Chapter 20: Euler Equations

20.1 a. Letting $y(x) = x^r \rightarrowtail y'(x) = rx^{r-1} \rightarrowtail y''(x) = r(r-1)x^{r-2}$,

yields

$$\begin{aligned} 0 &= x^2 y'' - 5xy' + 8y \\ &= x^2 r(r-1)x^{r-2} - 5xrx^{r-1} + 8x^r \\ &= \left[r^2 - r\right]x^r - 5rx^r + 8x^r \\ &= \left[r^2 - r - 5r + 8\right]x^r = \left[r^2 - 6r + 8\right]x^r \quad . \end{aligned}$$

So the indicial equation is

$$0 = r^2 - 6r + 8 \quad ,$$

which factors to

$$0 = (r-2)(r-4) \quad .$$

Thus, x^r is a solution to the differential equation if $r = 2$ or $r = 4$, and, consequently, the general solution to our differential equation is

$$y(x) = c_1 x^2 + c_2 x^4 \quad .$$

20.1 c. Letting $y(x) = x^r \rightarrowtail y'(x) = rx^{r-1} \rightarrowtail y''(x) = r(r-1)x^{r-2}$,

the differential equation becomes

$$\begin{aligned} 0 &= x^2 y'' - 2xy' \\ &= x^2 r(r-1)x^{r-2} - 2xrx^{r-1} \\ &= \left[r^2 - r\right]x^r - 2rx^r \\ &= \left[r^2 - r - 2r\right]x^r = \left[r^2 - 3r\right]x^r \quad . \end{aligned}$$

So the indicial equation is

$$0 = r^2 - 3r = r(r-3) = (r-0)(r-3) \quad ,$$

which means $r = 0$ and $r = 3$. Thus, two particular solutions to the differential equation are $x^0 = 1$ and x^3, and the general solution is

$$y(x) = c_1 \cdot 1 + c_2 x^3 \quad .$$

20.1 e. Letting $y(x) = x^r \rightarrowtail y'(x) = rx^{r-1} \rightarrowtail y''(x) = r(r-1)x^{r-2}$,

the differential equation becomes

$$\begin{aligned} 0 &= x^2 y'' - 5xy' + 9y \\ &= x^2 r(r-1)x^{r-2} - 5xrx^{r-1} + 9x^r \\ &= \left[r^2 - r - 5r + 9\right]x^r = \left[r^2 - 6r + 9\right]x^r \quad . \end{aligned}$$

So the indicial equation is

$$0 \;=\; r^2 \;-\; 6r + 9 \;=\; (r-3)^2 \quad ,$$

which only has $r = 3$ as a solution, leading to the one solution x^3 to the differential equation. As noted in Section 20.2, an appropriate second solution is obtained by either reduction of order, or, more simply, by multiplying the first solution, x^3, by $\ln|x|$. Thus, the general solution to the differential equation is

$$y(x) \;=\; c_1 x^3 \;+\; c_2 x^3 \ln|x| \quad .$$

20.1 g. Letting $\quad y(x) \;=\; x^r \;\rightarrowtail\; y'(x) \;=\; r x^{r-1} \;\rightarrowtail\; y''(x) \;=\; r(r-1)x^{r-2} \quad ,$

the differential equation becomes

$$0 \;=\; 4x^2 y'' \;+\; y \;=\; 4x^2 r(r-1)x^{r-2} \;+\; x^r \;=\; \left[4r^2 - 4r + 1\right] x^r \quad .$$

So the indicial equation is

$$0 \;=\; 4r^2 \;-\; 4r + 1 \;=\; (2r-1)^2 \quad ,$$

which only has $r = {}^1\!/_2$ as a solution. Thus, the general solution to the differential equation is

$$y(x) \;=\; c_1 x^{1/2} \;+\; c_2 x^{1/2} \ln|x| \quad .$$

20.1 i. Letting $\quad y(x) \;=\; x^r \;\rightarrowtail\; y'(x) \;=\; r x^{r-1} \;\rightarrowtail\; y''(x) \;=\; r(r-1)x^{r-2} \quad ,$

the differential equation becomes

$$\begin{aligned} 0 \;&=\; x^2 y'' \;-\; 5xy' \;+\; 13y \\ &=\; x^2 r(r-1)x^{r-2} \;-\; 5xrx^{r-1} \;+\; 13x^r \\ &=\; \left[r^2 - r - 5r + 13\right] x^r \;=\; \left[r^2 - 6r + 13\right] x^r \quad . \end{aligned}$$

So the indicial equation is

$$0 \;=\; r^2 \;-\; 6r + 13 \quad ,$$

the solutions of which are

$$r_\pm \;=\; \frac{-[-6] \pm \sqrt{[-6]^2 - 4 \cdot 13}}{2} \;=\; 3 \;\pm\; 2i \quad .$$

The corresponding particular solutions (with $x > 0$) to the differential equation are then

$$\begin{aligned} y_\pm(x) \;=\; x^{r_\pm} \;=\; x^{3\pm 2i} \;&=\; x^3 x^{\pm 2i} \\ &=\; x^3 e^{\ln\left(x^{\pm 2i}\right)} \\ &=\; x^3 e^{i 2 \ln|x|} \\ &=\; x^3 \left[\cos(2\ln|x|) + i \sin(2\ln|x|)\right] \\ &=\; \underbrace{x^3 \cos(2\ln|x|)}_{y_1(x)} + i\, \underbrace{x^3 \sin(2\ln|x|)}_{y_2(x)} \quad . \end{aligned}$$

So, in terms of real-valued functions, the general solution to the differential equation is

$$y(x) \;=\; c_1 y_1(x) \;+\; c_2 y_2(x) \;=\; c_1 x^3 \cos(2\ln|x|) \;+\; c_2 x^3 \sin(2\ln|x|) \quad .$$

20.1 k. Letting $y(x) = x^r \rightarrowtail y'(x) = rx^{r-1} \rightarrowtail y''(x) = r(r-1)x^{r-2}$,

the differential equation becomes

$$\begin{aligned} 0 &= x^2y'' + 5xy' + 29y \\ &= x^2r(r-1)x^{r-2} + 5xrx^{r-1} + 29x^r \\ &= \left[r^2 - r + 5r + 29\right]x^r = \left[r^2 + 4r + 29\right]x^r \quad . \end{aligned}$$

So the indicial equation is

$$0 = r^2 + 4r + 29 \quad ,$$

the solutions of which are

$$r_\pm = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 20}}{2} = -2 \pm 5i \quad .$$

The corresponding particular solutions (with $x > 0$) to the differential equation are then

$$\begin{aligned} y_\pm(x) = x^{r_\pm} = x^{-2\pm 5i} &= x^{-2}x^{\pm 5i} \\ &= x^{-2}e^{\ln\left(x^{\pm 5i}\right)} \\ &= x^{-2}e^{i5\ln|x|} \\ &= x^{-2}\left[\cos(5\ln|x|) + i\sin(5\ln|x|)\right] \\ &= \underbrace{x^{-2}\cos(5\ln|x|)}_{y_1(x)} + i\,\underbrace{x^{-2}\sin(5\ln|x|)}_{y_2(x)} \quad . \end{aligned}$$

So, in terms of real-valued functions, the general solution to the differential equation is

$$y(x) = c_1y_1(x) + c_2y_2(x) = c_1x^{-2}\cos(5\ln|x|) + c_2x^{-2}\sin(5\ln|x|) \quad .$$

20.1 m. Letting $y(x) = x^r \rightarrowtail y'(x) = rx^{r-1} \rightarrowtail y''(x) = r(r-1)x^{r-2}$,

the differential equation becomes

$$\begin{aligned} 0 &= 2x^2y'' + 5xy' + y \\ &= 2x^2r(r-1)x^{r-2} + 5xrx^{r-1} + x^r \\ &= \left[2r^2 - 2r + 5r + 1\right]x^r = \left[2r^2 + 3r + 1\right]x^r \quad . \end{aligned}$$

Writing out the indicial equation, and then continuing

$$2r^2 + 3r + 1 = 0$$

$$\hookrightarrow \qquad r = \frac{-3 \pm \sqrt{3^2 - 4 \cdot 2 \cdot 1}}{2 \cdot 2} = \frac{-3 \pm 1}{4}$$

$$\hookrightarrow \qquad r = -\frac{1}{2} \quad \text{and} \quad r = -1$$

$$\hookrightarrow \qquad y(x) = c_1x^{-1/2} + c_2x^{-1} \quad .$$

20.1 o. Letting $y(x) = x^r \rightarrowtail y'(x) = rx^{r-1} \rightarrowtail y''(x) = r(r-1)x^{r-2}$,

the differential equation becomes

$$0 = x^2y'' + xy' = x^2r(r-1)x^{r-2} + xrx^{r-1} = \left[r^2 - r + r\right]x^r = \left[r^2\right]x^r \quad .$$

Writing out the indicial equation, and then continuing

$$r^2 = 0 \rightarrowtail r = 0$$

$$\hookrightarrow \qquad y(x) = c_1x^{-0} + c_2x^0 \ln|x| = c_1 + c_2 \ln|x| \quad .$$

20.1 q. Letting $y(x) = x^r \rightarrowtail y'(x) = rx^{r-1} \rightarrowtail y''(x) = r(r-1)x^{r-2}$,

the differential equation becomes

$$\begin{aligned} 0 &= 4x^2y'' + 8xy' + 5y \\ &= 4x^2r(r-1)x^{r-2} + 8xrx^{r-1} + 5x^r \\ &= \left[4r^2 + 4r + 5r\right]x^r \quad . \end{aligned}$$

So the indicial equation is

$$0 = 4r^2 + 4r + 5r \quad ,$$

which means that

$$r = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 4 \cdot 5}}{2 \cdot 4} = -\frac{1}{2} \pm 1i \quad .$$

The corresponding particular solutions (with $x > 0$) to the differential equation are then

$$\begin{aligned} y_\pm(x) = x^{r_\pm} = x^{-1/2 \pm 1i} &= x^{-1/2}x^{\pm i} \\ &= x^{-1/2}e^{\ln\left(x^{\pm i}\right)} \\ &= x^{-1/2}e^{i \ln|x|} \\ &= x^{-1/2}\left[\cos(\ln|x|) + i \sin(\ln|x|)\right] \\ &= \underbrace{x^{-1/2}\cos(\ln|x|)}_{y_1(x)} + i\, \underbrace{x^{-1/2}\sin(\ln|x|)}_{y_2(x)} \quad . \end{aligned}$$

So, in terms of real-valued functions, the general solution to the differential equation is

$$y(x) = c_1y_1(x) + c_2y_2(x) = c_1x^{-1/2}\cos(\ln|x|) + c_2x^{-1/2}\sin(\ln|x|) \quad .$$

20.2 a. Letting $y(x) = x^r \rightarrowtail y'(x) = rx^{r-1} \rightarrowtail y''(x) = r(r-1)x^{r-2}$,

the differential equation becomes

$$\begin{aligned} 0 &= x^2y'' - 2xy' - 10y \\ &= x^2r(r-1)x^{r-2} - 2xrx^{r-1} - 10x^r \\ &= \left[r^2 - r - 2r - 10\right]x^r = \left[r^2 - 3r - 10\right]x^r \quad . \end{aligned}$$

Writing out the indicial equation, and then continuing

$$0 = r^2 - 3r - 10 = (r-5)(r+2)$$

$$\hookrightarrow \qquad r = 5 \quad \text{and} \quad r = -2$$

$$\hookrightarrow \qquad y(x) = c_1 x^5 + c_2 x^{-2} \qquad (\star)$$

$$\hookrightarrow \qquad y'(x) = 5c_1 x^4 - 2c_2 x^{-3} \quad .$$

Applying the initial data:

$$5 = y(1) = c_1 \cdot 1^5 + c_2 \cdot 1^{-2} = c_1 + c_2$$

and

$$4 = y'(1) = 5c_1 \cdot 1^4 - 2c_2 \cdot 1^{-3} = 5c_1 - 2c_2 \quad .$$

Solving for c_1 and c_2, and plugging the values back into formula ($\star$) for y:

$$5 = c_1 + c_2 \quad \text{and} \quad 4 = 5c_1 - 2c_2$$

$$\hookrightarrow \qquad c_1 = 5 - c_2 \quad \text{and} \quad 4 = 5[5 - c_2] - 2c_2 = 25 - 7c_2$$

$$\hookrightarrow \qquad c_1 = 5 - c_2 \quad \text{and} \quad c_2 = \frac{4-25}{-7} = 3$$

$$\hookrightarrow \qquad c_1 = 5 - 3 = 2 \quad \text{and} \quad c_2 = \frac{4-25}{-7} = 3$$

$$\hookrightarrow \qquad y(x) = 2x^5 + 3x^{-2} \quad .$$

20.2 c. Letting $y(x) = x^r \rightarrowtail y'(x) = rx^{r-1} \rightarrowtail y''(x) = r(r-1)x^{r-2}$,
the differential equation becomes

$$\begin{aligned} 0 &= x^2 y'' - 11xy' + 36y \\ &= x^2 r(r-1)x^{r-2} - 11xrx^{r-1} + 36x^r = \left[r^2 - 12r + 36\right] x^r \quad . \end{aligned}$$

Writing out the indicial equation, and then continuing

$$0 = r^2 - 12r + 36 = (r-6)^2$$

$$\hookrightarrow \qquad r = 6 \text{ is the only root.}$$

$$\hookrightarrow \qquad y(x) = c_1 x^6 + c_2 x^6 \ln|x| \qquad (\star)$$

$$\hookrightarrow \qquad y'(x) = 6c_1 x^5 + c_2 \left[6x^5 \ln|x| + x^5\right] \quad .$$

Applying the initial data:

$$\frac{1}{2} = y(1) = c_1 \cdot 1^6 + c_2 \cdot 1^6 \ln|1| = c_1$$

and

$$2 = y'(0) = 6c_1 \cdot 1^5 + c_2\left[6 \cdot 1^5 \ln|1| + 1^5\right] = 6c_1 + c_2 \quad .$$

Solving for c_1 and c_2, and plugging the values back into formula ($\star$) for y:

$$\frac{1}{2} = c_1 \qquad \text{and} \qquad 2 = 6c_1 + c_2$$

$$\hookrightarrow \qquad c_1 = \frac{1}{2} \qquad \text{and} \qquad c_2 = 2 - 6c_1 = 2 - 6\left[\frac{1}{2}\right] = -1$$

$$\hookrightarrow \qquad y(x) = \frac{1}{2}x^6 - x^6 \ln|x| \quad .$$

20.2 e. Letting $\quad y(x) = x^r \quad \rightarrowtail \quad y'(x) = rx^{r-1} \quad \rightarrowtail \quad y''(x) = r(r-1)x^{r-2} \quad ,$

the differential equation becomes

$$\begin{aligned} 0 &= x^2y'' - xy' + 2y \\ &= x^2r(r-1)x^{r-2} - xrx^{r-1} + 2x^r = \left[r^2 - 2r + 2\right]x^r \quad . \end{aligned}$$

Writing out the indicial equation, and then continuing

$$0 = r^2 - 2r + 2$$

$$\hookrightarrow \qquad r = \frac{-[-2] \pm \sqrt{[-2]^2 - 4 \cdot 2}}{2} = 1 \pm i$$

$$\hookrightarrow \qquad y_{\pm}(x) = x^{1\pm i} = \cdots = x\cos(\ln|x|) \pm x\sin(\ln|x|)$$

$$\hookrightarrow \qquad y(x) = c_1 x\cos(\ln|x|) + c_2 x\sin(\ln|x|) \quad . \qquad (\star)$$

Computing the derivative:

$$\begin{aligned} y'(x) &= \frac{d}{dx}\left[c_1 x\cos(\ln|x|) + c_2 x\sin(\ln|x|)\right] \\ &= c_1\left[\cos(\ln|x|) - x\sin(\ln|x|)\, x^{-1}\right] + c_2\left[\sin(\ln|x|) + x\cos(\ln|x|)\, x^{-1}\right] \\ &= c_1\left[\cos(\ln|x|) - \sin(\ln|x|)\right] + c_2\left[\sin(\ln|x|) + \cos(\ln|x|)\right] \quad . \end{aligned}$$

Applying the initial data:

$$3 = y(1) = c_1 1\cos(\ln|1|) + c_2 1\sin(\ln|1|) = c_1$$

and

$$\begin{aligned} 0 = y'(1) &= c_1\left[\cos(\ln|1|) - \sin(\ln|1|)\right] + c_2\left[\sin(\ln|1|) + \cos(\ln|1|)\right] \\ &= c_1 + c_2 \quad . \end{aligned}$$

Clearly $c_1 = 3$ and $c_2 = -c_1 = -3$. Plugging these values back into formula ($\star$) for y then yields

$$y(x) = 3x\cos(\ln|x|) - 3x\sin(\ln|x|)$$

20.4 a. Assuming $y = x^r$ and taking three derivatives, we get

$$y' = rx^{r-1} \quad \rightarrowtail \quad y'' = r(r-1)x^{r-2} \quad \rightarrowtail \quad y''' = r(r-1)(r-2)x^{r-3} \quad .$$

Plugging these into the differential equation:

$$\begin{aligned} 0 &= x^3 y''' + 2x^2 y'' - 4xy' + 4y \\ &= x^3 r(r-1)(r-2)x^{r-3} + 2x^2 r(r-1)x^{r-2} - 4xrx^{r-1} + 4x^r \\ &= \left[r^3 - 3r^2 + 2r\right]x^r + 2\left[r^2 - r\right]x^r - 4[r]x^r + 4x^r \\ &= \left[r^3 - r^2 - 4r + 4\right]x^r \quad . \end{aligned}$$

So the indicial equation is the third-degree polynomial equation

$$\underbrace{r^3 - r^2 - 4r + 4}_{p(x)} = 0 \quad .$$

To find the solutions, we'll first test to see if $r = 1$ is one root of $p(r)$:

$$p(1) = 1^3 - 1^2 - 4 \cdot 1 + 4 = 1 - 1 - 4 + 4 = 0 \quad .$$

Hence, $r = 1$ is a root, and $r - 1$ is a factor. Dividing out this factor:

$$\begin{array}{r|l} & r^2 \qquad\quad - 4 \\ \hline r - 1 & r^3 - r^2 - 4r + 4 \\ & \underline{-r^3 + r^2} \\ & \qquad\quad -4r + 4 \\ & \qquad\quad \underline{4r - 4} \\ & \qquad\qquad\quad 0 \end{array} \quad .$$

This means we can factor our indicial equation as follows:

$$\begin{aligned} 0 &= r^3 - r^2 - 4r + 4 \\ &= (r-1)\left(r^2 - 4\right) = (r-1)(r-2)(r+2) \quad . \end{aligned}$$

Thus, the solutions the indicial equation are the three distinct values

$$r = 1 \quad , \quad r = 2 \quad \text{and} \quad r = -2 \quad ,$$

and the corresponding general solution to the third-order differential equation is

$$y(x) = c_1 x + c_2 x^2 + c_3 x^{-2} \quad .$$

20.4 c. Assuming $y = x^r$ and taking three derivatives, we get

$$y' = rx^{r-1} \quad \rightarrowtail \quad y'' = r(r-1)x^{r-2} \quad \rightarrowtail \quad y''' = r(r-1)(r-2)x^{r-3} \quad ,$$

which, when plugged into the differential equation yields

$$\begin{aligned} 0 &= x^3 y''' - 5x^2 y'' + 14xy' - 18y \\ &= x^3 r(r-1)(r-2)x^{r-3} - 5x^2 r(r-1)x^{r-2} + 14xrx^{r-1} - 18x^r \end{aligned}$$

$$= \left[r^3 - 3r^2 + 2r\right]x^r \;-\; 5\left[r^2 - r\right]x^r \;+\; 14[r]x^r \;-\; 18x^r$$
$$= \left[r^3 - 8r^2 + 21r - 18\right]x^r \quad .$$

So the indicial equation is the third-degree polynomial equation

$$\underbrace{r^3 \;-\; 8r^2 \;+\; 21r \;-\; 18}_{p(x)} \;=\; 0 \quad .$$

To find the solutions, we'll compute $p(r)$ for different values of r until we find a value such that $p(r) = 0$:

$$p(1) \;=\; 1^3 \;-\; 8\cdot 1^2 \;+\; 21\cdot 1 \;-\; 18 \;=\; 1 \;-\; 8 \;+\; 21 \;-\; 18 \;=\; -4 \;\neq\; 0 \quad ,$$
$$p(2) \;=\; 2^3 \;-\; 8\cdot 2^2 \;+\; 21\cdot 2 \;-\; 18 \;=\; 8 \;-\; 32 \;+\; 42 \;-\; 18 \;=\; 0 \quad .$$

Hence, $r = 2$ is a root, and $r - 2$ is a factor. Dividing out this factor:

$$\begin{array}{r|l}
 & r^2 \;-6r \;+9 \\ \hline
r-2 & r^3 - 8r^2 + 21r - 18 \\
 & -r^3 + 2r^2 \\ \hline
 & \quad -6r^2 + 21r \\
 & \quad 6r^2 - 12r \\ \hline
 & \qquad 9r - 18 \\
 & \qquad -9r + 18 \\ \hline
 & \qquad\qquad 0
\end{array} \quad .$$

This means we can factor our indicial equation as follows:

$$0 \;=\; r^3 \;-\; 8r^2 \;+\; 21r \;-\; 18$$
$$\;=\; (r-2)\left(r^2 - -6r + 9\right) \;=\; (r-2)(r-3)^2 \quad .$$

Thus, the solutions the indicial equation are $r = 2$ (with multiplicity 1) and $r = 3$ (with multiplicity 2), and the corresponding general solution to the third-order differential equation is

$$y(x) \;=\; c_1x^2 \;+\; c_2x^3 \;+\; c_3x^3\ln|x| \quad .$$

20.4 e. Assuming $y = x^r$ and taking four derivatives, we get

$$y' \;=\; rx^{r-1} \quad\rightarrowtail\quad y'' \;=\; r(r-1)x^{r-2}$$

$$\hookrightarrow \quad y''' \;=\; r(r-1)(r-2) \quad\rightarrowtail\quad y^{(4)} \;=\; r(r-1)(r-2)(r-3)x^{r-4} \quad .$$

which, when plugged into the differential equation yields

$$0 \;=\; x^4y^{(4)} \;+\; 6x^3y''' \;+\; 15x^2y'' \;+\; 9xy' \;+\; 16y$$
$$\;=\; x^4r(r-1)(r-2)(r-3)x^{r-4} \;+\; 6x^3r(r-1)(r-2)x^{r-3}$$
$$\;+\; 15x^2r(r-1)x^{r-2} \;+\; 9xrx^{r-1} \;+\; 16x^r$$

$$= \left[r^4 - 6r^3 + 11r^2 - 6r\right]x^r \;+\; 6\left[r^3 - 3r^2 + 2r\right]x^r$$
$$+\; 15\left[r^2 - r\right]x^r \;+\; 9[r]x^r \;+\; 16x^r$$
$$= \left[r^4 + 0r^3 + 8r^2 + 0r + 16\right]x^r \quad .$$

So the indicial equation is

$$0 \;=\; r^4 \;+\; 8r^2 \;+\; 16 \;=\; \left(r^2 + 4\right)^2 \quad ,$$

which has solutions

$$r \;=\; \pm 2i \qquad \text{with multiplicity } 2 \quad ,$$

and the four corresponding real-valued solutions to our Euler equation are

$$\cos(2\ln|x|) \quad , \quad \sin(2\ln|x|) \quad , \quad \cos(2\ln|x|)\ln|x| \quad \text{and} \quad \sin(2\ln|x|)\ln|x| \quad .$$

The general solution, then, is

$$y \;=\; c_1\cos(2\ln|x|) \;+\; c_2\sin(2\ln|x|)$$
$$+\; c_4\cos(2\ln|x|)\ln|x| \;+\; c_4\sin(2\ln|x|)\ln|x| \quad .$$

20.4 g. Assuming $y = x^r$ and taking four derivatives, we get

$$y' \;=\; rx^{r-1} \quad \rightarrowtail \quad y'' \;=\; r(r-1)x^{r-2}$$

$$\hookrightarrow \quad y''' \;=\; r(r-1)(r-2) \quad \rightarrowtail \quad y^{(4)} \;=\; r(r-1)(r-2)(r-3)x^{r-4} \quad .$$

which, when plugged into the differential equation yields

$$0 \;=\; x^4y^{(4)} \;+\; 2x^3y''' \;+\; x^2y'' \;-\; xy' \;+\; y$$
$$= x^4r(r-1)(r-2)(r-3)x^{r-4} \;+\; 2x^3r(r-1)(r-2)x^{r-3}$$
$$+\; x^2r(r-1)x^{r-2} \;-\; xrx^{r-1} \;+\; x^r$$
$$= \left[r^4 - 6r^3 + 11r^2 - 6r\right]x^r \;+\; 2\left[r^3 - 3r^2 + 2r\right]x^r \;+\; \left[r^2 - r\right]x^r$$
$$-\; [r]x^r \;+\; x^r$$
$$= \left[r^4 - 4r^3 + 6r^2 - 4r + 1\right]x^r$$

So the indicial equation is the fourth-degree polynomial equation

$$\underbrace{r^4 \;-\; 4r^3 \;+\; 6r^2 \;-\; 4r \;+\; 1}_{p(x)} \;=\; 0 \quad .$$

Fortunately, it's easy to see that

$$p(1) \;=\; 1^4 \;-\; 4\cdot 1^3 \;+\; 6\cdot 1^2 \;-\; 4\cdot 1 \;+\; 1 \;=\; 0 \quad ,$$

telling us that $r = 1$ is a root, and $r - 1$ is a factor. Dividing out this factor,

$$
\begin{array}{r|r}
 & r^3 - 3r^2 + 3r - 1 \\
\hline
r - 1 & r^4 - 4r^3 + 6r^2 - 4r + 1 \\
 & -r^4 + r^3 \qquad\qquad\qquad\quad \\
\cline{2-2}
 & -3r^3 + 6r^2 \qquad\qquad\quad \\
 & 3r^3 - 3r^2 \qquad\qquad\quad \\
\cline{2-2}
 & 3r^2 - 4r \qquad\quad \\
 & -3r^2 + 3r \qquad\quad \\
\cline{2-2}
 & -r + 1 \\
 & r - 1 \\
\cline{2-2}
 & 0
\end{array}
\quad ,
$$

we find that

$$
p(r) = r^4 - 4r^3 + 6r^2 - 4r + 1 = (r-1)\big(\underbrace{r^3 - 3r^2 + 3r - 1}_{q(r)}\big) \quad .
$$

It is even more easy to see that

$$
q(1) = 1^3 - 3 \cdot 1^2 + 3 \cdot 1 - 1 = 0 \quad .
$$

So $r - 1$ is also a factor of $q(r)$. Dividing out that factor,

$$
\begin{array}{r|r}
 & r^2 - 2r + 1 \\
\hline
r - 1 & r^3 - 3r^2 + 3r - 1 \\
 & -r^3 + r^2 \qquad\qquad\quad \\
\cline{2-2}
 & -2r^2 + 3r \qquad\quad \\
 & 2r^2 - 2r \qquad\quad \\
\cline{2-2}
 & r - 1 \\
 & -r + 1 \\
\cline{2-2}
 & 0
\end{array}
\quad ,
$$

and we see that our indicial equation factors as follows:

$$
\begin{aligned}
0 &= r^4 - 4r^3 + 6r^2 - 4r + 1 \\
&= (r-1)\big(r^3 - 3r^2 + 3r - 1\big) \\
&= (r-1)(r-1)\big(r^2 - 2r + 1\big) \\
&= (r-1)(r-1)(r-1)(r-1) = (r-1)^4 \quad .
\end{aligned}
$$

Thus,

$$
y(x) = c_1 x + c_2 x \ln|x| + c_3 x (\ln|x|)^2 + c_4 x (\ln|x|)^3 \quad .
$$

20.6 a. Letting $\quad y = x^r \quad \rightarrowtail \quad y' = r x^{r-1} \quad \rightarrowtail \quad y'' = r(r-1)x^{r-2} \quad ,$

the differential equation becomes

$$
\begin{aligned}
0 &= \alpha x^2 y'' + \beta x y' + \gamma y \\
&= \alpha x^2 r(r-1)x^{r-2} + \beta x r x^{r-1} + \gamma x^r
\end{aligned}
$$

$$= \left[\alpha r^2 - \alpha r + \beta r + \gamma\right] x^r$$
$$= \left[\alpha r^2 r + (\beta - \alpha) r + \gamma\right] x^r \quad .$$

So the indicial equation is

$$\alpha r^2 r \ + \ (\beta - \alpha) r \ + \ \gamma \ = \ 0 \quad . \tag{$\star$}$$

20.6 b. Let $\quad y(x) = Y(t) \quad$ where $\quad x = e^t \quad$ and $\quad t = \ln|x| \quad .$
We will need to convert the derivatives in the given Euler equation to corresponding derivatives of Y . For convenience, let us first observe that

$$\frac{dt}{dx} = \frac{d \ln|x|}{dx} = \frac{1}{x} = e^{-t} \quad .$$

Using this and the chain rule, we then have

$$\frac{dy}{dx} = \frac{d}{dx} Y(t) = \frac{dt}{dx}\frac{d}{dt} Y(t) = e^{-t}\frac{dY}{dt}$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{dt}{dx}\frac{d}{dt}\left[e^{-t}\frac{dY}{dt}\right]$$
$$= e^{-t}\left[-e^{-t}\frac{dY}{dt} + e^{-t}\frac{d^2Y}{dt^2}\right] = e^{-2t}\left[\frac{d^2Y}{dt^2} - \frac{dY}{dt}\right] \quad .$$

Note that

$$xy' = x\frac{dy}{dx} = e^t \cdot e^{-t}\frac{dY}{dt} = \frac{dY}{dt}$$

and

$$x^2 y'' = x^2 \frac{d^2x}{dy^2} = \left(e^t\right)^2 e^{-2t}\left[\frac{d^2Y}{dt^2} - \frac{dY}{dt}\right] = \frac{d^2Y}{dt^2} - \frac{dY}{dt} \quad .$$

Applying the above to the given generic Euler equation gives us

$$0 = \alpha x^2 y'' + \beta x y' + \gamma y$$
$$= \alpha\left[\frac{d^2Y}{dt^2} - \frac{dY}{dt}\right] + \beta\frac{dY}{dt} + \gamma Y(t) \quad ,$$

which, after a trivial bit of algebra, becomes

$$\alpha\frac{d^2Y}{dt^2} + (\beta - \alpha)\frac{dY}{dt} + \gamma Y = 0 \quad ,$$

a second-order, constant cofficient differential equation with characteristic equation

$$\alpha r^2 \ + \ (\beta - \alpha) r \ + \ \gamma \ = \ 0 \quad . \tag{$\star\star$}$$

20.6 c. Just observe that equation ($\star$), the indicial equation for the Euler equation, and equation ($\star\star$), the characteristic equation for the corresponding constant coefficient equation, are identical.