2 \cdot 0] = \frac{4}{3}$$

and

$$y_1 = y_0 + \frac{2}{3t_0}[1 - x_0] = 0 + \frac{2}{3 \cdot 1}[1 - 2] = -\frac{2}{3}.$$

With  $k = 1$ :

$$t_2 = t_1 + \frac{1}{3} = \frac{4}{3} + \frac{1}{3} = \frac{5}{3},$$

$$x_2 = x_1 + \frac{1}{3t_1}[-x_1 + 2y_1] = \frac{4}{3} + \frac{1}{3 \cdot 4/3} \left[ -\frac{4}{3} + 2 \cdot \left( -\frac{2}{3} \right) \right] = \frac{2}{3}$$

and

$$y_2 = y_1 + \frac{2}{3t_1}[1 - x_1] = -\frac{2}{3} + \frac{2}{3 \cdot 4/3} \left[ 1 - \frac{4}{3} \right] = -\frac{5}{6}.$$

With  $k = 2$ :

$$t_3 = t_2 + \frac{1}{3} = \frac{5}{3} + \frac{1}{3} = 2,$$

$$x_3 = x_2 + \frac{1}{3t_2}[-x_2 + 2y_2] = \frac{2}{3} + \frac{1}{3 \cdot 5/3} \left[ -\frac{2}{3} + 2 \cdot \left( -\frac{5}{6} \right) \right] = \frac{1}{5}$$

and

$$y_3 = y_2 + \frac{2}{3t_2}[1 - x_2] = -\frac{5}{6} + \frac{2}{3 \cdot 5/3} \left[ 1 - \frac{2}{3} \right] = -\frac{7}{10}.$$

With  $k = 3$ :

$$t_4 = t_3 + \frac{1}{3} = 2 + \frac{1}{3} = \frac{7}{3},$$

$$x_4 = x_3 + \frac{1}{3t_3}[-x_3 + 2y_3]$$

$$= \frac{1}{5} + \frac{1}{3 \cdot 2} \left[ -\frac{1}{5} + 2 \cdot \left( -\frac{7}{10} \right) \right] = -\frac{1}{15}$$

and

$$y_4 = y_3 + \frac{2}{3t_3}[1 - x_3] = -\frac{7}{10} + \frac{2}{3 \cdot 2} \left[ 1 - \frac{1}{5} \right] = -\frac{13}{30}.$$

With  $k = 4$ :

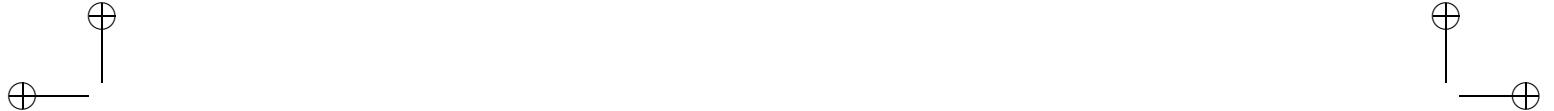
$$t_5 = t_4 + \frac{1}{3} = \frac{7}{3} + \frac{1}{3} = \frac{8}{3},$$

$$x_5 = x_4 + \frac{1}{3t_4}[-x_4 + 2y_4]$$

$$= -\frac{1}{15} + \frac{1}{3 \cdot 7/3} \left[ -\left( -\frac{1}{15} \right) + 2 \cdot \left( -\frac{13}{30} \right) \right] = -\frac{19}{105}$$

and

$$y_5 = y_4 + \frac{2}{3t_4}[1 - x_4] = -\frac{13}{30} + \frac{2}{3 \cdot 7/3} \left[ 1 - \left( -\frac{1}{15} \right) \right] = -\frac{9}{70}.$$



With  $k = 5$ :

$$\begin{aligned} t_6 &= t_5 + \frac{1}{3} = \frac{8}{3} + \frac{1}{3} = 3 , \\ x_6 &= x_5 + \frac{1}{3t_5}[-x_5 + 2y_5] \\ &= -\frac{19}{105} + \frac{1}{3 \cdot \frac{8}{3}} \left[ -\left( -\frac{19}{105} \right) + 2 \cdot \left( -\frac{9}{70} \right) \right] = -\frac{4}{21} \end{aligned}$$

and

$$y_6 = y_5 + \frac{2}{3t_5}[1 - x_5] = -\frac{9}{70} + \frac{2}{3 \cdot \frac{8}{3}} \left[ 1 - \left( -\frac{19}{105} \right) \right] = \frac{1}{6} .$$

In summary:

$k$	$t_k$	$x_k$	$y_k$
0	1	2	0
1	$\frac{4}{3}$	$\frac{4}{3}$	$-\frac{2}{3}$
2	$\frac{5}{3}$	$\frac{2}{3}$	$-\frac{5}{6}$
3	2	$\frac{1}{5}$	$-\frac{7}{10}$
4	$\frac{7}{3}$	$-\frac{1}{15}$	$-\frac{13}{30}$
5	$\frac{8}{3}$	$-\frac{19}{105}$	$-\frac{9}{70}$
6	3	$-\frac{4}{21}$	$\frac{1}{6}$

**40.5 a.** Letting  $x = y'$ ,

$$y'' - 2y' + y = 0 \implies y'' = 2y' - y \implies x' = 2x - y .$$

So, the system is

$$\begin{aligned} x' &= h(t, y, x) \\ y' &= x \end{aligned} \quad \text{with } h(t, y, x) = 2x - y .$$

From the initial data and given parameters, we have

$$t_0 = 0 , \quad y_0 = 1 , \quad x_0 = y'(0) = 1 , \quad \Delta t = \frac{1}{2} , \quad t_{\max} = 2$$

and, from  $t_{\max} = t_0 + N \Delta t$ ,

$$N = 4 .$$

Thus, the iterative steps are given by

$$t_{k+1} = t_k + \Delta t = t_k + \frac{1}{2} ,$$

$$x_{k+1} = x_k + \Delta t \cdot h(t_k, y_k, x_k) = x_k + \frac{1}{2} \cdot [2x_k - y_k]$$

and

$$y_{k+1} = y_k + \Delta t \cdot x_k = y_k + \frac{1}{2} \cdot x_k .$$

Repeatedly using these with  $k = 0, 1, \dots$ :

With  $k = 0$ :

$$t_1 = t_0 + \frac{1}{2} = 0 + \frac{1}{2} = \frac{1}{2} ,$$

$$x_1 = x_0 + \frac{1}{2} \cdot [2x_0 - y_0]$$

$$= 1 + \frac{1}{2} \cdot [2 \cdot 1 - 1] = \frac{3}{2}$$



and

$$\begin{aligned} y_1 &= y_0 + \frac{1}{2} \cdot x_0 \\ &= 1 + \frac{1}{2} \cdot 1 = \frac{3}{2} . \end{aligned}$$

With  $k = 1$ :

$$\begin{aligned} t_2 &= t_1 + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1 , \\ x_2 &= x_1 + \frac{1}{2} \cdot [2x_1 - y_1] \\ &= \frac{3}{2} + \frac{1}{2} \cdot \left[ 2 \cdot \frac{3}{2} - \frac{3}{2} \right] = \frac{9}{4} \end{aligned}$$

and

$$\begin{aligned} y_2 &= y_1 + \frac{1}{2} \cdot x_1 \\ &= \frac{3}{2} + \frac{1}{2} \cdot \frac{3}{2} = \frac{9}{4} . \end{aligned}$$

With  $k = 2$ :

$$\begin{aligned} t_3 &= t_2 + \frac{1}{2} = 1 + \frac{1}{2} = \frac{3}{2} (= t_{\max}) , \\ x_3 &= x_2 + \frac{1}{2} \cdot [2x_2 - y_2] \\ &= \frac{9}{4} + \frac{1}{2} \cdot \left[ 2 \cdot \frac{9}{4} - \frac{9}{4} \right] = \frac{27}{8} \end{aligned}$$

and

$$y_3 = y_2 + \frac{1}{2} \cdot x_2 = \frac{9}{4} + \frac{1}{2} \cdot \frac{9}{4} = \frac{27}{8} .$$

With  $k = 3$ :

$$\begin{aligned} t_4 &= t_3 + \frac{1}{2} = \frac{3}{2} + \frac{1}{2} = 2 , \\ x_4 &= x_3 + \frac{1}{2} \cdot [2x_3 - y_3] \\ &= \frac{27}{8} + \frac{1}{2} \cdot \left[ 2 \cdot \frac{27}{8} - \frac{27}{8} \right] = \frac{81}{16} \end{aligned}$$

and

$$\begin{aligned} y_4 &= y_3 + \frac{1}{2} \cdot x_3 \\ &= \frac{27}{8} + \frac{1}{2} \cdot \frac{27}{8} = \frac{81}{16} . \end{aligned}$$

In summary:

$k$	$t_k$	$y_k$
0	0	1
1	$\frac{1}{2}$	$\frac{3}{2}$
2	1	$\frac{9}{4}$
3	$\frac{3}{2}$	$\frac{27}{8}$
4	2	$\frac{81}{16}$

**40.5 c.** Letting  $x = y'$ ,

$$t^2 y'' - 6ty' + 10y = 0 \rightsquigarrow y'' = [6ty' - 10y]t^{-2}$$

$$\Leftrightarrow x' = [6tx - 10y]t^{-2} .$$



So, the system is

$$\begin{aligned} x' &= h(t, y, x) \\ y' &= x \end{aligned} \quad \text{with } h(t, y, x) = [6tx - 10y]t^{-2} .$$

From the initial data and given parameters, we have

$$t_0 = 1 , \quad y_0 = 1 , \quad x_0 = y'(1) = 2 , \quad \Delta t = \frac{1}{2} , \quad N = 4$$

and, from  $t_{\max} = t_0 + N\Delta t$ ,

$$t_{\max} = 3 .$$

Thus, the iterative steps are given by

$$\begin{aligned} t_{k+1} &= t_k + \Delta t = t_k + \frac{1}{2} , \\ x_{k+1} &= x_k + \Delta t \cdot h(t_k, y_k, x_k) \\ &= x_k + \frac{1}{2} \cdot [6t_k x_k - 10y_k] \cdot [t_k]^{-2} = x_k + [3t_k x_k - 5y_k] \cdot [t_k]^{-2} \end{aligned}$$

and

$$y_{k+1} = y_k + \Delta t \cdot x_k = y_k + \frac{1}{2} \cdot x_k .$$

Repeatedly using these with  $k = 0, 1, \dots$ :

With  $k = 0$ :

$$\begin{aligned} t_1 &= 1 + \frac{1}{2} = 1 + \frac{1}{2} = \frac{3}{2} , \\ x_1 &= x_0 + [3t_0 x_0 - 5y_0] \cdot [t_0]^{-2} \\ &= 2 + [3 \cdot 1 \cdot 2 - 5 \cdot 1] \cdot [1]^{-2} = 3 \end{aligned}$$

and

$$\begin{aligned} y_1 &= y_0 + \frac{1}{2} \cdot x_0 \\ &= 1 + \frac{1}{2} \cdot 2 = 2 . \end{aligned}$$

With  $k = 1$ :

$$\begin{aligned} t_2 &= t_1 + \frac{1}{2} = \frac{3}{2} + \frac{1}{2} = 2 , \\ x_2 &= x_1 + [3t_1 x_1 - 5y_1] \cdot [t_1]^{-2} \\ &= 3 + \left[ 3 \cdot \frac{3}{2} \cdot 3 - 5 \cdot 2 \right] \cdot \left[ \frac{3}{2} \right]^{-2} = \frac{41}{9} \end{aligned}$$

and

$$\begin{aligned} y_2 &= y_1 + \frac{1}{2} \cdot x_1 \\ &= 2 + \frac{1}{2} \cdot 3 = \frac{7}{2} . \end{aligned}$$

With  $k = 2$ :

$$\begin{aligned} t_3 &= t_2 + \frac{1}{2} = 2 + \frac{1}{2} = \frac{5}{2} , \\ x_3 &= x_2 + [3t_2 x_2 - 5y_2] \cdot [t_2]^{-2} \\ &= \frac{41}{9} + \left[ 3 \cdot 2 \cdot \frac{41}{9} - 5 \cdot \frac{7}{2} \right] \cdot [2]^{-2} = \frac{505}{72} \end{aligned}$$

and

$$y_3 = y_2 + \frac{1}{2} \cdot x_2 = \frac{7}{2} + \frac{1}{2} \cdot \frac{41}{9} = \frac{52}{9} .$$



With  $k = 3$ :

$$\begin{aligned} t_4 &= t_3 + \frac{1}{2} = \frac{5}{2} + \frac{1}{2} = 3 \quad (= t_{\max}) \quad , \\ x_4 &= x_3 + [3t_3x_3 - 5y_3] \cdot [t_3]^{-2} \\ &= \frac{505}{72} + \left[ 3 \cdot \frac{5}{2} \cdot \frac{505}{72} - 5 \cdot \frac{52}{9} \right] \cdot \left[ \frac{5}{2} \right]^{-2} = \frac{1297}{120} \end{aligned}$$

and

$$\begin{aligned} y_4 &= y_3 + \frac{1}{2} \cdot x_3 \\ &= \frac{52}{9} + \frac{1}{2} \cdot \frac{505}{72} = \frac{1337}{144} \quad . \end{aligned}$$

In summary:

$k$	$t_k$	$y_k$
0	1	1
1	$\frac{3}{2}$	2
2	2	$\frac{7}{2}$
3	$\frac{5}{2}$	$\frac{52}{9}$
4	3	$\frac{1337}{144}$