Chapter 28: The Inverse Laplace Transform

28.1 a. From the table: $\mathcal{L}^{-1}\left[\frac{1}{s-6}\right]\Big|_t = e^{6t}$.

28.1 c. $\mathcal{L}^{-1}\left[\frac{1}{s^2}\right]\Big|_t = \mathcal{L}^{-1}\left[\frac{1!}{s^{1+1}}\right]\Big|_t = t^1 = t$.

28.1 e. $\mathcal{L}^{-1}\left[\frac{5}{s^2+25}\right]\Big|_t = \mathcal{L}^{-1}\left[\frac{5}{s^2+5^2}\right]\Big|_t = \sin(5t)$.

28.2 a. $\mathcal{L}^{-1}\left[\frac{6}{s+2}\right]\Big|_t = \mathcal{L}^{-1}\left[6\cdot\frac{1}{s-(-2)}\right]\Big|_t = 6\mathcal{L}^{-1}\left[\frac{1}{s-(-2)}\right]\Big|_t = 6e^{-2t}$.

28.2 c.

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{3}{\sqrt{s}} - \frac{8}{s-4}\right]\Big|_t &= 3\mathcal{L}^{-1}\left[\frac{1}{\sqrt{s}}\right]\Big|_t - 8\mathcal{L}^{-1}\left[\frac{1}{s-4}\right]\Big|_t \\
&= 3\mathcal{L}^{-1}\left[\frac{1}{\sqrt{\pi}}\frac{\sqrt{\pi}}{\sqrt{s}}\right]\Big|_t - 8e^{4t} \\
&= \frac{3}{\sqrt{\pi}}\mathcal{L}^{-1}\left[\frac{\sqrt{\pi}}{\sqrt{s}}\right]\Big|_t - 8e^{4t} \\
&= \frac{3}{\sqrt{\pi}}\cdot\frac{1}{\sqrt{t}} - 8e^{4t} = \frac{3}{\sqrt{\pi t}} - 8e^{4t} .
\end{aligned}$$

28.2 e.

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{3s+1}{s^2+25}\right]\Big|_t &= \mathcal{L}^{-1}\left[\frac{3s}{s^2+25} + \frac{1}{s^2+25}\right]\Big|_t \\
&= \mathcal{L}^{-1}\left[3\frac{s}{s^2+5^2} + \frac{1}{5}\frac{5}{s^2+5^2}\right]\Big|_t \\
&= 3\mathcal{L}^{-1}\left[\frac{s}{s^2+5^2}\right]\Big|_t + \frac{1}{5}\mathcal{L}^{-1}\left[\frac{5}{s^2+5^2}\right]\Big|_t \\
&= 3\cos(5t) + \frac{1}{5}\sin(5t) .
\end{aligned}$$

28.3 a. From Exercise 27.3, we have $\mathcal{L}[t\sin(\omega t)]|_s = \frac{2\omega s}{\left(s^2+\omega^2\right)^2}$.

By the definition of the inverse Laplace transform and linearity, it then follows that

$$t\sin(\omega t) = \mathcal{L}^{-1}\left[\frac{2\omega s}{\left(s^2+\omega^2\right)^2}\right]\Big|_t = 2\omega\mathcal{L}^{-1}\left[\frac{s}{\left(s^2+\omega^2\right)^2}\right]\Big|_t .$$

Dividing through by 2ω then yields

$$\mathcal{L}^{-1}\left[\frac{s}{\left(s^2+\omega^2\right)^2}\right]\Big|_t = \frac{t}{2\omega}\sin(\omega t) .$$

28.4 a.

$$\mathcal{L}\big[y' + 9y\big]\big|_s = \mathcal{L}[0]|_s$$

$$\hookrightarrow \quad \mathcal{L}\big[y'\big]\big|_s + 9\mathcal{L}[y]|_s = 0$$

$$\hookrightarrow \quad [sY(s) - \underbrace{y(0)}_{4}] + 9Y(s) = 0$$

$$\hookrightarrow \quad (s+9)Y(s) - 4 = 0 \quad \rightarrowtail \quad Y(s) = \frac{4}{s+9} \quad .$$

Taking the inverse:

$$y(t) = \mathcal{L}^{-1}[Y(s)]|_t = \mathcal{L}^{-1}\left[\frac{4}{s+9}\right]\bigg|_t = 4\mathcal{L}^{-1}\left[\frac{1}{s-(-9)}\right]\bigg|_t = 4e^{-9t} \quad .$$

28.5 a. First, we must find the partial fraction expansion:

$$\frac{7s+5}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1}$$
$$= \frac{A(s-1)}{(s+2)(s-1)} + \frac{B(s+2)}{(s-1)(s+2)} = \frac{A(s-1)+B(s+2)}{(s+2)(s-1)} \quad .$$

Cutting out the middle and multiplying through by $(s+2)(s-1)$ then gives

$$A(s-1) + B(s+2) = 7s + 5 \quad .$$

Solving for A and B is easily done by, respectively, plugging $s = -2$ and $s = 1$ into the last equation and "solving":

$$A(-2-1) + B(-2+2) = 7(-2) + 5 \quad \rightarrowtail \quad A = \frac{7(-2)+5}{-3} = 3$$

and

$$A(1-1) + B(1+2) = 7 \cdot 1 + 5 \quad \rightarrowtail \quad B = \frac{7+5}{1+2} = 4 \quad .$$

Thus, the partial fraction expansion of our function is

$$\frac{7s+5}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1} = \frac{3}{s+2} + \frac{4}{s-1} \quad .$$

Now we can take the inverse transform:

$$\mathcal{L}^{-1}\left[\frac{7s+5}{(s+2)(s-1)}\right]\bigg|_s = \mathcal{L}^{-1}\left[\frac{3}{s+2} + \frac{4}{s-1}\right]\bigg|_s$$
$$= 3\mathcal{L}^{-1}\left[\frac{1}{s+2}\right]\bigg|_s + 4\mathcal{L}^{-1}\left[\frac{1}{s-1}\right]\bigg|_s = 3e^{-2t} + 4e^{t} \quad .$$

28.5 c.

$$\frac{1}{s^2-4} = \frac{1}{(s-2)(s+2)}$$
$$= \frac{A}{s-2} + \frac{B}{s+2}$$
$$= \frac{A(s+2)}{(s-2)(s+2)} + \frac{B(s-2)}{(s+2)(s-2)} = \frac{A(s+2)+B(s-2)}{(s-2)(s+2)} \quad .$$

So,

$$A(s+2) + B(s-2) = 1 \quad .$$

Letting $s = 2$ yields

$$A(2+2) + B(2-2) = 1 \quad \rightarrowtail \quad A = \frac{1}{4} \quad ,$$

while letting $s = -1$ yields

$$A(-2+2) + B(-2-2) = 1 \quad \rightarrowtail \quad B = -\frac{1}{4} \quad .$$

Thus,

$$\frac{1}{s^2-4} = \frac{A}{s-2} + \frac{B}{s+2} = \frac{1/4}{s-2} + \frac{-1/4}{s+2} \quad ,$$

and

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{1}{s^2-4}\right]\bigg|_t &= \mathcal{L}^{-1}\left[\frac{1/4}{s-2} + \frac{-1/4}{s+2}\right]\bigg|_t \\ &= \frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s-2}\right]\bigg|_t - \frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s+2}\right]\bigg|_t = \frac{1}{4}e^{2t} - \frac{1}{4}e^{-2t} \quad .\end{aligned}$$

28.5 e.

$$\begin{aligned}\frac{1}{s^3-4s^2} &= \frac{1}{s^2(s-4)} \\ &= \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s-4} = \frac{A(s-4)+Bs(s-4)+Cs^2}{s^2(s-4)} \quad .\end{aligned}$$

So

$$A(s-4) + Bs(s-4) + Cs^2 = 1 \quad .$$

Multiplying this out and gathering like terms yields

$$(B+C)s^2 + (A-4B)s - 4A = 1 = 0s^2 + 0s + 1 \quad ,$$

giving us the system

$$\begin{aligned} B + C &= 0 \\ A - 4B &= 0 \\ -4A &= 1 \end{aligned} \quad ,$$

which, in turn, means that

$$A = -\frac{1}{4} \quad , \quad B = \frac{A}{4} = -\frac{1}{16} \quad \text{and} \quad C = -B = \frac{1}{16} \quad .$$

So

$$\frac{1}{s^3-4s^2} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s-4} = -\frac{1/4}{s^2} - \frac{1/16}{s} + \frac{1/16}{s-4} \quad ,$$

and

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{1}{s^3-4s^2}\right]\bigg|_t &= \mathcal{L}^{-1}\left[-\frac{1/4}{s^2} - \frac{1/16}{s} + \frac{1/16}{s-4}\right]\bigg|_t \\ &= -\frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s^2}\right]\bigg|_t - \frac{1}{16}\mathcal{L}^{-1}\left[\frac{1}{s}\right]\bigg|_t + \frac{1}{16}\mathcal{L}^{-1}\left[\frac{1}{s-4}\right]\bigg|_t \\ &= -\frac{1}{4}t^1 - \frac{1}{16}\cdot 1 + \frac{1}{16}e^{4t} = \frac{1}{16}e^{4t} - \frac{1}{4}t - \frac{1}{16} \quad .\end{aligned}$$

28.5 g.

$$\frac{5s^2+6s-40}{(s+6)\left(s^2+16\right)} = \frac{A}{s+6} + \frac{Bs+C}{s^2+16} = \frac{A\left(s^2+16\right)+(Bs+C)(s+6)}{(s+6)\left(s^2+16\right)} \quad .$$

So

$$A\left(s^2+16\right)+(Bs+C)(s+6) = 5s^2+6s-40 \quad . \qquad (\star)$$

Multiplying this out and gathering like terms yields

$$(A+B)s^2 + (6B+C)s + (16A+6C) = 5s^2+6s-40 \quad ,$$

giving us the system

$$\begin{aligned} A + B &= 5 \\ 6B + C &= 6 \\ 16A + 6C &= 40 \end{aligned} \quad . \qquad (\star\star)$$

The first coefficient, A, is easily found after first letting $s = -6$ in equation $(\star)$:

$$A\left([-6]^2+16\right) + (B[-6]+C)(-6+6) = 5[-6]^2 + 6[-6] - 40$$

$$\hookrightarrow \qquad A = \frac{5[-6]^2+6[-6]-40}{[-6]^2+16} = 2 \quad .$$

From this and system $(\star\star)$, we then obtain

$$B = 5 - A = 5 - 2 = 3 \qquad \text{and} \qquad C = 6 - 6B = 6 - 6\cdot 3 = -12 \quad .$$

So,

$$\begin{aligned} \frac{5s^2+6s-40}{(s+6)\left(s^2+16\right)} &= \frac{A}{s+6} + \frac{Bs+C}{s^2+16} \\ &= \frac{2}{s+6} + \frac{3s-12}{s^2+16} = \frac{2}{s+6} + \frac{3s}{s^2+16} - \frac{12}{s^2+16} \quad , \end{aligned}$$

and

$$\begin{aligned} &\mathcal{L}^{-1}\left[\frac{5s^2+6s-40}{(s+6)\left(s^2+16\right)}\right]\Bigg|_t \\ &\quad = \mathcal{L}^{-1}\left[\frac{2}{s+6} + \frac{3s}{s^2+16} - \frac{12}{s^2+16}\right]\Bigg|_t \\ &\quad = 2\mathcal{L}^{-1}\left[\frac{1}{s+6}\right]\Bigg|_t + 3\mathcal{L}^{-1}\left[\frac{s}{s^2+16}\right]\Bigg|_t - 12\mathcal{L}^{-1}\left[\frac{1}{s^2+16}\right]\Bigg|_t \\ &\quad = 2e^{-6t} + 3\mathcal{L}^{-1}\left[\frac{s}{s^2+4^2}\right]\Bigg|_t - \frac{12}{4}\mathcal{L}^{-1}\left[\frac{4}{s^2+4^2}\right]\Bigg|_t \\ &\quad = 2e^{-6t} + 3\cos(4t) - 3\sin(4t) \quad . \end{aligned}$$

28.5 i.

$$\begin{aligned} \frac{6s^2+62s+92}{(s+1)\left(s^2+10s+21\right)} &= \frac{6s^2+62s+92}{(s+1)(s+3)(s+7)} \\ &= \frac{A}{s+1} + \frac{B}{s+3} + \frac{C}{s+7} \\ &= \frac{A(s+3)(s+7)+B(s+1)(s+7)+C(s+1)(s+3)}{(s+1)(s+3)(s+7)} \quad . \end{aligned}$$

So

$$A(s+3)(s+7) \;+\; B(s+1)(s+7) \;+\; C(s+1)(s+3) \;=\; 6s^2 \;+\; 62s \;+\; 92 \quad . \quad (\star)$$

Letting $s = -1$ in equation ($\star$):

$$A(-1+3)(-1+7) \;+\; B \cdot 0 \cdot (-1+7) \;+\; C \cdot 0 \cdot (-1+3)$$
$$= \; 6[-1]^2 \;+\; 62[-1] + 92$$

$$\hookrightarrow \qquad A \;=\; \frac{6[-1]^2 + 62[-1] + 92}{(-1+3)(-1+7)} \;=\; 3 \quad .$$

Letting $s = -3$ in equation ($\star$):

$$A \cdot 0 \cdot (-3+7) \;+\; B(-3+1)(-3+7) \;+\; C(-3+1) \cdot 0$$
$$= \; 6[-3]^2 \;+\; 62[-3] \;+\; 92$$

$$\hookrightarrow \qquad B \;=\; \frac{6[-3]^2 + 62[-3] + 92}{(-3+1)(-3+7)} \;=\; 5 \quad .$$

Letting $s = -7$ in equation ($\star$):

$$A(-7+3) \cdot 0 \;+\; B(-7+1) \cdot 0 \;+\; C(-7+1)(-7+3)$$
$$= \; 6[-7]^2 \;+\; 62[-7] \;+\; 92$$

$$\hookrightarrow \qquad C \;=\; \frac{6[-7]^2 + 62[-7] + 92}{(-7+1)(-7+3)} \;=\; -2 \quad .$$

Thus,

$$\frac{6s^2 + 62s + 92}{(s+1)\left(s^2 + 10s + 21\right)} \;=\; \frac{A}{s+1} \;+\; \frac{B}{s+3} \;+\; \frac{C}{s+7}$$
$$= \; \frac{3}{s+1} \;+\; \frac{5}{s+3} \;-\; \frac{2}{s+7} \quad ,$$

and

$$\mathcal{L}^{-1}\left[\frac{6s^2 + 62s + 92}{(s+1)\left(s^2 + 10s + 21\right)}\right]\Bigg|_t$$
$$= \; \mathcal{L}^{-1}\left[\frac{3}{s+1} \;+\; \frac{5}{s+3} \;-\; \frac{2}{s+7}\right]\Bigg|_t$$
$$= \; 3\mathcal{L}^{-1}\left[\frac{1}{s+1}\right]\Bigg|_t \;+\; 5\mathcal{L}^{-1}\left[\frac{1}{s+3}\right]\Bigg|_t \;-\; 2\mathcal{L}^{-1}\left[\frac{1}{s+7}\right]\Bigg|_t$$
$$= \; 3e^{-t} \;+\; 5e^{-3t} \;-\; 2e^{-7} \quad .$$

28.6 a. Taking the transform:

$$\mathcal{L}\big[y'' - 9y\big]\big|_s \;=\; \mathcal{L}[0]|_s$$

$$\hookrightarrow \qquad \mathcal{L}\big[y''\big]\big|_s \;-\; 9\mathcal{L}[y]|_s \;=\; 0$$

$$\hookrightarrow \quad \left[s^2 Y(s) - s \underbrace{y(0)}_{4} - \underbrace{y'(0)}_{9}\right] - 9Y(s) = 0$$

$$\hookrightarrow \quad \left(s^2 - 9\right) Y(s) - 4s - 9 = 0 \quad \rightarrowtail \quad Y(s) = \frac{4s+9}{s^2-9} \quad .$$

Finding the partial fraction expansion for Y and then taking the inverse transform:

$$\frac{4s+9}{s^2-9} = \frac{4s+9}{(s-3)(s+3)} = \frac{A}{s-3} + \frac{B}{s+3} = \frac{A(s+3)+B(s-3)}{(s-3)(s+3)} \quad .$$

So,

$$A(s+3) + B(s-3) = 4s + 9 \quad .$$

Letting $s = 3$ and $s = -3$, respectively, in this last equation:

$$A(3+3) + B \cdot 0 = 4 \cdot 3 + 9 \quad \rightarrowtail \quad A = \frac{4 \cdot 3 + 9}{3+3} = \frac{7}{2} \quad ,$$

and

$$A \cdot 0 + B(-3-3) = 4[-3] + 9 \quad \rightarrowtail \quad B = \frac{4[-3] + 9}{-3-3} = \frac{1}{2} \quad .$$

Thus,

$$Y(s) = \frac{A}{s-3} + \frac{B}{s+3} = \frac{7/2}{s-3} + \frac{1/2}{s+3} \quad ,$$

and

$$\begin{aligned} y(t) = \mathcal{L}^{-1}[Y(s)]|_t &= \mathcal{L}^{-1}\left[\frac{7/2}{s-3} + \frac{1/2}{s+3}\right]\Big|_t \\ &= \frac{7}{2}\mathcal{L}^{-1}\left[\frac{1}{s-3}\right]\Big|_t + \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s+3}\right]\Big|_t = \frac{7}{2}e^{3t} + \frac{1}{2}e^{-3t} \quad . \end{aligned}$$

28.6 c. Taking the transform:

$$\mathcal{L}\left[y'' + 8y' + 7y\right]\Big|_s = \mathcal{L}\left[165e^{4t}\right]\Big|_s$$

$$\hookrightarrow \quad \mathcal{L}\left[y''\right]\Big|_s + 8\mathcal{L}\left[y'\right]\Big|_s + 7\mathcal{L}[y]|_s = 165\mathcal{L}\left[e^{4t}\right]\Big|_s$$

$$\begin{aligned} \hookrightarrow \quad & \left[s^2 Y(s) - s \underbrace{y(0)}_{8} - \underbrace{y'(0)}_{1}\right] \\ & + 8[sY(s) - \underbrace{y(0)}_{8}] + 7Y(s) = \frac{165}{s-4} \end{aligned}$$

$$\hookrightarrow \quad \left(s^2 + 8s + 7\right) Y(s) - 8s - 65 = \frac{165}{s-4}$$

Solving for Y:

$$\begin{aligned} Y(s) &= \frac{165}{(s-4)\left(s^2+8s+7\right)} + \frac{8s+65}{s^2+8s+7} \\ &= \frac{165 + (8s+65)(s-4)}{(s-4)\left(s^2+8s+7\right)} = \frac{8s^2+33s-95}{(s-4)(s+7)(s+1)} \quad . \end{aligned}$$

Finding the partial fraction expansion of Y :

$$\begin{aligned}
Y(s) &= \frac{8s^2 + 33s - 95}{(s-4)(s+7)(s+1)} \\
&= \frac{A}{s-4} + \frac{B}{s+7} + \frac{C}{s+1} \\
&= \frac{A(s+7)(s+1) + B(s-4)(s+1) + C(s-4)(s+7)}{(s-4)(s+7)(s+1)} \quad ,
\end{aligned}$$

which requires that

$$A(s+7)(s+1) + B(s-4)(s+1) + C(s-4)(s+7) = 8s^2 + 33s - 95 \quad . \qquad (\star)$$

Letting $s = 4$ in equation $(\star)$:

$$A(4+7)(4+1) + B \cdot 0 \cdot (4+1) + C \cdot 0 \cdot (4+7) = 8 \cdot 4^2 + 33 \cdot 4 - 95$$

$$\hookrightarrow \qquad A = \frac{8 \cdot 4^2 + 33 \cdot 4 - 95}{(4+7)(4+1)} = 3 \quad .$$

Letting $s = -7$ in equation $(\star)$:

$$A \cdot 0 \cdot (-7+1) + B(-7-4)(-7+1) + C(-7-4) \cdot 0 = 8[-7]^2 + 33[-7] - 95$$

$$\hookrightarrow \qquad B = \frac{8[-7]^2 + 33[-7] - 95}{(-7-4)(-7+1)} = 1 \quad .$$

Letting $s = -1$ in equation $(\star)$:

$$A(-1+7) \cdot 0 + B(-1-4) \cdot 0 + C(-1-4)(-1+7) = 8[-1]^2 + 33[-1] - 95$$

$$\hookrightarrow \qquad C = \frac{8[-1]^2 + 33[-1] - 95}{(-1-4)(-1+7)} = 4 \quad .$$

So,

$$Y(s) = \frac{A}{s-4} + \frac{B}{s+7} + \frac{C}{s+1} = \frac{3}{s-4} + \frac{1}{s+7} + \frac{4}{s+1} \quad ,$$

and

$$\begin{aligned}
y(t) = \mathcal{L}^{-1}[Y(s)]|_t &= \mathcal{L}^{-1}\left[\frac{3}{s-4} + \frac{1}{s+7} + \frac{4}{s+1}\right]\bigg|_t \\
&= 3\mathcal{L}^{-1}\left[\frac{1}{s-4}\right]\bigg|_t + \mathcal{L}^{-1}\left[\frac{1}{s-(-7)}\right]\bigg|_t + 4\mathcal{L}^{-1}\left[\frac{1}{s-(-1)}\right]\bigg|_t \\
&= 3e^{4t} + e^{-7t} + 4e^{-t} \quad .
\end{aligned}$$

28.7 a. Using the translation identity, we have

$$\mathcal{L}^{-1}\left[\frac{1}{(s-7)^5}\right]\bigg|_t = \mathcal{L}^{-1}[F(s-7)]|_t = e^{7t} f(t) \qquad (\star)$$

where

$$F(s-7) = \frac{1}{(s-7)^5} \quad .$$

Replacing $s-7$ with X, this means

$$F(X) = \frac{1}{X^5} \quad .$$

Hence,

$$F(s) = \frac{1}{s^5}$$

and

$$f(t) = \mathcal{L}^{-1}[F(s)]\big|_t = \mathcal{L}^{-1}\left[\frac{1}{s^5}\right]\bigg|_t = \frac{1}{4!}\mathcal{L}^{-1}\left[\frac{4!}{s^{4+1}}\right]\bigg|_t = \frac{1}{24}t^4 \quad .$$

Thus, equation $(\star)$ becomes

$$\mathcal{L}^{-1}\left[\frac{1}{(s-7)^5}\right]\bigg|_t = \cdots = e^{7t}f(t) = e^{7t}\frac{1}{24}t^4 \quad .$$

28.7 c. Since the denominator does not factor, we "complete the square" of the denominator:

$$s^2 - 6s + 45 = \underbrace{s^2 - 2\cdot 3s + 3^2}_{(s-3)^2} - 3^2 + 45 = (s-3)^2 + 36 \quad .$$

So,

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{s}{s^2-6s+45}\right]\bigg|_t &= \mathcal{L}^{-1}\left[\frac{s}{(s-3)^2+36}\right]\bigg|_t \\ &= \mathcal{L}^{-1}[F(s-3)]\big|_t = e^{3t}f(t)\end{aligned} \qquad (\star)$$

with

$$F(s-3) = \frac{s}{(s-3)^2+36} \quad .$$

To get $F(s)$, first let $X = s-3$ (equivalently, $s = X+3$) in this equation:

$$F(X) = \frac{X+3}{X^2+36} \quad ,$$

which means

$$F(s) = \frac{s+3}{s^2+36} = \frac{s}{s^2+6^2} + \frac{3}{s^2+6^2} \quad .$$

Thus,

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}[F(s)]\big|_t \\ &= \mathcal{L}^{-1}\left[\frac{s}{s^2+6^2} + \frac{3}{s^2+6^2}\right]\bigg|_t \\ &= \mathcal{L}^{-1}\left[\frac{s}{s^2+6^2}\right]\bigg|_t + \frac{3}{6}\mathcal{L}^{-1}\left[\frac{6}{s^2+6^2}\right]\bigg|_t = \cos(6t) + \frac{1}{2}\sin(6t) \quad , \end{aligned}$$

and equation $(\star)$ continues as

$$\mathcal{L}^{-1}\left[\frac{s}{s^2-6s+45}\right]\bigg|_t = \cdots = e^{3t}f(t) = e^{3t}\left[\cos(6t) + \frac{1}{2}\sin(6t)\right] \quad .$$

28.7 e. Since the denominator factors as

$$s^2 + 8s + 16 = (s+4)^2 = \big(s - [-4]\big)^2 \quad ,$$

we can apply the translation identity as follows:

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+8s+16}\right]\Bigg|_t = \mathcal{L}^{-1}\left[\frac{1}{\big(s-[-4]\big)^2}\right]\Bigg|_t \qquad (\star)$$
$$= F(s-[-4]) = e^{-4t} f(t) \quad .$$

with

$$F(s-[-4]) = \frac{1}{\big(s-[-4]\big)^2} \quad .$$

Letting $X = s - [-4]$ this becomes

$$F(X) = \frac{1}{X^2} \quad .$$

So,

$$F(s) = \frac{1}{s^2} \quad , \qquad f(t) = \mathcal{L}^{-1}[F(s)]|_t = \mathcal{L}^{-1}\left[\frac{1}{s^2}\right]\Bigg|_t = t \quad ,$$

and equation $(\star)$ becomes

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+8s+16}\right]\Bigg|_t = \cdots = e^{-4t} f(t) = e^{-4t} t = te^{-4t} \quad .$$

28.7 g. Since

$$s^2 + 12s + 40 = s^2 + 12s + 36 + 4 = (s+6)^2 + 4 \quad ,$$

we have

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+12s+40}\right]\Bigg|_t = \mathcal{L}^{-1}\left[\frac{1}{(s+6)^2+4}\right]\Bigg|_t \qquad (\star)$$
$$= F(s+6) = F(s-[-6]) = e^{-6t} f(t)$$

with

$$F(s+6) = \frac{1}{(s+6)^2+4} \quad .$$

Letting $X = s + 6$, this becomes

$$F(X) = \frac{1}{X^2+4} \quad .$$

Hence,

$$F(s) = \frac{1}{s^2+4} = \frac{1}{s^2+2^2} \quad ,$$

$$f(t) = \mathcal{L}^{-1}[F(s)]|_t = \mathcal{L}^{-1}\left[\frac{1}{s^2+2^2}\right]\Bigg|_t = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{s^2+2^2}\right]\Bigg|_t = \frac{1}{2}\sin(2t) \quad ,$$

and equation $(\star)$ becomes

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+12s+40}\right]\Bigg|_t = \cdots = e^{-6t} f(t) = e^{-6t}\left[\frac{1}{2}\sin(2t)\right] = \frac{1}{2}\sin(2t)e^{-6t} \quad .$$

28.8 a.

$$\mathcal{L}\big[y'' - 8y' + 17y\big]\big|_t = \mathcal{L}[0]|_t$$

$$\hookrightarrow \quad \mathcal{L}\big[y''\big]\big|_t - 8\mathcal{L}\big[y'\big]\big|_t + 17\mathcal{L}[y]|_t = 0$$

$$\hookrightarrow \quad \big[s^2 Y(s) - s\underbrace{y(0)}_{3} - \underbrace{y'(0)}_{12}\big] - 8[sY(s) - \underbrace{y(0)}_{3}] + 17Y(s) = 0$$

$$\hookrightarrow \quad \left(s^2 - 8s + 17\right) Y(s) - 3s + 12 = 0 \quad .$$

So,

$$Y(s) = \frac{3s-12}{s^2-8s+17} = \frac{3s-12}{s^2-8s+16+1} = \frac{3s-12}{(s-4)^2+1} \quad ,$$

and

$$y(t) = \mathcal{L}^{-1}\left[\frac{3s-12}{(s-4)^2+1}\right]\Bigg|_t = \mathcal{L}^{-1}[F(s-4)]|_t = e^{4t}f(t) \qquad (\star)$$

with

$$F(s-4) = \frac{3s-12}{(s-4)^2+1} \quad .$$

Letting $X = s - 4$ (equivalently, $s = X + 4$) we have

$$F(X) = \frac{3[X+4]-12}{X^2+1} = \frac{3X}{X^2+1} \quad .$$

Thus,

$$F(s) = \frac{3s}{s^2+1} \quad ,$$

$$f(t) = \mathcal{L}^{-1}\left[\frac{3s}{s^2+1}\right]\Bigg|_t = 3\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right]\Bigg|_t = 3\cos(t) \quad ,$$

and

$$y(t) = \cdots = e^{4t}f(t) = 3\cos(t)e^{4t} \quad .$$

28.8 c.

$$\mathcal{L}\big[y'' + 6y' + 13y\big]\big|_t = \mathcal{L}[0]|_t$$

$$\hookrightarrow \quad \mathcal{L}\big[y''\big]\big|_t + 6\mathcal{L}\big[y'\big]\big|_t + 13\mathcal{L}[y]|_t = 0$$

$$\hookrightarrow \quad \big[s^2 Y(s) - s\underbrace{y(0)}_{2} - \underbrace{y'(0)}_{8}\big] + 6[sY(s) - \underbrace{y(0)}_{2}] + 13Y(s) = 0$$

$$\hookrightarrow \quad \left(s^2 + 6s + 13\right) Y(s) - 2s - 20 = 0 \quad .$$

So,

$$Y(s) = \frac{2s+20}{s^2+6s+13} = \frac{2s+20}{s^2+6s+9+4} = \frac{2s+20}{(s+3)^2+4} = \frac{2s+20}{\left(s-[-3]\right)^2+4} \quad ,$$

and

$$y(t) = \mathcal{L}^{-1}\left[\frac{2s+20}{\left(s-[-3]\right)^2+4}\right]\bigg|_t = \mathcal{L}^{-1}[F(s-[-3])]|_t = e^{-3t}f(t) \qquad (\star)$$

with

$$F(s-[-3]) = \frac{2s+20}{\left(s-[-3]\right)^2+4} \quad .$$

Letting $X = s-[-3] = s+3$ (equivalently, $s = X-3$), we have

$$F(X) = \frac{2[X-3]+20}{X^2+4} = \frac{2X+14}{X^2+4} \quad .$$

Thus,

$$F(s) = \frac{2s+14}{s^2+4} = \frac{2s}{s^2+2^2} + \frac{7\cdot 2}{s^2+2^2} \quad ,$$

$$f(t) = \mathcal{L}^{-1}\left[\frac{2s}{s^2+2^2} + \frac{7\cdot 2}{s^2+2^2}\right]\bigg|_t$$
$$= 2\mathcal{L}^{-1}\left[\frac{s}{s^2+2^2}\right]\bigg|_t + 7\mathcal{L}^{-1}\left[\frac{2}{s^2+2^2}\right]\bigg|_t = 2\cos(2t) + 7\sin(2t) \quad ,$$

and

$$y(t) = \cdots = e^{-3t}f(t) = e^{-3t}\left[2\cos(2t)+7\sin(2t)\right] \quad .$$

28.9 a.

$$\mathcal{L}\left[y''\right]\big|_t = \mathcal{L}\left[e^t\sin(t)\right]\big|_s$$

$$\hookrightarrow \quad s^2Y(s) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0} = \mathcal{L}[e^{1t}\underbrace{\sin(t)}_{f(t)}]|_s$$

$$\hookrightarrow \quad s^2Y(s) = F(s-1)$$

where

$$F(s) = \mathcal{L}[f(t)]|_s = \mathcal{L}[\sin(t)]|_s = \frac{1}{s^2+1} \quad .$$

So, the last line containing Y becomes

$$s^2Y(s) = F(\underbrace{s-1}_{X}) = F(X) = \frac{1}{X^2+1} = \frac{1}{(s-1)^2+1} \quad .$$

Solving for Y and starting to find the corresponding partial fraction expansion leads to

$$Y(s) = \frac{1}{s^2\left((s-1)^2+1\right)}$$
$$= \frac{A}{s^2} + \frac{B}{s} + \frac{Cs+D}{(s-1)^2+1}$$
$$= \frac{A\left((s-1)^2+1\right) + Bs\left((s-1)^2+1\right) + (Cs+D)s^2}{s^2\left((s-1)^2+1\right)} \quad ,$$

which requires that

$$A\big((s-1)^2+1\big) \;+\; Bs\big((s-1)^2 \;+\; 1\big) \;+\; (Cs+D)s^2 \;=\; 1 \quad .$$

Multiplying this out and gathering like terms then yields

$$\begin{aligned} s^3 \;+\; [A-2B+D]s^2 \;+\; [-2A+2B]s \;+\; 2A \;&=\; 1 \\ &=\; 0s^3 \;+\; 0s^2 \;+\; 0s \;+\; 1 \quad , \end{aligned}$$

which, in turn, requires that

$$\begin{aligned} B \;+\; C \;&=\; 0 \quad , \\ A \;-\; 2B \;+\; D \;&=\; 0 \quad , \\ -2A \;+\; 2B \;&=\; 0 \quad , \end{aligned}$$

and

$$2A \;=\; 1 \quad .$$

From this it follows that

$$A \;=\; \tfrac{1}{2} \quad , \quad B \;=\; A \;=\; \tfrac{1}{2} \quad , \quad D \;=\; 2B \;-\; A \;=\; \tfrac{1}{2} \quad , \quad C \;=\; -B \;=\; -\tfrac{1}{2} \quad ,$$

and, thus, the partial fraction expansion of Y is

$$\begin{aligned} Y(s) \;&=\; \frac{A}{s^2} \;+\; \frac{B}{s} \;+\; \frac{Cs+D}{(s-1)^2+1} \\ &=\; \frac{1}{2}\left(\frac{1}{s^2} \;+\; \frac{1}{s} \;+\; \frac{-s+1}{(s-1)^2+1}\right) \;=\; \frac{1}{2}\left(\frac{1}{s^2} \;+\; \frac{1}{s} \;-\; \frac{s-1}{(s-1)^2+1}\right) \quad . \end{aligned}$$

Consequently,

$$\begin{aligned} y(t) \;&=\; \mathcal{L}^{-1}\left[\frac{1}{2}\left(\frac{1}{s^2} \;+\; \frac{1}{s} \;-\; \frac{s-1}{(s-1)^2+1}\right)\right]\bigg|_t \\ &=\; \frac{1}{2}\left(\mathcal{L}^{-1}\left[\frac{1}{s^2}\right]\bigg|_t + \mathcal{L}^{-1}\left[\frac{1}{s}\right]\bigg|_t - \mathcal{L}^{-1}\left[\frac{s-1}{(s-1)^2+1}\right]\bigg|_t\right) \\ &=\; \frac{1}{2}\left(t+1-\mathcal{L}^{-1}[G(s-1)]|_t\right) \;=\; \frac{1}{2}\left(t+1-e^{1t}g(t)\right) \end{aligned}$$

where

$$G(s-1) \;=\; \frac{s-1}{(s-1)^2+1} \quad .$$

Letting $X = s-1$, this becomes

$$G(X) \;=\; \frac{X}{X^2+1} \quad .$$

Hence,

$$g(t) \;=\; \mathcal{L}^{-1}[G(s)]|_t \;=\; \mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right]\bigg|_t \;=\; \cos(t) \quad ,$$

and the last formula above for y becomes

$$y(t) \;=\; \frac{1}{2}\left(t+1-e^{1t}g(t)\right) \;=\; \frac{1}{2}\left(t+1-e^{t}\cos(t)\right) \quad .$$

28.9 c.

$$\mathcal{L}\big[y'' - 9y\big]\big|_s = \mathcal{L}\Big[24e^{-3t}\Big]\Big|_s$$

$$\hookrightarrow \quad \mathcal{L}\big[y''\big]\big|_s - \mathcal{L}[9y]|_s = 24\mathcal{L}\Big[e^{-3t}\Big]\Big|_s$$

$$\hookrightarrow \quad \Big[s^2 Y(s) - s\underbrace{y(0)}_{6} - \underbrace{y'(0)}_{2}\Big] - 9Y(s) = \frac{24}{s - [-3]t}$$

$$\hookrightarrow \quad \left(s^2 - 9\right) Y(s) - 6s - 2 = \frac{24}{s+3}$$

So,

$$\begin{aligned} Y(s) &= \frac{24}{(s+3)\left(s^2-9\right)} + \frac{6s+2}{s^2-9} \\ &= \frac{24 + (6s+2)(s+3)}{(s+3)\left(s^2-9\right)} \\ &= \frac{6s^2 + 20s + 30}{(s+3)^2(s-3)} \\ &= \frac{A}{(s+3)^2} + \frac{B}{s+3} + \frac{C}{s-3} = \frac{A(s-3) + B(s+3)(s-3) + C(s+3)^2}{(s+3)^2(s-3)} \quad . \end{aligned}$$

This requires that

$$A(s-3) + B(s+3)(s-3) + C(s+3)^2 = 6s^2 + 20s + 30 \quad . \qquad (\star)$$

Multiplying this out and gathering like terms yields

$$[B+C]s^2 + [A+6C]s + [-3A - 9B + 9C] = 6s^2 + 20s + 30 \quad ,$$

which, in turn, yields the system

$$\begin{aligned} B + C &= 6 \\ A + 6C &= 20 \\ -3A - 9B + 9C &= 30 \end{aligned} \quad . \qquad (\star\star)$$

Letting $s = -3$ in $(\star)$:

$$A(-3-3) + B\cdot 0\cdot(-3-3) + C\cdot 0^2 = 6[-3]^2 + 20[-3] + 30$$

$$\hookrightarrow \quad A = \frac{6[-3]^2 + 20[-3] + 30}{-3-3} = -4 \quad .$$

Letting $s = 3$ in $(\star)$:

$$A\cdot 0 + B(3+3)\cdot 0 + C(3+3)^2 = 6[3]^2 + 20[3] + 30$$

$$\hookrightarrow \quad C = \frac{2[3]^2 + 20[3] + 30}{(3+3)^2} = 4 \quad .$$

Using these values with the first equation in system $(\star\star)$:

$$B = 6 - C = 6 - 4 = 2 \quad .$$

Thus,

$$Y(s) = \frac{A}{(s+3)^2} + \frac{B}{s+3} + \frac{C}{s-3} = -\frac{4}{(s+3)^2} + \frac{2}{s+3} + \frac{4}{s-3} \quad ,$$

and

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[-\frac{4}{(s+3)^2} + \frac{2}{s+3} + \frac{4}{s-3}\right]\Big|_t \\ &= -4\mathcal{L}^{-1}\left[\frac{1}{(s+3)^2}\right]\Big|_t + 2\mathcal{L}^{-1}\left[\frac{1}{s+3}\right]\Big|_t + 4\mathcal{L}^{-1}\left[\frac{1}{s-3}\right]\Big|_t \\ &= -4\mathcal{L}^{-1}\left[F(s-[-3])\right]\big|_t + 2e^{-3t} + 4e^{3t} \\ &= -4e^{-3t}f(t) + 2e^{-3t} + 4e^{3t} \end{aligned}$$

with

$$F(s-[-3]) = F(s+3) = \frac{1}{(s+3)^2} \quad \rightarrowtail \quad F(X) = \frac{1}{X^2}$$

$$\hookrightarrow \qquad f(t) = \mathcal{L}^{-1}[F(s)]|_t = \mathcal{L}^{-1}\left[\frac{1}{s^2}\right]\Big|_y = t \quad .$$

So,

$$\begin{aligned} y(t) &= -4e^{-3t}f(t) + 2e^{-3t} + 4e^{3t} \\ &= -4e^{-3t}t + 2e^{-3t} + 4e^{3t} = 2e^{-3t} - 4te^{-3t} + 4e^{3t} \quad . \end{aligned}$$

28.10 a. From Theorem 27.5 on page 489 and the definition of the inverse transform, we have

$$\mathcal{L}^{-1}\left[\frac{F(s)}{s}\right]\Big|_t = \int_0^t f(\tau)\,d\tau \qquad \text{with} \quad f(\tau) = \mathcal{L}^{-1}[F(s)]|_\tau \quad .$$

In this problem,

$$\frac{1}{s(s^2+9)} = \frac{F(s)}{s} \qquad \text{with} \quad F(s) = \frac{1}{s^2+9} \quad .$$

So,

$$f(\tau) = \mathcal{L}\left[\frac{1}{s^2+9}\right]\Big|_\tau = \frac{1}{3}\mathcal{L}\left[\frac{3}{s^2+3^2}\right]\Big|_\tau = \frac{1}{3}\sin(3\tau) \quad ,$$

and

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{1}{s(s^2+9)}\right]\Big|_t = \mathcal{L}^{-1}\left[\frac{F(s)}{s}\right]\Big|_t &= \int_0^t f(\tau)\,d\tau \\ &= \int_0^t \frac{1}{3}\sin(3\tau)\,d\tau = \frac{1}{9}[1-\cos(3t)] \quad . \end{aligned}$$

28.10 c. In this problem,

$$\frac{1}{s(s-3)^2} = \frac{F(s)}{s} \qquad \text{with} \quad F(s) = \frac{1}{(s-3)^2} \quad .$$

So,

$$f(\tau) = \mathcal{L}\left[\frac{1}{(s-3)^2}\right]\Big|_\tau = \mathcal{L}[F(s-3)]|_\tau = e^{3t}f(t) = \cdots = \tau e^{3\tau} \quad ,$$

and

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{1}{s(s-3)^2}\right]\bigg|_t &= \mathcal{L}^{-1}\left[\frac{F(s)}{s}\right]\bigg|_t \\
&= \int_0^t f(\tau)\,d\tau \\
&= \int_0^t \tau e^{3\tau}\,d\tau \\
&= \frac{1}{3}\tau e^{3\tau}\bigg|_0^t - \frac{1}{3}\int_0^t e^{3\tau}\,d\tau \\
&= \frac{1}{3}te^{3t} - \frac{1}{9}\left[e^{3t}-1\right] = \frac{1}{9}\left[1+(3t-1)e^{3t}\right] \quad .
\end{aligned}$$