Chapter 24: Variation of Parameters

24.1 a. Here, $y_1(x) = x$ and $y_2(x) = x^2$. So the solution is given by

$$y(x) = y_1 u + y_2 v = xu + x^2 v \qquad (\star)$$

where $u = u(x)$ and $v = v(x)$ satisfy the two equations

$$y_1 u' + y_2 v' = 0 \qquad \rightarrowtail \qquad xu' + x^2 v' = 0$$

and

$$y_1{}' u' + y_2{}' v' = \frac{g}{a} \qquad \rightarrowtail \qquad u' + 2xv' = \frac{3\sqrt{x}}{x^2} = 3x^{-3/2} \quad .$$

Solving for u' and v':

$$xu' + x^2 v' = 0 \qquad \text{and} \qquad u' + 2xv' = 3x^{-3/2}$$

$$\hookrightarrow \qquad u' = -xv' \qquad \text{and} \qquad -xv' + 2xv' = 3x^{-3/2}$$

$$\hookrightarrow \qquad u' = -xv' \qquad \text{and} \qquad v' = \frac{3x^{-3/2}}{x} = 3x^{-5/2}$$

$$\hookrightarrow \qquad u' = -x\left[3x^{-5/2}\right] = -3x^{-3/2} \qquad \text{and} \qquad v' = 3x^{-5/2} \quad .$$

Integrating, we get

$$u = \int u'\,dx = -\int 3x^{-3/2}\,dx = 6x^{-1/2} + c_1 \quad ,$$

and

$$v = \int v'\,dx = \int 3x^{-5/2}\,dx = -2x^{-3/2} + c_2 \quad .$$

Plugging back into formula $(\star)$ for y then yields

$$\begin{aligned} y(x) &= xu + x^2 v \\ &= x\left[6x^{-1/2} + c_1\right] + x^2\left[-2x^{-3/2} + c_2\right] \\ &= 6x^{1/2} + c_1 x - 2x^{1/2} + c_2 x^2 = 4\sqrt{x} + c_1 x + c_2 x^2 \quad . \end{aligned}$$

24.1 c. Here, $y_1(x) = \cos(2x)$ and $y_2(x) = \sin(2x)$. So the solution is given by

$$y(x) = y_1 u + y_2 v = \cos(2x)u + \sin(2x)v \qquad (\star)$$

where $u = u(x)$ and $v = v(x)$ satisfy the system

$$\begin{aligned} y_1 u' + y_2 v' &= 0 \\ y_1{}' u' + y_2{}' v' &= \frac{g}{a} \end{aligned} \quad ,$$

which, in this case, is

$$\begin{aligned} \cos(2x)u' + \sin(2x)v' &= 0 \\ -2\sin(2x)u' + 2\cos(2x)v' &= \frac{\csc(2x)}{1} = \frac{1}{\sin(2x)} \end{aligned} \quad .$$

From the first equation in this system, we get

$$v' = -\frac{\cos(2x)}{\sin(2x)}u' \quad .$$

Plugging this into the second equation and continuing:

$$-2\sin(2x)u' + 2\cos(2x)\left[-\frac{\cos(2x)}{\sin(2x)}u'\right] = \frac{1}{\sin(2x)}$$

$$\hookrightarrow \qquad -2\left[\sin(2x) + \frac{\cos^2(2x)}{\sin(2x)}\right]u' = \frac{1}{\sin(2x)}$$

$$\hookrightarrow \qquad -2\frac{\sin^2(2x)+\cos^2(2x)}{\sin(2x)}u' = \frac{1}{\sin(2x)}$$

$$\hookrightarrow \qquad -2\frac{1}{\sin(2x)}u' = \frac{1}{\sin(2x)}$$

$$\hookrightarrow \qquad u' = -\frac{1}{2} \quad .$$

Hence, also,

$$v' = -\frac{\cos(2x)}{\sin(2x)}u' = \frac{\cos(2x)}{2\sin(2x)} \quad .$$

Integrating, we get

$$u = \int u'\,dx = -\int \tfrac{1}{2}\,dx = -\tfrac{1}{2}x + c_1 \quad ,$$

and

$$v = \int v'\,dx = \int \frac{\cos(2x)}{2\sin(2x)}\,dx = \tfrac{1}{4}\ln|\sin(2x)| + c_2 \quad .$$

Plugging back into formula ($\star$) for y then yields

$$\begin{aligned} y(x) &= \cos(2x)u + \sin(2x)v \\ &= \cos(2x)\left[-\tfrac{1}{2}x + c_1\right] + \sin(2x)\left[\tfrac{1}{4}\ln|\sin(2x)| + c_2\right] \\ &= -\tfrac{1}{2}x\cos(2x) + \tfrac{1}{4}\sin(2x)\ln|\sin(2x)| + c_1\cos(2x) + c_2\sin(2x) \quad . \end{aligned}$$

24.1 e. Here, $y_1(x) = e^{2x}$ and $y_2(x) = xe^{2x}$. So the solution is given by

$$y(x) = y_1 u + y_2 v = e^{2x}u + xe^{2x}v \qquad (\star)$$

where $u = u(x)$ and $v = v(x)$ satisfy the system

$$\begin{aligned} y_1 u' + y_2 v' &= 0 \\ {y_1}'u' + {y_2}'v' &= \frac{g}{a} \end{aligned} \quad ,$$

which, in this case, is

$$\begin{aligned} e^{2x}u' + xe^{2x}v' &= 0 \\ 2e^{2x}u' + [1+2x]e^{2x}v' &= \frac{\left[24x^2+2\right]e^{2x}}{1} = \left[24x^2+2\right]e^{2x} \end{aligned} \quad ,$$

and which, after dividing out the e^{2x}, further simplifies to

$$\begin{aligned} u' + xv' &= 0 \\ 2u' + [1+2x]v' &= 24x^2 + 2 \end{aligned} \quad .$$

Solving for u' and v':

$$u' + xv' = 0 \quad \text{and} \quad 2u' + [1+2x]v' = 24x^2 + 2$$

$$\hookrightarrow \quad u' = -xv' \quad \text{and} \quad 2[-xv'] + [1+2x]v' = 24x^2 + 2$$

$$\hookrightarrow \quad u' = -xv' \quad \text{and} \quad v' = 24x^2 + 2$$

$$\hookrightarrow \quad u' = -x[24x^2 + 2] = -24x^3 - 2x \quad \text{and} \quad v' = 24x^2 + 2 \quad .$$

Integrating, we get

$$u = \int u'\,dx = -\int \left[24x^3 + 2x\right] dx = -6x^4 - x^2 + c_1 \quad ,$$

and

$$v = \int v'\,dx = \int \left[24x^2 + 2\right] dx = 8x^3 + 2x + c_2 \quad .$$

Plugging back into formula (⋆) for y then yields

$$\begin{aligned} y(x) &= e^{2x}\left[-6x^4 - x^2 + c_1\right] + xe^{2x}\left[8x^3 + 2x + c_2\right] \\ &= \left[2x^4 + x^2 + c_1 x + c_2 x\right]e^{2x} \quad . \end{aligned}$$

24.1 g. The solution is given by

$$y(x) = y_1 u + y_2 v = xu + x^{-1}v \qquad (\star)$$

where $u = u(x)$ and $v = v(x)$ satisfy

$$y_1 u' + y_2 v' = 0 \quad \rightarrowtail \quad xu' + x^{-1}v' = 0$$

and

$$y_1{}' u' + y_2{}' v' = \frac{g}{a} \quad \rightarrowtail \quad u' - x^{-2}v' = \frac{\sqrt{x}}{x^2} = x^{-3/2} \quad .$$

Solving for u' and v':

$$xu' + x^{-1}v' = 0 \quad \text{and} \quad u' - x^{-2}v' = x^{-3/2}$$

$$\hookrightarrow \quad u' = -x^{-2}v' \quad \text{and} \quad -x^{-2}v' - x^{-2}v' = x^{-3/2}$$

$$\hookrightarrow \quad u' = -x^{-2}v' \quad \text{and} \quad v' = -\tfrac{1}{2}x^2 \cdot x^{-3/2} = -\tfrac{1}{2}x^{1/2}$$

$$\hookrightarrow \quad u' = -x^{-2}\left[-\tfrac{1}{2}x^{1/2}\right] = \tfrac{1}{2}x^{-3/2} \quad \text{and} \quad v' = = -\tfrac{1}{2}x^{1/2} \quad .$$

Integrating, we get

$$u = \int u'\,dx = \int \tfrac{1}{2}x^{-3/2}\,dx = -x^{-1/2} + c_1 \quad ,$$

and

$$v = \int v'\,dx = -\int \tfrac{1}{2}x^{1/2}\,dx = -\tfrac{1}{3}x^{3/2} + c_2 \quad .$$

Plugging back into formula (⋆) for y then yields

$$\begin{aligned} y(x) &= x\left[-x^{-1/2} + c_1\right] + x^{-1}\left[-\tfrac{1}{3}x^{3/2} + c_2\right] \\ &= -\tfrac{4}{3}x^{1/2} + c_1x + c_2x^2 = c_1x + c_2x^2 - \tfrac{4}{3}\sqrt{x} \quad . \end{aligned}$$

24.1 i. The solution is given by

$$y(x) = y_1u + y_2v = x^2u + x^2\ln|x|\,v \tag{$\star$}$$

where $u = u(x)$ and $v = v(x)$ satisfy the system

$$\begin{aligned} y_1u' + y_2v' &= 0 \\ y_1{}'u' + y_2{}'v' &= \frac{g}{a} \end{aligned} \quad ,$$

which, in this case, is

$$\begin{aligned} x^2u' + x^2\ln|x|\,v' &= 0 \\ 2xu' + [2x\ln|x| + x]\,v' &= \frac{x^2}{x^2} = 1 \end{aligned} \quad .$$

From the first equation in this system, we get

$$u' = -\ln|x|\,v' \quad .$$

Plugging this into the second equation and continuing:

$$\begin{aligned} 2x[-\ln|x|] + [2x\ln|x| + x]\,v' &= 1 \\ \hookrightarrow \qquad xv' &= 1 \\ \hookrightarrow \qquad v' &= \frac{1}{x} \quad . \end{aligned}$$

Hence, also,

$$u' = -\ln|x|\,v' = -x^{-1}\ln|x| \quad .$$

Integrating (using integration by parts to compute the integral for u), we get

$$u = \int u'\,dx = -\int x^{-1}\ln|x|\,dx = -\tfrac{1}{2}(\ln|x|)^2 + c_1 \quad ,$$

and

$$v = \int v'\,dx = \int \frac{1}{x}\,dx = \ln|x| + c_2 \quad .$$

Plugging back into formula (⋆) for y then yields

$$\begin{aligned} y(x) &= x^2\left[-\tfrac{1}{2}(\ln|x|)^2 + c_1\right] + x^2\ln|x|\,[\ln|x| + c_2] \\ &= \tfrac{1}{2}x^2(\ln|x|)^2 + c_1x^2 + c_2x^2\ln|x| \quad . \end{aligned}$$

24.1 k. The solution is given by

$$y(x) = y_1 u + y_2 v = x^2 u + x^{-1} v \qquad (\star)$$

where $u = u(x)$ and $v = v(x)$ satisfy the system

$$\begin{aligned} y_1 u' + y_2 v' &= 0 \\ {y_1}' u' + {y_2}' v' &= \frac{g}{a} \end{aligned} ,$$

which, in this case, is

$$\begin{aligned} x^2 u' + x^{-1} v' &= 0 \\ 2xu' - x^{-2} v' &= \frac{1}{x^2(x-2)} \end{aligned} .$$

From the first equation in this system, we get

$$u' = -x^{-3} v' \quad .$$

Plugging this into the second equation and continuing:

$$2x\left[-x^{-3}v'\right] - x^{-2}v' = \frac{1}{x^2(x-2)}$$

$$\hookrightarrow \qquad -3x^{-2}v' = \frac{1}{x^2(x-2)}$$

$$\hookrightarrow \qquad v' = -\frac{1}{3(x-2)} \quad .$$

Hence, also,

$$u' = -x^{-3}v' = \frac{1}{3x^3(x-2)} \quad ,$$

which can be expanded via a partial fractions to

$$\begin{aligned} u' = \frac{1}{3x^3(x-2)} &= \frac{A}{x^3} + \frac{B}{x^2} + \frac{C}{x} + \frac{D}{x-2} \\ &= \cdots \\ &= \frac{-1/6}{x^3} + \frac{-1/12}{x^2} + \frac{-1/24}{x} + \frac{1/24}{x-2} \quad . \end{aligned}$$

Integrating, we get

$$\begin{aligned} u = \int u'\,dx &= \int \left[\frac{-1/6}{x^3} + \frac{-1/12}{x^2} + \frac{-1/24}{x} + \frac{1/24}{x-2}\right] dx \\ &= \frac{1}{12}x^{-2} + \frac{1}{12}x^{-1} - \frac{1}{24}\ln|x| + \frac{1}{24}\ln|x-2| + c_1 \end{aligned}$$

and

$$v = \int v'\,dx = \int \frac{-1}{3(x-2)}\,dx = -\frac{1}{3}\ln|x-2| + c_2 \quad .$$

Plugging back into formula $(\star)$ for y then yields

$$\begin{aligned} y(x) &= x^2\left[\frac{1}{12}x^{-2} + \frac{1}{12}x^{-1} - \frac{1}{24}\ln|x| + \frac{1}{24}\ln|x-2| + c_1\right] \\ &\quad + x^{-1}\left[-\frac{1}{3}\ln|x-2|\right] \\ &= \frac{1}{12}[x+1] - \frac{1}{24}x^2\ln|x| + \frac{1}{24}x^2\ln|x-2| + c_1x^2 \\ &\quad - \frac{1}{3}x^{-1}\ln|x-2| + c_2x^{-1} \quad . \end{aligned}$$

24.1 m. The solution is given by

$$y(x) = y_1 u + y_2 v = x^{-1}u + x^{-1}e^{-2x}v \qquad (\star)$$

where $u = u(x)$ and $v = v(x)$ satisfy the system

$$\begin{aligned} y_1 u' + y_2 v' &= 0 \\ y_1{}' u' + y_2{}' v' &= \frac{g}{a} \end{aligned} \quad ,$$

which, in this case, is

$$\begin{aligned} x^{-1}u' + x^{-1}e^{-2x}v' &= 0 \\ -x^{-2}u' - \left[x^{-2} + 2x^{-1}\right]e^{-2x}v' &= \frac{8e^{-2x}}{x} = 8x^{-1}e^{2x} \end{aligned} \quad ,$$

and which further simplifies to

$$\begin{aligned} u' + e^{-2x}v' &= 0 \\ u' + [1+2x]\,e^{-2x}v' &= -8xe^{2x} \end{aligned} \quad .$$

From the first equation in this system, we get

$$u' = -e^{-2x}v' \quad .$$

Plugging this into the second equation and continuing:

$$\left[-e^{-2x}v'\right] + [1+2x]\,e^{-2x}v' = -8xe^{2x}$$

$$\hookrightarrow \qquad 2xe^{-2x}v' = -8xe^{2x}$$

$$\hookrightarrow \qquad v' = -4e^{4x} \quad .$$

Hence, also,

$$u' = -e^{-2x}v' = -e^{-2x}\left[-4e^{4x}\right] = 4e^{2x} \quad .$$

Integrating, we get

$$u = \int u'\,dx = \int 4e^{2x}\,dx = 2e^{2x} + c_1 \quad ,$$

and

$$v = \int v'\,dx = -\int 4e^{4x}\,dx = -e^{4x} + c_2 \quad .$$

Plugging back into formula $(\star)$ for y then yields

$$\begin{aligned} y(x) &= x^{-1}\left[2e^{2x} + c_1\right] + x^{-1}e^{-2x}\left[-e^{4x} + c_2\right] \\ &= x^{-1}e^{2x} + c_1 x^{-1} + c_2 x^{-1}e^{-2x} \quad . \end{aligned}$$

24.2 a. First, we must find y_h, the general solution to the corresponding homogeneous equation,

$$x^2 y'' - 2xy' - 4y = 0 \quad .$$

Since this is an Euler equation, we try a solution of the form $y(x) = x^r$:

$$\begin{aligned} 0 &= x^2y'' - 2xy' - 4y \\ &= x^2\left[x^r\right]'' - 2x\left[x^r\right]' - 4\left[x^r\right] \\ &= x^2\left[r(r-1)x^{r-2}\right] - 2x\left[rx^{r-1}\right] - 4\left[x^r\right] \\ &= x^r\left[r(r-1) - 2r - 4\right] = x^r\left[r^2 - 3r - 4\right] \quad . \end{aligned}$$

So,

$$0 = r^2 - 3r - 4 = (r+1)(r-4)$$

$$\hookrightarrow \qquad r = -1 \quad \text{and} \quad r = 4$$

$$\hookrightarrow \qquad y_h(x) = c_1x^{-1} + c_2x^4 \quad .$$

Thus, to solve the given nonhomogeneous differential equation using variation of parameters, we set

$$y(x) = x^{-1}u + x^4v \tag{$\star$}$$

where u and v satisfy

$$x^{-1}u' + x^4v' = 0$$

and

$$-x^{-2}u' + 4x^3v' = \frac{10/x}{x^2} = 10x^{-3} \quad ,$$

which we can write more simply as the system

$$\begin{aligned} u' + x^5v' &= 0 \\ -u' + 4x^5v' &= 10x^{-1} \end{aligned} \quad .$$

Adding these two equations together and solving:

$$-5x^5v' = -10x^{-1} \quad \rightarrowtail \quad v' = 2x^{-6} \quad .$$

This with the first equation in the system then yields

$$u' = -x^5v' = -x^5\left[2x^{-6}\right] = -2x^{-1} \quad .$$

Integrating:

$$u(x) = \int u'\,dx = -\int 2x^{-1}\,dx = -2\ln|x| + c_1 \quad ,$$

and

$$v(x) = \int v'\,dx = \int 2x^{-6}\,dx = -\frac{2}{5}x^{-5} + c_2 \quad .$$

Plugging back into formula ($\star$) for y :

$$\begin{aligned} y(x) &= x^{-1}\left[-2\ln|x| + c_1\right] + x^4\left[-\frac{2}{5}x^{-5} + c_2\right] \\ &= -2x^{-1}\ln|x| + \left[c_1 - \frac{2}{5}\right]x^{-1} + c_2x^4 \\ &= -2x^{-1}\ln|x| + Ax^{-1} + Bx^4 \quad . \end{aligned} \tag{$\star\star$}$$

This is the general solution to the differential equation. Computing it's derivative, we get

$$y'(x) = 2x^{-2}\ln|x| - 2x^{-2} - Ax^{-2} + 4Bx^3 \quad .$$

Applying the initial conditions:

$$3 = y(1) = -2x^{-1}\ln|1| + A \cdot 1^{-1} + B \cdot 1^4 = A + B$$

and

$$\begin{aligned} -15 = y'(1) &= 2 \cdot 1^{-2}\ln|x| - 2 \cdot 1^{-2} - A \cdot 1^{-2} + 4B \cdot 1^3 \\ &= -2 - A + 4B \quad . \end{aligned}$$

That is,

$$A + B = 3 \qquad \text{and} \qquad -A + 4B = -13 \quad .$$

Solving this simple system yields $A = 5$ and $B = -2$, which, plugged back into formula ($\star\star$) for y gives our final answer:

$$y(x) = -2x^{-1}\ln|x| + 5x^{-1} - 2x^4 \quad .$$

24.3 a. Here, $y_1(x) = 1$, $y_2(x) = e^{2x}$ and $y_3(x) = e^{-2x}$. So the solution is given by

$$y(x) = y_1 u + y_2 v + y_3 w = 1 \cdot u + e^{2x} v + e^{-2x} w \tag{$\star$}$$

where $u = u(x)$, $v = v(x)$ and $w = w(x)$ satisfy the system

$$\begin{aligned} y_1 u' + y_2 v' + y_3 w' &= 0 \\ y_1{}' u' + y_2{}' v' + y_3{}' w' &= 0 \\ y_1{}'' u' + y_2{}'' v' + y_3{}'' w' &= \frac{g}{a} \end{aligned} \quad ,$$

which, in this case, is

$$\begin{aligned} 1u' + e^{2x} v' + e^{-2x} w' &= 0 \\ 0u' + 2e^{2x} v' - 2e^{-2x} w' &= 0 \\ 0u' + 4e^{2x} v' + 4e^{-2x} w' &= \frac{30e^{3x}}{1} = 30e^{3x} \end{aligned} \quad ,$$

and which, after dividing out common factors in each equation, reduces to

$$\begin{aligned} u' + e^{2x} v' + e^{-2x} w' &= 0 && \text{(S1)} \\ e^{2x} v' - e^{-2x} w' &= 0 \quad . && \text{(S2)} \\ e^{2x} v' + e^{-2x} w' &= \frac{15}{2} e^{3x} && \text{(S3)} \end{aligned}$$

This is easily solved by adding or subtracting the equations. In particular, subtracting (S3) from (S1) yields

$$u' = -\frac{15}{2} e^{3x} \quad ,$$

adding (S2) and (S3) together gives

$$2e^{2x} v' = \frac{15}{2} e^{3x} \quad \rightarrowtail \quad v' = \frac{15}{4} e^{x} \quad ,$$

and subtracting (S2) from (S3) gives

$$2e^{-2x}w' = \frac{15}{2}e^{3x} \quad \rightarrowtail \quad w' = \frac{15}{4}e^{5x} \quad .$$

Integrating, we obtain

$$u(x) = \int u'\,dx = -\int \frac{15}{2}e^{3x}\,dx = -\frac{5}{2}e^{3x} + c_1 \quad ,$$

$$v(x) = \int v'\,dx = \int \frac{15}{4}e^{x}\,dx = \frac{15}{4}e^{x} + c_2$$

and

$$w(x) = \int w'\,dx = -\int \frac{15}{4}e^{5x}\,dx = \frac{3}{4}e^{5x} + c_3 \quad .$$

The final answer is then given by plugging these back into formula ($\star$) for y:

$$\begin{aligned} y(x) &= u + e^{2x}v + e^{-2x}w \\ &= \left[-\frac{5}{2}e^{3x} + c_1\right] + e^{2x}\left[\frac{15}{4}e^{x} + c_2\right] + e^{-2x}\left[\frac{3}{4}e^{5x} + c_3\right] \\ &= \left[-\frac{5}{2} + \frac{15}{4} + \frac{3}{4}\right]e^{3x} + c_1 + c_2e^{2x} + c_3e^{-2x} \\ &= 2e^{3x} + c_1 + c_2e^{2x} + c_3e^{-2x} \quad . \end{aligned}$$

24.4 a. Here, $y_1(x) = x$, $y_2(x) = x^2$ and $y_3(x) = x^3$. So the solution is given by

$$y(x) = y_1u + y_2v + y_3w = xu + x^2v + x^3w$$

where $u = u(x)$, $v = v(x)$ and $w = w(x)$ satisfy the system

$$\begin{aligned} y_1u' + y_2v' + y_3w' &= 0 \\ {y_1}'u' + {y_2}'v' + {y_3}'w' &= 0 \\ {y_1}''u' + {y_2}''v' + {y_3}''w' &= \frac{g}{a} \end{aligned} \quad ,$$

which, in this case, is

$$\begin{aligned} xu' + x^2v' + x^3w' &= 0 \\ 1u' + 2xv' + 3x^2w' &= 0 \\ 0u' + 2v' + 6xw' &= \frac{e^{-x}}{x^3} = x^{-3}e^{-x} \end{aligned} \quad .$$

24.4 c. Here, $y_1(x) = e^{3x}$, $y_2(x) = e^{-3x}$, $y_3(x) = \cos(3x)$ and $y_4(x) = \sin(3x)$. So

$$y(x) = e^{3x}u_1 + e^{-3x}u_2 + \cos(3x)u_3 + \sin(3x)u_4$$

where

$$\begin{aligned} y_1{u_1}' + y_2{u_2}' + y_3{u_3}' + y_4{u_4}' &= 0 \\ {y_1}'{u_1}' + {y_2}'{u_2}' + {y_3}'{u_3}' + {y_4}'{u_4}' &= 0 \\ {y_1}''{u_1}' + {y_2}''{u_2}' + {y_3}''{u_3}' + {y_4}''{u_4}' &= 0 \\ {y_1}'''{u_1}' + {y_2}'''{u_2}' + {y_3}'''{u_3}' + {y_4}'''{u_4}' &= \frac{g}{a} \end{aligned} \quad ,$$

which, in this case, is

$$\begin{aligned}
e^{3x}u_1{}' + e^{-3x}u_2{}' + \cos(3x)u_3{}' + \sin(3x)u_4{}' &= 0\\
3e^{3x}u_1{}' - 3e^{-3x}u_2{}' - 3\sin(3x)u_3{}' + 3\cos(3x)u_4{}' &= 0\\
9e^{3x}u_1{}' + 9e^{-3x}u_2{}' - 9\cos(3x)u_3{}' - 9\sin(3x)u_4{}' &= 0\\
27e^{3x}u_1{}' - 27e^{-3x}u_2{}' + 27\sin(3x)u_3{}' - 27\cos(3x)u_4{}' &= \frac{\sinh(x)}{1}
\end{aligned} \quad .$$

Dividing out common factors in each equation, this reduces to

$$\begin{aligned}
e^{3x}u_1{}' + e^{-3x}u_2{}' + \cos(3x)u_3{}' + \sin(3x)u_4{}' &= 0\\
e^{3x}u_1{}' - e^{-3x}u_2{}' - \sin(3x)u_3{}' + \cos(3x)u_4{}' &= 0\\
e^{3x}u_1{}' + e^{-3x}u_2{}' - \cos(3x)u_3{}' - \sin(3x)u_4{}' &= 0\\
e^{3x}u_1{}' - e^{-3x}u_2{}' + \sin(3x)u_3{}' - \cos(3x)u_4{}' &= \frac{1}{27}\sinh(x)
\end{aligned} \quad .$$

24.6. Recall that, for any sufficiently continuous function g,

$$\int_{x_0}^{x_0} g(s)\,ds = 0 \qquad \text{and} \qquad \frac{d}{dx}\int_{x_0}^{x} g(s)\,ds = g(x) \quad .$$

Letting $x = x_0$ in formula (24.15) yields

$$\begin{aligned}
y_p(x_0) &= -y_1(x_0)\int_{x_0}^{x_0}\frac{y_2(s)f(s)}{W(s)}\,ds + y_2(x_0)\int_{x_0}^{x_0}\frac{y_1(s)f(s)}{W(s)}\,ds\\
&= -y_1(x_0)\cdot 0 + y_2(x_0)\cdot 0 = 0 \quad ,
\end{aligned}$$

verifying the claim that $y_p(x_0) = 0$.

Differentiating formula (24.15) yields

$$\begin{aligned}
y_p{}'(x) &= \frac{d}{dx}\left[-y_1(x)\int_{x_0}^{x}\frac{y_2(s)f(s)}{W(s)}\,ds + y_2(x)\int_{x_0}^{x}\frac{y_1(s)f(s)}{W(s)}\,ds\right]\\
&= -y_1{}'(x)\int_{x_0}^{x}\frac{y_2(s)f(s)}{W(s)}\,ds - y_1(x)\frac{d}{dx}\int_{x_0}^{x}\frac{y_2(s)f(s)}{W(s)}\,ds\\
&\qquad + y_2{}'(x)\int_{x_0}^{x}\frac{y_1(s)f(s)}{W(s)}\,ds + y_2(x)\frac{d}{dx}\int_{x_0}^{x}\frac{y_1(s)f(s)}{W(s)}\,ds\\
&= -y_1{}'(x)\int_{x_0}^{x}\frac{y_2(s)f(s)}{W(s)}\,ds - y_1(x)\frac{y_2(x)f(x)}{W(x)}\\
&\qquad + y_2{}'(x)\int_{x_0}^{x}\frac{y_1(s)f(s)}{W(s)}\,ds + y_2(x)\frac{y_1(x)f(x)}{W(x)}\\
&= -y_1{}'(x)\int_{x_0}^{x}\frac{y_2(s)f(s)}{W(s)}\,ds + y_2{}'(x)\int_{x_0}^{x}\frac{y_1(s)f(s)}{W(s)}\,ds\\
&\qquad - \underbrace{[y_1(x)y_2(x) - y_2(x)y_1(x)]}_{0}\frac{f(x)}{W(x)}\\
&= -y_1{}'(x)\int_{x_0}^{x}\frac{y_2(s)f(s)}{W(s)}\,ds + y_2{}'(x)\int_{x_0}^{x}\frac{y_1(s)f(s)}{W(s)}\,ds \quad .
\end{aligned}$$

Thus,

$$\begin{aligned} {y_p}'(x_0) &= -{y_1}'(x_0) \int_{x_0}^{x_0} \frac{y_2(s) f(s)}{W(s)} \, ds \; + \; {y_2}'(x_0) \int_{x_0}^{x_0} \frac{y_1(s) f(s)}{W(s)} \, ds \\ &= -{y_1}'(x_0) \cdot 0 \; + \; {y_2}'(x_0) \cdot 0 \; = \; 0 \quad , \end{aligned}$$

confirming that ${y_p}'(x_0) = 0$.