Chapter 21: Nonhomogeneous Equations in General

21.1 a. Plugging $y = e^{3x}$ into the differential equation, we get

$$g(x) = y'' + y = \left[e^{3x}\right]'' + e^{3x} = 9e^{3x} + e^{3x} = 10e^{3x} \quad .$$

21.1 b. Plugging $y = e^{3x}$ into the differential equation, we get

$$\begin{aligned} g(x) &= x^2 y'' - 4y = x^2\left[e^{3x}\right]'' - 4e^{3x} \\ &= x^2 9e^{3x} - 4e^{3x} = \left(9x^2 - 4\right)e^{3x} \quad . \end{aligned}$$

21.1 c. Plugging $y = e^{3x}$ into the differential equation, we get

$$\begin{aligned} g(x) &= y^{(3)} - 4y' + 5y = \left[e^{3x}\right]^{(3)} - 4\left[e^{3x}\right]' + 5e^{3x} \\ &= 27e^{3x} - 4 \cdot 3e^{3x} + 5e^{3x} = 20e^{3x} \quad . \end{aligned}$$

21.3 a. Plugging $y = \sin(x)$ into the differential equation, we get

$$g(x) = y'' + y = [\sin(x)]'' + \sin(x) = -\sin(x) + \sin(x) = 0 \quad ,$$

telling us that g cannot be a nonzero function. So the answer is "No, because $y'' + y = 0$ when $y(x) = \sin(x)$".

21.3 b.

$$y = x\sin(x) \quad \rightarrowtail \quad y' = \sin(x) + x\cos(x)$$

$$\hookrightarrow \qquad y'' = 2\cos(x) - x\sin(x) \quad .$$

So, if $y = x\sin(x)$, then

$$g(x) = y'' + y = 2\cos(x) - x\sin(x) + x\sin(x) = 2\cos(x) \quad .$$

21.5 a. Plugging $y = 3e^{2x}$ into the left side of the differential equation, we get

$$y'' + 4y = \left[3e^{2x}\right]'' + 4\left[3e^{2x}\right] = 3 \cdot 2^2 e^{2x} + 12e^{2x} = 24e^{2x} \quad ,$$

verifying that $y_p = 3e^{2x}$ is one solution to the given nonhomogeneous differential equation.

21.5 b. The corresponding homogeneous differential equation is

$$y'' + 4y = 0 \quad ,$$

Writing out the corresponding characteristic equation, and then continuing:

$$r^2 + 4 = 0 \quad \rightarrowtail \quad r = \pm\sqrt{-4} = \pm 2i$$

$$\hookrightarrow \qquad y_{\pm}(x) = e^{\pm 2i} = \underbrace{\cos(2x)}_{y_1(x)} \pm \underbrace{\sin(2x)}_{y_2(x)} \quad .$$

So the general solution to the corresponding homogeneous equation is

$$y_h(x) = c_1 \cos(2x) + c_2 \sin(2x) \quad .$$

21.5 c. A general solution to any nonhomogeneous linear differential equation is constructed by adding a particular solution y_p to the general solution y_h of the corresponding homogeneous differential equation. In this case (using the y_p and y_h just found),

$$y(x) = y_p(x) + y_h(x) = 3e^{2x} + c_1 \cos(2x) + c_2 \sin(2x) \quad .$$

21.5 d. For the initial-value problems, we need to use the general solution just obtained,

$$y(x) = 3e^{2x} + c_1 \cos(2x) + c_2 \sin(2x) \tag{$\star$}$$

and its derivative

$$y'(x) = 6e^{2x} - 2c_1 \sin(2x) + 2c_2 \cos(2x)$$

evaluated at $x = 0$,

$$y(0) = 3e^{2 \cdot 0} + c_1 \cos(2 \cdot 0) + c_2 \sin(2 \cdot 0) = 3 + c_1$$

and

$$y'(0) = 6e^{2 \cdot 0} - 2c_1 \sin(2 \cdot 0) + 2c_2 \cos(2 \cdot 0) = 6 + 2c_2 \quad .$$

21.5 d i. Using the initial conditions with the above formulas for $y(0)$ and $y'(0)$, and then formula ($\star$) for $y(x)$:

$$6 = y(0) = 3 + c_1 \qquad \text{and} \qquad 6 = y'(0) = 6 + 2c_2$$

$$\hookrightarrow \qquad c_1 = 6 - 3 = 3 \qquad \text{and} \qquad c_2 = \frac{6-6}{2} = 0$$

$$\hookrightarrow \quad y(x) = 3e^{2x} + 3\cos(2x) + 0\sin(2x) = 3e^{2x} + 3\cos(2x) \quad .$$

21.5 d ii. Using the initial conditions with the above formulas for $y(0)$ and $y'(0)$, and then formula ($\star$) for $y(x)$:

$$-2 = y(0) = 3 + c_1 \qquad \text{and} \qquad 2 = y'(0) = 6 + 2c_2$$

$$\hookrightarrow \qquad c_1 = -2 = 3 = -5 \qquad \text{and} \qquad c_2 = \frac{2-6}{2} = -2$$

$$\hookrightarrow \qquad y(x) = 3e^{2x} - 5\cos(2x) - 2\sin(2x) \quad .$$

21.7 a. Plugging $y = -4$ into the left side of the differential equation, we get

$$y'' - 9y = [-4]'' - 9[-4] = 0 + 36 = 36 \quad ,$$

verifying that $y_p = -4$ is one solution to the given nonhomogeneous differential equation.

21.7 b. The corresponding homogeneous differential equation is

$$y'' - 9y = 0 \quad .$$

Writing out and solving the characteristic equation, and then writing out the resulting general solution y_h to the homogeneous equation:

$$r^2 - 9 = 0 \quad \rightarrowtail \quad r = \pm\sqrt{9} = \pm 3$$

$$\hookrightarrow \qquad y_h(x) = c_1 e^{3x} + c_2 e^{-3x} \quad .$$

Adding this to the particular solution y_p just obtained above then yields the general solution to the given nonhomogeneous differential equation,

$$y(x) = y_p(x) + y_h(x) = -4 + c_1 e^{3x} + c_2 e^{-3x} \quad .$$

21.7 c. From the last part, we know the general solution is

$$y(x) = -4 + c_1 e^{3x} + c_2 e^{-3x} \quad . \tag{$\star$}$$

Taking its derivative yields

$$y'(x) = 0 + 3c_1 e^{3x} - 3c_2 e^{-3x} \quad .$$

Applying the initial conditions:

$$8 = y(0) = -4 + c_1 e^{3\cdot 0} + c_2 e^{-3\cdot 0} = 4 + c_1 + c_2$$

and

$$6 = y'(0) = 3c_1 e^{3\cdot 0} - 3c_2 e^{-3\cdot 0} = 3c_1 - 3c_2 \quad .$$

Solving for the constants and plugging back into formula ($\star$) for y:

$$8 = -4 + c_1 + c_2 \qquad \text{and} \qquad 6 = 3c_1 - 3c_2$$

$$\hookrightarrow \quad c_1 = 12 - c_2 \qquad \text{and} \qquad 2 = c_1 - c_2 = 12 - c_2 - c_2$$

$$\hookrightarrow \quad c_1 = 12 - c_2 \qquad \text{and} \qquad c_2 = \frac{12-2}{2} = 5$$

$$\hookrightarrow \quad c_1 = 12 - 5 = 7 \qquad \text{and} \qquad c_2 = 5$$

$$\hookrightarrow \quad y(x) = -4 + 7e^{3x} + 5e^{-3x} \quad .$$

21.9 a. $y = xe^{5x} \quad \rightarrowtail \quad y' = e^{5x} + 5xe^{5x} \quad \rightarrowtail \quad y'' = 2\cdot 5e^{5x} + 25xe^{5x} \quad .$

So, plugging $y = xe^{5x}$ into the left side of the differential equation yields

$$\begin{aligned} y'' - 3y' - 10y &= \left[2\cdot 5e^{5x} + 25xe^{5x}\right] - 3\left[e^{5x} + 5xe^{5x}\right] - 10xe^{5x} \\ &= [10 + 25x - 3 - 15x - 10x]\, e^{5x} = 7e^{5x} \quad . \end{aligned}$$

21.9 b. The corresponding homogeneous differential equation is

$$y'' - 3y' - 10y = 0 \quad .$$

Writing out and solving the characteristic equation, and then writing out the resulting general solution y_h to the homogeneous equation:

$$0 = r^2 - 3r - 10 = (r-5)(r+2)$$

$$\hookrightarrow \qquad r = 5 \quad \text{and} \quad r = -2$$

$$\hookrightarrow \qquad y_h(x) = c_1 e^{5x} + c_2 e^{-2x} \quad .$$

Adding this to the particular solution y_p just obtained above then yields the general solution to the given nonhomogeneous differential equation,

$$y(x) = y_p(x) + y_h(x) = xe^{5x} + c_1 e^{5x} + c_2 e^{-2x} \quad .$$

21.9 c. From the last part, we know the general solution is

$$y(x) = xe^{5x} + c_1 e^{5x} + c_2 e^{-2x} \quad . \tag{$\star$}$$

Taking its derivative yields

$$y'(x) = e^{5x} + 5xe^{5x} + 5c_1 e^{5x} - 2c_2 e^{-2x} \quad .$$

Applying the initial conditions:

$$12 = y(0) = 0e^{5\cdot 0} + c_1 e^{5\cdot 0} + c_2 e^{-2\cdot 0} = c_1 + c_2$$

and

$$-2 = y'(0) = e^{5\cdot 0} + 5\cdot 0e^{5\cdot 0} + 5c_1 e^{5\cdot 0} - 2c_2 e^{-2\cdot 0} = 1 + 5c_1 - 2c_2 \quad .$$

Solving for the constants and then plugging them back into formula ($\star$) for y :

$$12 = c_1 + c_2 \quad \text{and} \quad -2 = 1 + 5c_1 - 2c_2$$

$$\hookrightarrow \qquad c_1 = 12 - c_2 \quad \text{and} \quad -3 = 5[12 - c_2] - 2c_2 = 60 - 7c_2$$

$$\hookrightarrow \qquad c_1 = 12 - c_2 \quad \text{and} \quad c_2 = \frac{60+3}{7} = 9$$

$$\hookrightarrow \qquad c_1 = 12 - 9 = 3 \quad \text{and} \quad c_2 = 9$$

$$\hookrightarrow \qquad y(x) = xe^{5x} + 3e^{5x} + 9e^{-2x} \quad .$$

21.11 a. $y = 5x + 2 \quad \rightarrowtail \quad y' = 5 \quad \rightarrowtail \quad y'' = 0 \quad .$

So, plugging $y = 5x + 2$ into the left side of the differential equation yields

$$\begin{aligned} x^2 y'' - 4xy' + 6y &= x^2[0] - 4x[5] + 6[5x+2] \\ &= -20x + 30x + 12 = 10x + 12 \quad . \end{aligned}$$

21.11 b. The corresponding homogeneous differential equation is

$$x^2 y'' - 4xy' + 6y = 0 \quad .$$

This is an Euler equation. To solve it, we must first find and solve the corresponding indicial equation obtained by plugging $y = x^r$ into the homogeneous differential equation:

$$\begin{aligned} 0 &= x^2 \left[x^r\right]'' - 4x \left[x^r\right]' + 6x^r \\ &= x^2 r(r-1)x^{r-2} - 4xrx^{r-1} + 6x^r \\ &= \left[r^2 - r - 4r + 6\right] x^2 = \left[r^2 - 5r + 6\right] x^2 \quad . \end{aligned}$$

Dividing out x^r leaves the indicial equation. Writing that equation down and continuing until we obtain the solution y_h to the homogeneous differential equation:

$$0 = r^2 - 5r + 6 = (r-2)(r-3)$$

$$\hookrightarrow \qquad r = 2 \quad \text{and} \quad r = 3$$

$$\hookrightarrow \qquad y_h(x) = c_1 x^2 + c_2 x^3 \quad .$$

Adding this to the particular solution y_p just obtained above then yields the general solution to the given nonhomogeneous differential equation,

$$y(x) = y_p(x) + y_h(x) = 5x + 2 + c_1 x^2 + c_2 x^3 \quad .$$

21.11 c. From the last part, we know the general solution is

$$y(x) = 5x + 2 + c_1 x^2 + c_2 x^3 \quad . \qquad (\star)$$

Taking its derivative yields

$$y'(x) = 5 + 2c_1 x + 3c_2 x^2 \quad .$$

Applying the initial conditions:

$$6 = y(1) = 5 \cdot 1 + 2 + c_1 \cdot 1^2 + c_2 \cdot 1^3 = 7 + c_1 + c_2$$

and

$$8 = y'(1) = 5 + 2c_1 \cdot 1 + 3c_2 \cdot 1^2 = 5 + 2c_1 + 3c_2 \quad .$$

Solving for the constants and then plugging them back into formula $(\star)$ for y:

$$6 = 7 + c_1 + c_2 \quad \text{and} \quad 8 = 5 + 2c_1 + 3c_2$$

$$\hookrightarrow \qquad c_1 = -1 - c_2 \quad \text{and} \quad 3 = 2[-1 - c_2] + 3c_2 = -2 + c_2$$

$$\hookrightarrow \qquad c_1 = -1 - c_2 \quad \text{and} \quad c_2 = 3 + 2 = 5$$

$$\hookrightarrow \qquad c_1 = -1 - 5 = -6 \quad \text{and} \quad c_2 = 5$$

$$\hookrightarrow \qquad y(x) = 5x + 2 - 5x^2 + 5x^3 \quad .$$

21.13 a. Being a little more explicit than necessary:

$$\begin{aligned} y'' - 3y' - 10y = e^{4x} &= -\frac{1}{6}\left[-6e^{4x}\right] \\ &= -\frac{1}{6}g_2(x) \\ &= -\frac{1}{6}\left[y_1{}'' - 3y_1{}' - 10y_1\right] \\ &= \left[-\frac{1}{6}y_1\right]'' - 3\left[-\frac{1}{6}y_1\right]' - 10\left[-\frac{1}{6}y_1\right] \end{aligned}$$

So, one solution is

$$y_p(x) = -\frac{1}{6}y_1 = -\frac{1}{6}e^{4x} \quad .$$

21.13 b. We have: $y'' - 3y' - 10y = e^{5x} = \frac{1}{7}\left[7e^{5x}\right] = \frac{1}{7}g_2(x)$.

So, by the principle of superposition, one solution is

$$y_p(x) = \frac{1}{7}y_2 = \frac{1}{7}xe^{5x} \quad .$$

21.13 c.

$$\begin{aligned} y'' - 3y' - 10y &= -18e^{4x} + 14e^{5x} \\ &= 3\left[-6e^{4x}\right] + 2\left[7e^{5x}\right] \\ &= 3g_1(x) + 2g_2(x) \quad . \end{aligned}$$

So, by the principle of superposition, one solution is

$$y_p(x) = 3y_1 + 2y_2 = 3e^{4x} + 2xe^{5x} \quad .$$

21.13 d. Directly applying the principle of superposition:

$$y'' - 3y' - 10y = 35e^{5x} + 12e^{4x} = 5[\underbrace{7e^{5x}}_{g_2(x)}] - 2[\underbrace{-6e^{4x}}_{g_1(x)}]$$

$$\hookrightarrow \qquad y_p(x) = 5y_2(x) - 2y_1(x) = 5xe^{5x} - 2e^{4x} \quad .$$

21.15 a i. With $y = x^2$,

$$\begin{aligned} g(x) = x^2y'' - 7xy' + 15y &= x^2\left[x^2\right]'' - 7x\left[x^2\right]' + 15x^2 \\ &= x^2[2] - 7x[2x] + 15x^2 = 3x^2 \quad . \end{aligned}$$

21.15 a ii. With $y = x$,

$$\begin{aligned} g(x) = x^2y'' - 7xy' + 15y &= x^2[x]'' - 7x[x]' + 15x \\ &= x^2[0] - 7x[1] + 15x = 8x \quad . \end{aligned}$$

21.15 a iii. With $y = 1$,

$$g(x) = x^2y'' - 7xy' + 15y = x^2[1]'' - 7x[1]' + 15[1]$$
$$= x^2[0] - 7x[0] + 15 = 15 \quad .$$

21.15 b.

$$x^2y'' - 7xy' + 15y = x^2 = \frac{1}{3}\left[3x^2\right]$$

$$\hookrightarrow \qquad y_p(x) = \frac{1}{3}\left[x^2\right] = \frac{1}{3}x^2 \quad .$$

21.15 c.

$$x^2y'' - 7xy' + 15y = 4x^2 + 2x + 3 = \frac{4}{3}\left[3x^2\right] + \frac{1}{4}[8x] + \frac{1}{5}[15]$$

$$\hookrightarrow \qquad y_p(x) = \frac{4}{3}\left[x^2\right] + \frac{1}{4}[x] + \frac{1}{5}[1] = \frac{4}{3}x^2 + \frac{1}{4}x + \frac{1}{5} \quad .$$