

Chapter 31: Delta Functions

31.1 a. $|\mathcal{I}| = |m[v_{\text{after}} - v_{\text{before}}]| = |0.145[45 - 0]| = 6.525 \text{ (kg}\cdot\text{meter/sec)} .$

31.2 a. Solving the equation $\mathcal{I} = m[v_{\text{after}} - v_{\text{before}}]$ for v_{after} we get

$$v_{\text{after}} = \frac{\mathcal{I}}{m} + v_{\text{before}} .$$

In particular, assuming $m = 2$ and $v_{\text{before}} = -10$:

$$\text{If } \mathcal{I} = 60, \text{ then } v_{\text{after}} = \frac{60}{2} - 10 = 20 \text{ (meter/sec)} .$$

$$\text{If } \mathcal{I} = 100, \text{ then } v_{\text{after}} = \frac{100}{2} - 10 = 40 \text{ (meter/sec)} .$$

$$\text{If } \mathcal{I} = 20, \text{ then } v_{\text{after}} = \frac{20}{2} - 10 = 0 \text{ (meter/sec)} .$$

31.2 c. Solving the equation $\mathcal{I} = m[v_{\text{after}} - v_{\text{before}}]$ for m we get

$$m = \frac{\mathcal{I}}{v_{\text{after}} - v_{\text{before}}} .$$

In particular, assuming $\mathcal{I} = 30$ and $v_{\text{before}} = -10$:

$$\text{If } v_{\text{before}} = -10 \text{ and } v_{\text{after}} = 50, \text{ then } m = \frac{30}{50 - (-10)} = \frac{1}{2} \text{ (kg)} .$$

$$\text{If } v_{\text{before}} = 0 \text{ and } v_{\text{after}} = 15, \text{ then } m = \frac{30}{15 - 0} = 2 \text{ (kg)} .$$

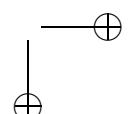
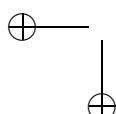
31.3 a. $\int_0^\infty t^2 \delta_4(t) dt = 4^2 = 16 .$

31.3 c. $\int_0^\infty \cos(t) \delta(t) dt = \int_0^\infty \cos(t) \delta_0(t) dt = \cos(0) = 1 .$

31.3 e. $\int_0^\infty t^2 \text{rect}_{(1,4)}(t) \delta_3(t) dt = 3^2 \text{rect}_{(1,4)}(3) = 9 \cdot 1 = 9 .$

31.5. Keeping in mind Theorem 31.1, and the fact that $\alpha \geq 0$ and $t > 0$, we have

$$\begin{aligned} g * \delta_\alpha(t) &= \delta_\alpha * g(t) = \int_0^t \delta_\alpha(x) g(t-x) dx \\ &= \int_0^\infty \delta_\alpha(x) g(t-x) \text{rect}_{(0,t)}(x) dx \\ &= g(t-\alpha) \text{rect}_{(0,t)}(\alpha) \\ &= g(t-\alpha) \cdot \begin{cases} 1 & \text{if } 0 < \alpha < t \\ 0 & \text{if } t < \alpha \end{cases} \\ &= g(t-\alpha) \cdot \begin{cases} 0 & \text{if } t < \alpha \\ 1 & \text{if } \alpha < t \end{cases} = g(t-\alpha) \text{step}_\alpha(t) . \end{aligned}$$





31.6 a.

$$\begin{aligned} \mathcal{L}[y']|_s &= \mathcal{L}[3\delta_2(t)]|_s \\ \leftrightarrow sY(s) - 0 &= 3\mathcal{L}[\delta_2(t)]|_s = 3e^{-2t} \\ \leftrightarrow Y(s) &= 3\frac{e^{-2s}}{s} \\ \leftrightarrow y(t) &= \mathcal{L}^{-1}[Y(s)]|_t = \mathcal{L}^{-1}\left[3\frac{e^{-2s}}{s}\right]|_t = 3\text{step}_2(t) . \end{aligned}$$

31.6 c.

$$\begin{aligned} \mathcal{L}[y'']|_s &= \mathcal{L}[\delta_3(t)]|_s \\ \leftrightarrow s^2Y(s) - s \cdot 0 - 0 &= e^{-3t} \\ \leftrightarrow Y(s) &= \frac{e^{-3s}}{s^2} \\ \leftrightarrow y(t) &= \mathcal{L}^{-1}[Y(s)]|_t = \mathcal{L}^{-1}\left[e^{-3s}\frac{1}{s^2}\right]|_t = f(t-3)\text{step}_3(t) \end{aligned}$$

where

$$f(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\right]|_t = t .$$

Thus,

$$f(X) = X , \quad f(t-3) = t-3$$

and

$$y(t) = f(t-3)\text{step}_3(t) = (t-3)\text{step}_3(t) .$$

31.6 e.

$$\begin{aligned} \mathcal{L}[y' + 2y]|_s &= \mathcal{L}[4\delta_1(t)]|_s \\ \leftrightarrow [sY(s) - 0] + 2Y(s) &= 4e^{-s} \\ \leftrightarrow Y(s) &= 4e^{-s}\frac{1}{s+2} \\ \leftrightarrow y(t) &= \mathcal{L}^{-1}\left[4e^{-1-s}\frac{1}{s+2}\right]|_t = 4f(t-1)\text{step}_1(t) \end{aligned}$$

where

$$f(t) = \mathcal{L}^{-1}\left[\frac{1}{s+2}\right]|_t = e^{-2t} .$$

So,

$$f(X) = e^{-2X} , \quad f(t-1) = e^{-2(t-1)}$$

and

$$y(t) = 4f(t-1)\text{step}_1(t) = 4e^{-2(t-1)}\text{step}_1(t) .$$

31.6 g.

$$\mathcal{L}[y'' + y]|_s = \mathcal{L}[-2\delta_{\pi/2}(t)]|_s$$

$$\leftrightarrow \left[s^2Y(s) - s \cdot 0 - 0\right] + Y(s) = -2e^{-\pi s/2}$$





$$\begin{aligned} \hookrightarrow & Y(s) = -2e^{-\pi s/2} \frac{1}{s^2 + 1} \\ \hookrightarrow & y(t) = \mathcal{L}^{-1} \left[-2e^{-\pi s/2} \frac{1}{s^2 + 1} \right] \Big|_t = -2f\left(t - \frac{\pi}{2}\right) \text{step}_{\pi/2}(t) \end{aligned}$$

where

$$f(t) = \mathcal{L}^{-1} \left[\frac{1}{s^2 + 1} \right] \Big|_t = \sin(t) .$$

So,

$$-2f\left(t - \frac{\pi}{2}\right) = -2 \sin\left(t - \frac{\pi}{2}\right) = 2 \cos(t)$$

and

$$y(t) = -2f\left(t - \frac{\pi}{2}\right) \text{step}_{\pi/2}(t) = \cos(t) \text{step}_{\pi/2}(t) .$$

31.7 a.

$$\begin{aligned} & \mathcal{L}[y' + 3y] \Big|_s = \mathcal{L}[\delta_2(t)] \Big|_s \\ \hookrightarrow & [sY(s) - \underbrace{y(0)}_2] + 3Y(s) = e^{-2s} \\ \hookrightarrow & (s + 3)Y(s) - 2 = e^{-2s} \\ \hookrightarrow & Y(s) = \frac{2}{s + 3} + e^{-2s} \frac{1}{s + 3} . \end{aligned}$$

Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left[\frac{2}{s + 3} + e^{-2s} \frac{1}{s + 3} \right] \Big|_t \\ &= 2\mathcal{L}^{-1} \left[\frac{1}{s + 3} \right] \Big|_t + \mathcal{L}^{-1} \left[e^{-2s} \frac{1}{s + 3} \right] \Big|_t = 2f(t) + f(t - 2) \text{step}_2(t) \end{aligned}$$

where

$$f(t) = \mathcal{L}^{-1} \left[\frac{1}{s + 3} \right] \Big|_t = e^{-3t} .$$

So,

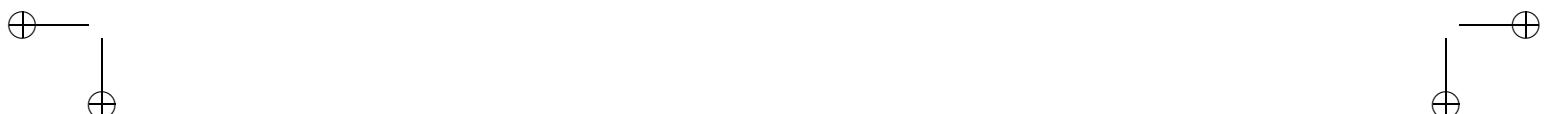
$$f(X) = e^{-3X} , \quad f(t - 2) = e^{-3(x-2)}$$

and

$$y(t) = 2f(t) + f(t - 2) \text{step}_2(t) = 2e^{-3t} + e^{-3(t-2)} \text{step}_2(t) .$$

31.7 c.

$$\begin{aligned} & \mathcal{L}[y'' + 3y'] \Big|_s = \mathcal{L}[\delta_1(t)] \Big|_s \\ \hookrightarrow & [s^2Y(s) - s\underbrace{y(0)}_0 - \underbrace{y'(s)}_1] \\ & + 3[sY(s) - \underbrace{y(0)}_0] = e^{-s} \\ \hookrightarrow & (s^2 + 3s)Y(s) - 2 = e^{-s} \\ \hookrightarrow & Y(s) = \frac{2}{s^2 + 3s} + e^{-s} \frac{1}{s^2 + 3s} . \end{aligned}$$





Thus,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left[\frac{2}{s^2 + 3s} + e^{-s} \frac{1}{s^2 + 3s} \right] \Big|_t \\ &= 2\mathcal{L}^{-1} \left[\frac{1}{s^2 + 3s} \right] \Big|_t + \mathcal{L}^{-1} \left[e^{-1s} \frac{1}{s^2 + 3s} \right] \Big|_t = 2f(t) + f(t-1) \text{step}_1(t) \end{aligned}$$

where

$$\begin{aligned} f(t) &= \mathcal{L}^{-1} \left[\frac{1}{s^2 + 3s} \right] \Big|_t = \mathcal{L}^{-1} \left[\frac{1}{s(s+3)} \right] \Big|_t \\ &= \dots \quad (\text{Use partial fractions or convolution.}) \\ &= \frac{1}{3} [1 - e^{-3t}] . \end{aligned}$$

So,

$$f(X) = \frac{1}{3} [1 - e^{-3t}] , \quad f(t-1) = \frac{1}{3} [1 - e^{-3(t-1)}]$$

and

$$y(t) = 2f(t) + f(t-2) \text{step}_2(t) = \frac{2}{3} [1 - e^{-3t}] + \frac{1}{3} [1 - e^{-3(t-1)}] \text{step}_1(t) .$$

31.7 e.

$$\mathcal{L}[y'' - 16y] \Big|_s = \mathcal{L}[\delta_{10}(t)] \Big|_s$$

$$\hookrightarrow [s^2 Y(s) - s \cdot 0 - 0] - 16Y(s) = e^{-10s}$$

$$\hookrightarrow Y(s) = e^{-10s} \frac{1}{s^2 + 16}$$

$$\hookrightarrow y(t) = \mathcal{L} \left[e^{-10s} \frac{1}{s^2 + 16} \right] \Big|_t = f(t-10) \text{step}_{10}(t)$$

where

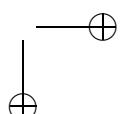
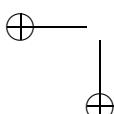
$$\begin{aligned} f(t) &= \mathcal{L}^{-1} \left[\frac{1}{s^2 + 16} \right] \Big|_t = \mathcal{L}^{-1} \left[\frac{1}{(s-4)(s+4)} \right] \Big|_t \\ &= \dots \quad (\text{Use partial fractions or convolution.}) \\ &= \frac{1}{8} [e^{4t} - e^{-4t}] . \end{aligned}$$

So,

$$f(X) = \frac{1}{8} [e^{4X} - e^{-4X}] , \quad f(t-10) = \frac{1}{8} [e^{4(t-10)} - e^{-4(t-10)}]$$

and

$$y(t) = f(t-10) \text{step}_{10}(t) = \frac{1}{8} [e^{4(t-10)} - e^{-4(t-10)}] \text{step}_{10}(t) .$$





$$\text{31.7 g.} \quad \mathcal{L}[y'' + 4y' - 12y]|_s = \mathcal{L}[\delta(t)]|_s$$

$$\begin{aligned} \leftrightarrow & \quad \left[s^2 Y(s) - s \cdot 0 - 0 \right] \\ & + 4[sY(s) - 0] - 12Y(s) = 1 \\ \leftrightarrow & \quad Y(s) = \frac{1}{s^2 + 4s - 12} . \end{aligned}$$

So, using either partial fractions or convolution, we find that

$$y(t) = \mathcal{L}^{-1} \left[\frac{1}{s^2 + 4s - 12} \right]_t = \mathcal{L}^{-1} \left[\frac{1}{(s-2)(s+6)} \right]_t = \frac{1}{8} [e^{2t} - e^{-6t}] .$$

$$\text{31.7 i.} \quad \mathcal{L}[y'' + 6y' + 9y]|_s = \mathcal{L}[\delta_4(t)]|_s$$

$$\begin{aligned} \leftrightarrow & \quad \left[s^2 Y(s) - s \cdot 0 - 0 \right] \\ & + 6[sY(s) - 0] + 9Y(s) = e^{-4s} \\ \leftrightarrow & \quad Y(s) = e^{-4s} \frac{1}{s^2 + 6s + 9} \\ \leftrightarrow & \quad y(t) = \mathcal{L} \left[e^{-4s} \frac{1}{s^2 + 6s + 9} \right]_t = f(t-4) \text{step}_4(t) \end{aligned}$$

where

$$\begin{aligned} f(t) &= \mathcal{L}^{-1} \left[\frac{1}{s^2 + 6s + 9} \right]_t \\ &= \mathcal{L}^{-1} \left[\frac{1}{(s+3)^2} \right]_t \\ &= \dots \quad (\text{Use the first translation identity or convolution.}) \\ &= te^{-3t} . \end{aligned}$$

So,

$$f(X) = Xe^{-3X}, \quad f(t-4) = (t-4)e^{-3(t-4)}$$

and

$$y(t) = f(t-4) \text{step}_4(t) = (t-4)e^{-3(t-4)} \text{step}_4(t) .$$

$$\text{31.7 k.} \quad \mathcal{L}[y''' + 9y']|_s = \mathcal{L}[\delta_1(t)]|_s$$

$$\begin{aligned} \leftrightarrow & \quad \left[s^3 Y(s) - s^2 \cdot 0 - s \cdot 0 - 0 \right] \\ & + 9[sY(s) - 0] = e^{-s} \end{aligned}$$





$$\hookrightarrow Y(s) = e^{-s} \frac{1}{s^3 + 9s}$$

$$\hookrightarrow y(t) = \mathcal{L} \left[e^{-1s} \frac{1}{s^3 + 9s} \right]_t = f(t-1) \text{step}_1(t)$$

where

$$\begin{aligned} f(t) &= \mathcal{L}^{-1} \left[\frac{1}{s^3 + 9s} \right]_t = \mathcal{L}^{-1} \left[\frac{1}{s(s^2 + 9)} \right]_t \\ &= \dots \quad (\text{Use partial fractions or convolution.}) \\ &= \frac{1}{9} [1 - \cos(3t)] . \end{aligned}$$

So,

$$f(X) = \frac{1}{9} [1 - \cos(3t)] , \quad f(t-1) = \frac{1}{9} [1 - \cos(3[t-1])]$$

and

$$y(t) = f(t-1) \text{step}_1(t) = \frac{1}{9} [1 - \cos(3[t-1])] \text{step}_1(t) .$$

