Chapter 31: Delta Functions

31.1 a. $|\mathcal{I}| = |m[v_{\text{after}} - v_{\text{before}}]| = |0.145[45 - 0]| = 6.525 \quad (\text{kg·meter/sec}) \quad .$

31.2 a. Solving the equation $\mathcal{I} = m[v_{\text{after}} - v_{\text{before}}]$ for v_{after} we get

$$v_{\text{after}} = \frac{\mathcal{I}}{m} + v_{\text{before}} \quad .$$

In particular, assuming $m = 2$ and $v_{\text{before}} = -10$:

If $\mathcal{I} = 60$, then $v_{\text{after}} = \frac{60}{2} - 10 = 20 \quad (\text{meter/sec}) \quad .$

If $\mathcal{I} = 100$, then $v_{\text{after}} = \frac{100}{2} - 10 = 40 \quad (\text{meter/sec}) \quad .$

If $\mathcal{I} = 20$, then $v_{\text{after}} = \frac{20}{2} - 10 = 0 \quad (\text{meter/sec}) \quad .$

31.2 c. Solving the equation $\mathcal{I} = m[v_{\text{after}} - v_{\text{before}}]$ for m we get

$$m = \frac{\mathcal{I}}{v_{\text{after}} - v_{\text{before}}} \quad .$$

In particular, assuming $\mathcal{I} = 30$ and $v_{\text{before}} = -10$:

If $v_{\text{before}} = -10$ and $v_{\text{after}} = 50$, then $m = \frac{30}{50 - (-10)} = \frac{1}{2} \quad (\text{kg}) \quad .$

If $v_{\text{before}} = 0$ and $v_{\text{after}}x = 15$, then $m = \frac{30}{15 - 0} = 2 \quad (\text{kg}) \quad .$

31.3 a.
$$\int_0^\infty t^2 \delta_4(t)\, dt = 4^2 = 16 \quad .$$

31.3 c.
$$\int_0^\infty \cos(t)\, \delta(t)\, dt = \int_0^\infty \cos(t)\, \delta_0(t)\, dt = \cos(0) = 1 \quad .$$

31.3 e.
$$\int_0^\infty t^2 \operatorname{rect}_{(1,4)}(t)\delta_3(t)\, dt = 3^2 \operatorname{rect}_{(1,4)}(3) = 9 \cdot 1 = 9 \quad .$$

31.5. Keeping in mind Theorem 31.1, and the fact that $\alpha \geq 0$ and $t > 0$, we have

$$\begin{aligned}
g * \delta_\alpha(t) = \delta_\alpha * g(t) &= \int_0^t \delta_\alpha(x) g(t-x)\, dx \\
&= \int_0^\infty \delta_\alpha(x) g(t-x) \operatorname{rect}_{(0,t)}(x)\, dx \\
&= g(t-\alpha) \operatorname{rect}_{(0,t)}(\alpha) \\
&= g(t-\alpha) \cdot \begin{Bmatrix} 1 & \text{if} & 0 < \alpha < t \\ 0 & \text{if} & t < \alpha \end{Bmatrix} \\
&= g(t-\alpha) \cdot \begin{Bmatrix} 0 & \text{if} & t < \alpha \\ 1 & \text{if} & \alpha < t \end{Bmatrix} = g(t-\alpha) \operatorname{step}_\alpha(t) \quad .
\end{aligned}$$

31.6 a.
$$\mathcal{L}\big[y'\big]\big|_s = \mathcal{L}[3\delta_2(t)]|_s$$
$$\hookrightarrow \quad sY(s) - 0 = 3\mathcal{L}[\delta_2(t)]|_s = 3e^{-2t}$$
$$\hookrightarrow \quad Y(s) = 3\frac{e^{-2s}}{s}$$
$$\hookrightarrow \quad y(t) = \mathcal{L}^{-1}[Y(s)]|_t = \mathcal{L}^{-1}\left[3\frac{e^{-2s}}{s}\right]\Big|_t = 3\,\text{step}_2(t) \quad .$$

31.6 c.
$$\mathcal{L}\big[y''\big]\big|_s = \mathcal{L}[\delta_3(t)]|_s$$
$$\hookrightarrow \quad s^2Y(s) - s\cdot 0 - 0 = e^{-3t}$$
$$\hookrightarrow \quad Y(s) = \frac{e^{-3s}}{s^2}$$
$$\hookrightarrow \quad y(t) = \mathcal{L}^{-1}[Y(s)]|_t = \mathcal{L}^{-1}\left[e^{-3s}\frac{1}{s^2}\right]\Big|_t = f(t-3)\,\text{step}_3(t)$$

where
$$f(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\right]\Big|_t = t \quad .$$

Thus,
$$f(X) = X \quad , \quad f(t-3) = t - 3$$

and
$$y(t) = f(t-3)\,\text{step}_3(t) = (t-3)\,\text{step}_3(t) \quad .$$

31.6 e.
$$\mathcal{L}\big[y' + 2y\big]\big|_s = \mathcal{L}[4\delta_1(t)]|_s$$
$$\hookrightarrow \quad [sY(s) - 0] + 2Y(s) = 4e^{-s}$$
$$\hookrightarrow \quad Y(s) = 4e^{-s}\frac{1}{s+2}$$
$$\hookrightarrow \quad y(t) = \mathcal{L}^{-1}\left[4e^{-1\cdot s}\frac{1}{s+2}\right]\Big|_t = 4f(t-1)\,\text{step}_1(t)$$

where
$$f(t) = \mathcal{L}^{-1}\left[\frac{1}{s+2}\right]\Big|_t = e^{-2t} \quad .$$

So,
$$f(X) = e^{-2X} \quad , \quad f(t-1) = e^{-2(t-1)}$$

and
$$y(t) = 4f(t-1)\,\text{step}_1(t) = 4e^{-2(t-1)}\,\text{step}_1(t) \quad .$$

31.6 g.
$$\mathcal{L}\big[y'' + y\big]\big|_s = \mathcal{L}\big[-2\delta_{\pi/2}(t)\big]\big|_s$$
$$\hookrightarrow \quad \Big[s^2Y(s) - s\cdot 0 - 0\Big] + Y(s) = -2e^{-\pi s/2}$$

$$\hookrightarrow \qquad Y(s) = -2e^{-\pi s/2}\frac{1}{s^2+1}$$

$$\hookrightarrow \qquad y(t) = \mathcal{L}^{-1}\left[-2e^{-\pi s/2}\frac{1}{s^2+1}\right]\bigg|_t = -2f\left(t-\frac{\pi}{2}\right)\text{step}_{\pi/2}(t)$$

where

$$f(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right]\bigg|_t = \sin(t) \quad .$$

So,

$$-2f\left(t-\frac{\pi}{2}\right) = -2\sin\left(t-\frac{\pi}{2}\right) = 2\cos(t)$$

and

$$y(t) = -2f\left(t-\frac{\pi}{2}\right)\text{step}_{\pi/2}(t) = \cos(t)\,\text{step}_{\pi/2}(t) \quad .$$

31.7 a.

$$\mathcal{L}\big[y' + 3y\big]\big|_s = \mathcal{L}[\delta_2(t)]|_s$$

$$\hookrightarrow \qquad [sY(s) - \underbrace{y(0)}_{2}] + 3Y(s) = e^{-2s}$$

$$\hookrightarrow \qquad (s+3)Y(s) - 2 = e^{-2s}$$

$$\hookrightarrow \qquad Y(s) = \frac{2}{s+3} + e^{-2s}\frac{1}{s+3} \quad .$$

Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[\frac{2}{s+3} + e^{-2s}\frac{1}{s+3}\right]\bigg|_t \\ &= 2\mathcal{L}^{-1}\left[\frac{1}{s+3}\right]\bigg|_t + \mathcal{L}^{-1}\left[e^{-2s}\frac{1}{s+3}\right]\bigg|_t = 2f(t) + f(t-2)\,\text{step}_2(t) \end{aligned}$$

where

$$f(t) = \mathcal{L}^{-1}\left[\frac{1}{s+3}\right]\bigg|_t = e^{-3t} \quad .$$

So,

$$f(X) = e^{-3X} \quad , \quad f(t-2) = e^{-3(x-2)}$$

and

$$y(t) = 2f(t) + f(t-2)\,\text{step}_2(t) = 2e^{-3t} + e^{-3(t-2)}\,\text{step}_2(t) \quad .$$

31.7 c.

$$\mathcal{L}\big[y'' + 3y'\big]\big|_s = \mathcal{L}[\delta_1(t)]|_s$$

$$\hookrightarrow \qquad \big[s^2Y(s) - s\underbrace{y(0)}_{0} - \underbrace{y'(s)}_{1}\big]$$

$$+ 3[SY(s) - \underbrace{y(0)}_{0} = e^{-s}$$

$$\hookrightarrow \qquad (s^2+3s)Y(s) - 2 = e^{-s}$$

$$\hookrightarrow \qquad Y(s) = \frac{2}{s^2+3s} + e^{-s}\frac{1}{s^2+3s} \quad . \quad .$$

Thus,

$$y(t) = \mathcal{L}^{-1}\left[\frac{2}{s^2+3s} + e^{-s}\frac{1}{s^2+3s}\right]\bigg|_t$$

$$= 2\mathcal{L}^{-1}\left[\frac{1}{s^2+3s}\right]\bigg|_t + \mathcal{L}^{-1}\left[e^{-1s}\frac{1}{s^2+3s}\right]\bigg|_t = 2f(t) + f(t-1)\,\text{step}_1(t)$$

where

$$f(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2+3s}\right]\bigg|_t = \mathcal{L}^{-1}\left[\frac{1}{s(s+3)}\right]\bigg|_t$$

$$= \cdots \quad \text{(Use partial fractions or convolution.)}$$

$$= \frac{1}{3}\left[1 - e^{-3t}\right] \quad .$$

So,

$$f(X) = \frac{1}{3}\left[1 - e^{-3t}\right] \quad , \quad f(t-1) = \frac{1}{3}\left[1 - e^{-3(t-1)}\right]$$

and

$$y(t) = 2f(t) + f(t-2)\,\text{step}_2(t) = \frac{2}{3}\left[1 - e^{-3t}\right] + \frac{1}{3}\left[1 - e^{-3(t-1)}\right]\text{step}_1(t) \quad .$$

31.7 e.

$$\mathcal{L}\left[y'' - 16y\right]\big|_s = \mathcal{L}[\delta_{10}(t)]|_s$$

$$\hookrightarrow \quad \left[s^2Y(s) - s\cdot 0 - 0\right] - 16Y(s) = e^{-10s}$$

$$\hookrightarrow \quad Y(s) = e^{-10s}\frac{1}{s^2+16}$$

$$\hookrightarrow \quad y(t) = \mathcal{L}\left[e^{-10s}\frac{1}{s^2-16}\right]\bigg|_t = f(t-10)\,\text{step}_{10}(t)$$

where

$$f(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2-16}\right]\bigg|_t = \mathcal{L}^{-1}\left[\frac{1}{(s-4)(s+4)}\right]\bigg|_t$$

$$= \cdots \quad \text{(Use partial fractions or convolution.)}$$

$$= \frac{1}{8}\left[e^{4t} - e^{-4t}\right] \quad .$$

So,

$$f(X) = \frac{1}{8}\left[e^{4X} - e^{-4X}\right] \quad , \quad f(t-10) = \frac{1}{8}\left[e^{4(t-10)} - e^{-4(t-10)}\right]$$

and

$$y(t) = f(t-10)\,\text{step}_{10}(t) = \frac{1}{8}\left[e^{4(t-10)} - e^{-4(t-10)}\right]\text{step}_{10}(t) \quad .$$

31.7 g.
$$\mathcal{L}\big[y'' + 4y' - 12y\big]\big|_s = \mathcal{L}[\delta(t)]|_s$$

$$\hookrightarrow \quad \Big[s^2Y(s) - s\cdot 0 - 0\Big] + 4[sY(s) - 0] - 12Y(s) = 1$$

$$\hookrightarrow \quad Y(s) = \frac{1}{s^2 + 4s - 12} \quad .$$

So, using either partial fractions or convolution, we find that

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2 + 4s - 12}\right]\bigg|_t = \mathcal{L}^{-1}\left[\frac{1}{(s-2)(s+6)}\right]\bigg|_t = \frac{1}{8}\left[e^{2t} - e^{-6t}\right] \quad .$$

31.7 i.
$$\mathcal{L}\big[y'' + 6y' + 9y\big]\big|_s = \mathcal{L}[\delta_4(t)]|_s$$

$$\hookrightarrow \quad \Big[s^2Y(s) - s\cdot 0 - 0\Big] + 6[sY(s) - 0] + 9Y(s) = e^{-4s}$$

$$\hookrightarrow \quad Y(s) = e^{-4s}\frac{1}{s^2 + 6s + 9}$$

$$\hookrightarrow \quad y(t) = \mathcal{L}\left[e^{-4s}\frac{1}{s^2 + 6s + 9}\right]\bigg|_t = f(t-4)\,\text{step}_4(t)$$

where

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left[\frac{1}{s^2 + 6s + 9}\right]\bigg|_t \\ &= \mathcal{L}^{-1}\left[\frac{1}{(s+3)^2}\right]\bigg|_t \\ &= \cdots \quad \text{(Use the first translation identity or convolution.)} \\ &= te^{-3t} \quad . \end{aligned}$$

So,

$$f(X) = Xe^{-3X} \quad , \quad f(t-4) = (t-4)e^{-3(t-4)}$$

and

$$y(t) = f(t-4)\,\text{step}_4(t) = (t-4)e^{-3(t-4)}\,\text{step}_4(t) \quad .$$

31.7 k.
$$\mathcal{L}\big[y''' + 9y'\big]\big|_s = \mathcal{L}[\delta_1(t)]|_s$$

$$\hookrightarrow \quad \Big[s^3Y(s) - s^2\cdot 0 - s\cdot 0 - 0\Big] + 9[sY(s) - 0] = e^{-s}$$

$$\hookrightarrow \qquad Y(s) = e^{-s}\frac{1}{s^3+9s}$$

$$\hookrightarrow \qquad y(t) = \mathcal{L}\left[e^{-1s}\frac{1}{s^3+9s}\right]\bigg|_t = f(t-1)\,\text{step}_1(t)$$

where

$$f(t) = \mathcal{L}^{-1}\left[\frac{1}{s^3+9s}\right]\bigg|_t = \mathcal{L}^{-1}\left[\frac{1}{s\left(s^2+9\right)}\right]\bigg|_t$$

$$= \cdots \quad \text{(Use partial fractions or convolution.)}$$

$$= \frac{1}{9}\big[1-\cos(3t)\big] \quad .$$

So,

$$f(X) = \frac{1}{9}\big[1-\cos(3t)\big] \quad , \quad f(t-1) = \frac{1}{9}\big[1-\cos(3[t-1])\big]$$

and

$$y(t) = f(t-1)\,\text{step}_1(t) = \frac{1}{9}\big[1-\cos(3[t-1])\big]\,\text{step}_1(t) \quad .$$