**Chapter 16: Verifying the Big Theorems and an Introduction to Differential Operators**

**16.2 a.**  By inspection:   $L = \dfrac{d^2}{dx^2} + 5\dfrac{d}{dx} + 6$ .

**16.2 b i.**    $L[\sin(x)] = \dfrac{d^2}{dx^2}\big[\sin(x)\big] + 5\dfrac{d}{dx}\big[\sin(x)\big] + 6\big[\sin(x)\big]$

$= -\sin(x) + 5\cos(x) + 6\sin(x) = 5\sin(x) + 5\cos(x)$   .

**16.2 b iii.**    $L\big[e^{-3x}\big] = \dfrac{d^2}{dx^2}\big[e^{-3x}\big] + 5\dfrac{d}{dx}\big[e^{-3x}\big] + 6\big[e^{-3x}\big]$

$= 9e^{-3x} + 5\big[-3e^{-3x}\big] + 6e^{-3x} = [9 - 15 + 6]\,e^{-3x} = 0$   .

**16.2 c.**   Since the differential equation can be written as

$$L[y] = 0 \quad,$$

and we have

$$L\big[e^{-3x}\big] = 0 \quad,$$

it follows that  $y(x) = e^{-3x}$  is a solution to the differential equation.

**16.3 a.**  By inspection:    $L = \dfrac{d^2}{dx^2} - 5\dfrac{d}{dx} + 9$ .

**16.3 b i.**    $L[\sin(x)] = \dfrac{d^2}{dx^2}\big[\sin(x)\big] - 5\dfrac{d}{dx}\big[\sin(x)\big] + 9\big[\sin(x)\big]$

$= -\sin(x) - 5\cos(x) + 9\sin(x) = 8\sin(x) - 5\cos(x)$   .

**16.3 b iii.**                          $L\big[e^{2x}\big] = \dfrac{d^2}{dx^2}\big[e^{2x}\big] - 5\dfrac{d}{dx}\big[e^{2x}\big] + 9\big[e^{2x}\big]$

$= 4e^{2x} - 5\cdot 2e^{2x} + 9e^{2x} = 3e^{2x}$   .

**16.4 a.**  By inspection:   $L = x^2\dfrac{d^2}{dx^2} + 5x\dfrac{d}{dx} + 6$ .

**16.4 b i.**                    $L[\sin(x)] = x^2\dfrac{d^2}{dx^2}\big[\sin(x)\big] + 5x\dfrac{d}{dx}\big[\sin(x)\big] + 6\big[\sin(x)\big]$

$= -x^2\sin(x) + 5x\cos(x) + 6\sin(x)$

$= \big(6 - x^2\big)\sin(x) + 5x\cos(x)$   .

**16.4 b iii.**    $L\big[x^3\big] = x^2\dfrac{d^2}{dx^2}\big[x^3\big] + 5x\dfrac{d}{dx}\big[x^3\big] + 6\big[x^3\big]$

$= x^2\big[3\cdot 2x\big] + 5x\big[3x^2\big] + 6x^3 = 6x^3 + 15x^3 + 6x^3 = 27x^3$   .

**16.5 a.**  By inspection:   $L = \dfrac{d^3}{dx^3} - \sin(x)\dfrac{d}{dx} + \cos(x)$   .

**16.5 b i.** 
$$L[\sin(x)] = \frac{d^3}{dx^3}\big[\sin(x)\big] - \sin(x)\frac{d}{dx}\big[\sin(x)\big] + \cos(x)\big[\sin(x)\big]$$
$$= -\cos(x) - \sin(x)\cos(x) + \cos(x)\sin(x) = -\cos(x) \quad .$$

**16.5 b iii.** 
$$L[x^2] = \frac{d^3}{dx^3}[x^2] - \sin(x)\frac{d}{dx}[x^2] + \cos(x)\big[x^2\big]$$
$$= 0 - \sin(x)\cdot 2x + \cos(x)\cdot x^2 = x^2\cos(x) - 2x\sin(x) \quad .$$

**16.6 a.** Let $\phi$ be any sufficiently differentiable function. Then

$$L_2L_1[\phi] = L_2[L_1[\phi]] = L_2\left[\frac{d\phi}{dx} + x\phi\right]$$
$$= \frac{d}{dx}\left[\frac{d\phi}{dx} + x\phi\right] - x\left[\frac{d\phi}{dx} + x\phi\right]$$
$$= \frac{d^2\phi}{dx^2} + \left[\phi + x\frac{d\phi}{dx}\right] - x\frac{d\phi}{dx} - x^2\phi$$
$$= \frac{d^2\phi}{dx^2} + \left(1 - x^2\right)\phi \quad .$$

So, $\quad L_2L_1 = \dfrac{d^2}{dx^2} + \left(1 - x^2\right) \quad .$

On the other hand,

$$L_1L_2[\phi] = L_1[L_2[\phi]] = L_1\left[\frac{d\phi}{dx} - x\phi\right]$$
$$= \frac{d}{dx}\left[\frac{d\phi}{dx} - x\phi\right] + x\left[\frac{d\phi}{dx} - x\phi\right]$$
$$= \frac{d^2\phi}{dx^2} - \left[\phi + x\frac{d\phi}{dx}\right] + x\frac{d\phi}{dx} - x^2\phi$$
$$= \frac{d^2\phi}{dx^2} - \left(1 + x^2\right)\phi \quad .$$

So, $\quad L_1L_2 = \dfrac{d^2}{dx^2} - \left(1 + x^2\right) \quad .$

**16.6 c.** Let $\phi$ be any sufficiently differentiable function. Then

$$L_2L_1[\phi] = L_2[L_1[\phi]] = L_2\left[x\frac{d\phi}{dx} + 3\phi\right]$$
$$= \frac{d}{dx}\left[x\frac{d\phi}{dx} + 3\phi\right] + 2x\left[x\frac{d\phi}{dx} + 3\phi\right]$$
$$= \left[\frac{d\phi}{dx} + x\frac{d^2\phi}{dx^2}\right] + 3\frac{d\phi}{dx} + 2x^2\frac{d\phi}{dx} + 6x\phi$$
$$= x\frac{d^2\phi}{dx^2} + \left(4 + 2x^2\right)\frac{d\phi}{dx} + 6x\phi \quad .$$

So, $\quad L_2L_1 = x\dfrac{d^2}{dx^2} + \left(4 + 2x^2\right)\dfrac{d}{dx} + 6x \quad .$

On the other hand,

$$L_1 L_2[\phi] = L_1[L_2[\phi]] = L_1\left[\frac{d\phi}{dx} + 2x\phi\right]$$

$$= x\frac{d}{dx}\left[\frac{d\phi}{dx} + 2x\phi\right] + 3\left[\frac{d\phi}{dx} + 2x\phi\right]$$

$$= x\frac{d^2\phi}{dx^2} + x\left[2\phi + 2x\frac{d\phi}{dx}\right] + 3\frac{d\phi}{dx} + 6x\phi$$

$$= x\frac{d^2\phi}{dx^2} + \left(3 + 2x^2\right)\frac{d\phi}{dx} + 8x\phi \quad .$$

So,    $L_1 L_2 = x\dfrac{d^2}{dx^2} + \left(3 + 2x^2\right)\dfrac{d}{dx} + 8x$  .

**16.6 e.**  Let $\phi$ be any sufficiently differentiable function. Then

$$L_2 L_1[\phi] = L_2[L_1[\phi]] = L_2\left[\frac{d^2\phi}{dx^2}\right] = x^3\frac{d^2\phi}{dx^2} \quad .$$

So,    $L_2 L_1 = x^3\dfrac{d^2}{dx^2}$  .

On the other hand,

$$L_1 L_2[\phi] = L_1[L_1[\phi]] = L_2[x^3\phi]$$

$$= \frac{d^2}{dx^2}[x^3\phi]$$

$$= \frac{d}{dx}\left[\frac{d}{dx}[x^3\phi]\right]$$

$$= \frac{d}{dx}\left[3x^2\phi + x^3\frac{d\phi}{dx}\right]$$

$$= \left[6x\phi + 3x^2\frac{d\phi}{dx}\right] + \left[3x^2\frac{d\phi}{dx} + x^3\frac{d^2\phi}{dx^2}\right]$$

$$= x^3\frac{d^2\phi}{dx^2} + 6x^2\frac{d\phi}{dx} + 6x\phi \quad .$$

So,    $L_1 L_2 = x^3\dfrac{d^2}{dx^2} + 6x^2\dfrac{d}{dx} + 6x$  .

**16.7 a.**  For any sufficiently differentiable function $\phi$,

$$\left(\frac{d}{dx} + 2\right)\left(\frac{d}{dx} + 3\right)[\phi] = \left(\frac{d}{dx} + 2\right)\left[\frac{d\phi}{dx} + 3\phi\right]$$

$$= \frac{d}{dx}\left[\frac{d\phi}{dx} + 3\phi\right] + 2\left[\frac{d\phi}{dx} + 3\phi\right]$$

$$= \frac{d^2\phi}{dx^2} + 3\frac{d\phi}{dx} + 2\frac{d\phi}{dx} + 6\phi$$

$$= \frac{d^2\phi}{dx^2} + 5\frac{d\phi}{dx} + 6\phi \quad .$$

So,

$$\left(\frac{d}{dx} + 2\right)\left(\frac{d}{dx} + 3\right) = \frac{d^2}{dx^2} + 5\frac{d}{dx} + 6 \quad .$$

**16.7 c.** For any sufficiently differentiable function $\phi$,

$$\begin{aligned}
\left(x\frac{d}{dx} + 4\right)\left(\frac{d}{dx} + \frac{1}{x}\right)[\phi] &= \left(x\frac{d}{dx} + 4\right)\left[\frac{d\phi}{dx} + \frac{1}{x}\phi\right] \\
&= x\frac{d}{dx}\left[\frac{d\phi}{dx} + \frac{1}{x}\phi\right] + 4\left[\frac{d\phi}{dx} + \frac{1}{x}\phi\right] \\
&= x\frac{d^2\phi}{dx^2} + x\left[\frac{-1}{x^2}\phi + \frac{1}{x}\frac{d\phi}{dx}\right] + 4\frac{d\phi}{dx} + \frac{4}{x}\phi \\
&= x\frac{d^2\phi}{dx^2} + 5\frac{d\phi}{dx} + \frac{3}{x}\phi \quad .
\end{aligned}$$

So,

$$\left(x\frac{d}{dx} + 4\right)\left(\frac{d}{dx} + \frac{1}{x}\right) = x\frac{d^2}{dx^2} + 5\frac{d}{dx} + \frac{3}{x} \quad .$$

**16.7 e.** For any sufficiently differentiable function $\phi$,

$$\begin{aligned}
\left(\frac{d}{dx} + \frac{1}{x}\right)\left(\frac{d}{dx} + 4x\right)[\phi] &= \left(\frac{d}{dx} + \frac{1}{x}\right)\left[\frac{d\phi}{dx} + 4x\phi\right] \\
&= \frac{d}{dx}\left[\frac{d\phi}{dx} + 4x\phi\right] + \frac{1}{x}\left[\frac{d\phi}{dx} + 4x\phi\right] \\
&= \frac{d^2\phi}{dx^2} + \left[4\phi + 4x\frac{d\phi}{dx}\right] + \frac{1}{x}\frac{d\phi}{dx} + 4\phi \\
&= \frac{d^2\phi}{dx^2} + \left(4x + \frac{1}{x}\right)\frac{d\phi}{dx} + 8\phi \quad .
\end{aligned}$$

So,

$$\left(\frac{d}{dx} + \frac{1}{x}\right)\left(\frac{d}{dx} + 4x\right) = \frac{d^2}{dx^2} + \left(4x + \frac{1}{x}\right)\frac{d}{dx} + 4 \quad .$$

**16.7 g.** For any sufficiently differentiable function $\phi$,

$$\begin{aligned}
\left(\frac{d}{dx} + x^2\right)\left(\frac{d^2}{dx^2} + \frac{d}{dx}\right)[\phi] &= \left(\frac{d}{dx} + x^2\right)\left[\frac{d^2\phi}{dx^2} + \frac{d\phi}{dx}\right] \\
&= \frac{d}{dx}\left[\frac{d^2\phi}{dx^2} + \frac{d\phi}{dx}\right] + x^2\left[\frac{d^2\phi}{dx^2} + \frac{d\phi}{dx}\right] \\
&= \frac{d^3\phi}{dx^3} + \frac{d^2\phi}{dx^2} + x^2\frac{d^2\phi}{dx^2} + x^2\frac{d\phi}{dx} \\
&= \frac{d^3\phi}{dx^3} + \left(1 + x^2\right)\frac{d^2\phi}{dx^2} + x^2\frac{d\phi}{dx} \quad .
\end{aligned}$$

So,

$$\left(\frac{d}{dx} + x^2\right)\left(\frac{d^2}{dx^2} + \frac{d}{dx}\right) = \frac{d^3}{dx^3} + \left(1 + x^2\right)\frac{d^2}{dx^2} + x^2\frac{d}{dx} \quad .$$

**16.9.** *Verifying the factorization:* For any sufficiently differentiable function $\phi$,

$$\left(\frac{d}{dx} - x\right)\left(\frac{d}{dx} + 2x\right)[\phi] = \left(\frac{d}{dx} - x\right)\left[\frac{d\phi}{dx} + 2x\phi\right]$$

$$= \frac{d}{dx}\left[\frac{d\phi}{dx} + 2x\phi\right] - x\left[\frac{d\phi}{dx} + 2x\phi\right]$$

$$= \frac{d^2\phi}{dx^2} + \left[2\phi + 2x\frac{d\phi}{dx}\right] - x\frac{d\phi}{dx} - 2x^2\phi$$

$$= \frac{d^2\phi}{dx^2} + x\frac{d\phi}{dx} + \left(2 - 2x^2\right)\phi \quad .$$

So,

$$\frac{d^2}{dx^2} + x\frac{d}{dx} + \left(2 - 2x^2\right) = \left(\frac{d}{dx} - x\right)\left(\frac{d}{dx} + 2x\right) \quad .$$

*Finding the solution:* From the factoring above, we know the differential equation can be written as

$$\left(\frac{d}{dx} - x\right)\left(\frac{d}{dx} + 2x\right)[y] = 0 \quad ,$$

and from Theorem 16.6 on page 312 of the text, we know that any solution $y$ to

$$\left(\frac{d}{dx} + 2x\right)[y] = 0$$

also satisfies the previous equation. Rewriting the last equation in the more traditional form

$$\frac{dy}{dx} + 2xy = 0$$

we see that we have a simple first-order linear and separable differential equation. Using the integrating factor

$$\mu = \mu(x) = e^{\int 2x\,dx} = e^{x^2}$$

we have

$$e^{x^2}\left[\frac{dy}{dx} + 2xy\right] = e^{x^2} \cdot 0$$

$$\hookrightarrow \qquad\qquad \frac{d}{dx}\left[e^{x^2}y\right] = 0$$

$$\hookrightarrow \qquad\qquad e^{x^2}y = c$$

$$\hookrightarrow \qquad\qquad y = y(x) = c^{-x^2} \quad .$$