Chapter 15: General Solutions to Homogeneous Linear Differential Equations

15.2 a. Verifying that $\{y_1, y_2\}$ is a fundamental solution set: We have

$$y_1(x) = \cos(2x) \quad \rightarrowtail \quad y_1{}'(x) = -2\sin(2x) \quad \rightarrowtail \quad y_1{}''(x) = -4\cos(2x) \quad ,$$

and

$$y_2(x) = \sin(2x) \quad \rightarrowtail \quad y_2{}'(x) = 2\cos(2x) \quad \rightarrowtail \quad y_2{}''(x) = -4\sin(2x) \quad .$$

Thus,

$$y_1{}'' + 4y_1 = -4\cos(2x) + 4\cos(2x) = 0 \quad ,$$

and

$$y_2{}'' + 4y_2 = -4\sin(2x) + 4\sin(2x) = 0 \quad ,$$

verifying that $\cos(2x)$ and $\sin(2x)$ are solutions to the given differential equation. Also, it should be obvious that neither is a constant multiple of each other. Hence, $\{\cos(2x), \sin(2x)\}$ is a fundamental set of solutions for the given differential equation.

Solving the initial-value problem: Set

$$y(x) = A\cos(2x) + B\sin(2x) \quad . \qquad (\star)$$

Applying the initial conditions and using the above derivatives, we have

$$2 = y(0) = A\cos(2\cdot 0) + B\sin(2\cdot 0) = A\cdot 1 + B\cdot 0 = A \quad ,$$

and

$$6 = y'(0) = -2A\sin(2\cdot 0) + 2B\cos(2\cdot 0) - -2A\cdot 0 + 2B\cdot 1 = 2B \quad .$$

So the solution to the initial-value problem is given by formula $(\star)$ with $A = 2$ and $B = {}^6\!/_2 = 3$; that is,

$$y(x) = 2\cos(2x) + 3\sin(2x) \quad .$$

15.2 c. Verifying that $\{y_1, y_2\}$ is a fundamental solution set: We have

$$y_1(x) = e^{2x} \quad \rightarrowtail \quad y_1{}'(x) = 2e^{2x} \quad \rightarrowtail \quad y_1{}''(x) = 4e^{2x} \quad ,$$

and

$$y_2(x) = e^{-3x} \quad \rightarrowtail \quad y_2{}'(x) = -3e^{-3x} \quad \rightarrowtail \quad y_2{}''(x) = 9e^{-3x} \quad .$$

Thus,

$$y_1{}'' + y_1{}' - 6y_1 = 4e^{2x} + 2e^{2x} - 6e^{2x} = [4+2-6]e^{2x} = 0 \quad ,$$

and

$$y_2{}'' + y_2{}' - 6y_2 = 9e^{3x} - 3e^{3x} - 6e^{3x} = [9-3-6]e^{3x} = 0 \quad ,$$

verifying that e^{2x} and e^{-3x} are solutions to the given differential equation. Also, it should be obvious that neither is a constant multiple of each other. Hence, $\{e^{2x}, e^{-3x}\}$ is a fundamental set of solutions for the given differential equation.

Solving the initial-value problem: Set

$$y(x) = Ae^{2x} + Be^{-3x} \quad . \qquad (\star)$$

Applying the initial conditions and using the above derivatives, we have

$$8 = y(0) = Ae^{2\cdot 0} + Be^{-3\cdot 0} = A\cdot 1 + B\cdot 1 = A + B \quad ,$$

and

$$-9 = y'(0) = 2Ae^{2\cdot 0} - 3Be^{-3\cdot 0} - 2A\cdot 1 - 3B\cdot 1 = 2A - 3B \quad ,$$

giving us the algebraic system

$$A + B = 8 \qquad \text{and} \qquad 2A - 3B = -9 \quad ,$$

which can be solved many ways. For now, we'll just solve the first equation for $B = 8 - A$, plug that into the second equation, obtaining

$$2A - 3(8 - A) = -9 \quad \rightarrowtail \quad 5A = 15 \quad \rightarrowtail \quad A = \frac{15}{5} = 3 \quad ,$$

and then plug that result back into the formula for B. So the solution to the initial-value problem is given by formula ($\star$) with $A = 3$ and $B = 8 - A = 8 - 3 = 5$; that is,

$$y(x) = 3e^{2x} + 4e^{-3x} \quad .$$

15.2 e. Verifying that $\{y_1, y_2\}$ is a fundamental solution set: We have

$$y_1(x) = x^2 \quad \rightarrowtail \quad {y_1}'(x) = 2x \quad \rightarrowtail \quad {y_1}''(x) = 2 \quad ,$$

and

$$y_2(x) = x^3 \quad \rightarrowtail \quad {y_2}'(x) = 3x^2 \quad \rightarrowtail \quad {y_2}''(x) = 6x \quad .$$

Thus,

$$\begin{aligned} x^2{y_1}'' - 4x{y_1}' + 6y_1 &= x^2[2] - 4x\,[2x] + 6\left[x^2\right] \\ &= [2 - 8 + 6]x^2 = 0 \quad , \end{aligned}$$

and

$$\begin{aligned} x^2{y_2}'' - 4x{y_2}' + 6y_2 &= x^2[6x] - 4x\left[3x^2\right] + 6\left[x^3\right] \\ &= [6 - 12 + 6]x^3 = 0 \quad , \end{aligned}$$

verifying that x^2 and x^3 are solutions to the given differential equation. Also, it should be obvious that neither is a constant multiple of each other. Hence, $\{x^2, x^3\}$ is a fundamental set of solutions for the given differential equation.

Solving the initial-value problem: Set

$$y(x) = Ax^2 + Bx^3 \quad . \qquad (\star)$$

Applying the initial conditions and using the above derivatives, we have

$$0 = y(1) = A\cdot 1^2 + B\cdot 2^3 = A + B \quad ,$$

and

$$4 = y'(1) = A\cdot 2\cdot 1 + B\cdot 3\cdot 1^2 = 2A + 3B \quad .$$

So, $$A + B = 0 \quad \text{and} \quad 2A + 3B = 4$$

$$\hookrightarrow \quad B = -A \quad \text{and} \quad 4 = 2A + 3B = 2A + 3(-A) = -A$$

$$\hookrightarrow \quad B = -A = 4 \quad \text{and} \quad A = -4 \quad .$$

So the solution to the initial-value problem is given by formula ($\star$) with $A = -4$ and $B = 4$; that is,
$$y(x) = -4x^2 + 4x^3 \quad .$$

15.2 g. Verifying that $\{y_1, y_2\}$ is a fundamental solution set: We have

$$y_1(x) = x \quad \rightarrowtail \quad y_1{}'(x) = 1 \quad \rightarrowtail \quad y_1{}''(x) = 0 \quad ,$$

and
$$y_2(x) = x \ln|x| \quad \rightarrowtail \quad y_2{}'(x) = \ln|x| + 1 \quad \rightarrowtail \quad y_2{}''(x) = x^{-1} \quad .$$

Thus,
$$\begin{aligned} x^2 y_1{}'' - x y_1{}' + y_1 &= x^2[0] - x\,[1] + [x] \\ &= -x + x = 0 \quad , \end{aligned}$$

and
$$\begin{aligned} x^2 y_2{}'' - x y_2{}' + y_2 &= x^2[x^{-1}] - x\,[\ln|x| + 1] + [x \ln|x|] \\ &= x - x\ln|x| - x + x\ln|x| = 0 \quad , \end{aligned}$$

verifying that x and $x \ln|x|$ are solutions to the given differential equation. Also, it should be obvious that neither is a constant multiple of each other. Hence, $\{x, x\ln|x|\}$ is a fundamental set of solutions for the given differential equation (on $(0, \infty)$).

Solving the initial-value problem: Set
$$y(x) = Ax + Bx \ln|x| \quad . \tag{$\star$}$$

Applying the initial conditions and using the above derivatives, we have
$$5 = y(1) = A[1] + B[1 \ln|1|] = A + B \cdot 0 = A \quad ,$$

and
$$3 = y'(1) = = A[1] + B[\ln|1| + 1] = A + B \cdot 1 = A + B \quad .$$

So, $$A = 5 \quad \text{and} \quad A + B = 3$$

$$\hookrightarrow \quad A = 5 \quad \text{and} \quad B = 3 - A = 3 - 5 = -2 \quad .$$

So the solution to the initial-value problem is given by formula ($\star$) with $A = 5$ and $B = -2$; that is,
$$y(x) = 5x - 2x \ln|x| \quad .$$

15.2 i. Verifying that $\{y_1, y_2\}$ is a fundamental solution set: We have

$$y_1(x) = x^2 - 1 \quad \rightarrowtail \quad y_1{}'(x) = 2x \quad \rightarrowtail \quad y_1{}''(x) = 2 \quad ,$$

and

$$y_2(x) \;=\; x + 1 \quad\rightarrowtail\quad {y_2}'(x) \;=\; 1 \quad\rightarrowtail\quad {y_2}''(x) \;=\; 0 \quad .$$

Thus,

$$\begin{aligned}(x+1)^2 {y_1}'' \;-\; 2(x+1){y_1}' \;+\; 2y_1 \\ &= (x+1)^2\,[2] \;-\; 2(x+1)\,[2x] \;+\; 2\left[x^2-1\right] \\ &= \left[2x^2+4x+2\right] \;-\; \left[4x^2+4x\right] \;+\; \left[2x^2-2\right] \\ &= [2-4+2]x^2 \;+\; [4-4]x \;+\; [2-2] = 0 \quad ,\end{aligned}$$

and

$$\begin{aligned}(x+1)^2 {y_2}'' \;-\; 2(x+1){y_2}' \;+\; 2y_2 \\ &= (x+1)^2\,[0] \;-\; 2(x+1)\,[1] \;+\; 2\,[x+1] \\ &= 2(x+1) \;-\; 2(x+1) \;=\; 0 \quad ,\end{aligned}$$

verifying that $x^2 - 1$ and $x + 1$ are solutions to the given differential equation. Also, it should be obvious that neither is a constant multiple of each other. Hence, $\{x^2 - 1, x + 1\}$ is a fundamental set of solutions for the given differential equation.

Solving the initial-value problem: Set

$$y(x) \;=\; A\left[x^2 - 1\right] \;+\; B\,[x+1] \quad . \tag{$\star$}$$

Applying the initial conditions and using the above derivatives, we have

$$0 \;=\; y(0) \;=\; A[0^2 - 1] \;+\; B[0+1] \;=\; -A \;+\; B \quad ,$$

and

$$4 \;=\; y'(0) \;=\;=\; A[2\cdot 0] \;+\; B[1] \;=\; B \quad .$$

So,

$$-A \;+\; B \;=\; 0 \qquad \text{and} \qquad B \;=\; 4$$

$$\hookrightarrow \qquad A \;=\; B \;=\; 4 \qquad \text{and} \qquad B \;=\; 4 \quad .$$

So the solution to the initial-value problem is given by formula ($\star$) with $A = 4$ and $B = 4$; that is,

$$y(x) \;=\; 4\left[x^2 - 1\right] \;+\; 4\,[x+1] \;=\; 4x^2 \;-\; 4 \;+\; 4x \;+\; 4 \;=\; 4x^2 \;+\; 4x \quad .$$

15.3 a. The equation is

$$ay'' \;+\; by' \;+\; cy \;=\; 0 \qquad \text{with} \quad a \;=\; x^2 \quad , \quad b \;=\; -4x \quad \text{and} \quad c \;=\; 6 \quad .$$

Each coefficient is continuous on $(-\infty, \infty)$, but the first, a is 0 if and only if $x = 0$. So the interval must not contain $x - 0$, and the largest such interval that also contains $x_0 = 1$ is $(0, \infty)$.

15.3 b. In this case,

$$y(x) = c_1 x^2 + c_2 x^3 \quad \text{and} \quad y'(x) = 2c_1 x + 3c_2 x^2 \quad .$$

Applying the initial conditions, we get

$$0 = y(0) = c_1 \cdot 0^2 + c_2 \cdot 0^3 = 0 \quad ,$$

and

$$-4 = y'(0) = 2c_1 \cdot 0 + 3c_2 \cdot 0^2 = 0 \quad .$$

So, no matter what c_1 and c_2 are, $y = c_1 x^2 + c_2 x^3$ and its derivative will always be 0 when $x = 0$, and, hence c_1 and c_2 cannot be chosen so that $y(0)$ or $y'(0)$ is nonzero.

Theorem 15.3 requires that the point x_0 at which initial values are given be in an interval (α, β) over which the coefficients of the differential equation are continuous with the first one (the $a = x^2$, here) never being zero. Hence, the theorem requires that the coefficients be continuous and $a \neq 0$ at the point x_0 at which initial values are given. As noted in the first part of this exercise, while the coefficients are continuous at $x = 0$, the first coefficient is zero at $x = 0$. So Theorem 15.3 does not apply here.

15.5 a. Verifying that $\{y_1, y_2, y_3\}$ is a fundamental solution set: We have

$$y_1(x) = 1 \quad \rightarrowtail \quad y_1{}'(x) = 0 \quad \rightarrowtail \quad y_1{}''(x) = 0 \quad \rightarrowtail \quad y_1{}'''(x) = 0 \quad ,$$

and

$$y_2(x) = \cos(2x) \quad \rightarrowtail \quad y_2{}'(x) = -2\sin(2x)$$

$$\hookrightarrow \qquad y_2{}''(x) = -4\cos(2x) \quad \rightarrowtail \quad y_2{}'''(x) = 8\sin(2x) \quad ,$$

and

$$y_3(x) = \sin(2x) \quad \rightarrowtail \quad y_3{}'(x) = 2\cos(2x)$$

$$\hookrightarrow \qquad y_3{}''(x) = -4\sin(2x) \quad \rightarrowtail \quad y_3{}'''(x) = -8\cos(2x) \quad .$$

Thus,

$$y_1{}''' + 4y_1{}' = 0 + 4 \cdot 0 = 0 \quad ,$$

$$y_2{}''' + 4y_2{}' = 8\sin(2x) + 4[-2\sin(2x)] = [8 - 8]\sin(2x) = 0 \quad ,$$

and

$$y_3{}''' + 4y_3{}' = -8\cos(2x) + 4[2\cos(2x)] = [-8 + 8]\cos(2x) = 0 \quad ,$$

verifying that 1, $\cos(2x)$ and $\sin(2x)$ are solutions to the given differential equation. To confirm that they form a fundamental set of solutions for this third-order equation, we must show that they form a linearly independent set. To do that, first form the corresponding Wronskian,

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1{}' & y_2{}' & y_3{}' \\ y_1{}'' & y_2{}'' & y_3{}'' \end{vmatrix} = \begin{vmatrix} 1 & \cos(2x) & \sin(2x) \\ 0 & -2\sin(2x) & 2\cos(2x) \\ 0 & -4\cos(2x) & -4\sin(2x) \end{vmatrix} \quad .$$

Plugging in a convenient value for x, say $x = {}^{\pi}/_{4}$ so that $2x = {}^{\pi}/_{2}$, we have

$$W\left(\frac{\pi}{4}\right) = \begin{vmatrix} 1 & \cos\left(\frac{\pi}{2}\right) & \sin\left(\frac{\pi}{2}\right) \\ 0 & -2\sin\left(\frac{\pi}{2}\right) & 2\cos\left(\frac{\pi}{2}\right) \\ 0 & -4\cos\left(\frac{\pi}{2}\right) & -4\sin\left(\frac{\pi}{2}\right) \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{vmatrix} = 8 \neq 0 \quad .$$

Since the Wronskian is nonzero at one point, Theorem 15.6 assures us that

$$\{1, \cos(2x)\,, \sin(2x)\}$$

is linearly independent and is a fundamental set of solutions for this differential equation.

Solving the initial-value problem: Set

$$y(x) = A \cdot 1 + B\cos(2x) + C\sin(2x) \quad . \tag{$\star$}$$

Applying the initial conditions and using the above derivatives, we have

$$3 = y(0) = A \cdot 1 + B\cos(2 \cdot 0) + C\sin(2 \cdot 0) = A + B \quad ,$$

$$8 = y'(0) = A \cdot 0 + B[-2\sin(2 \cdot 0)] + C[2\cos(2 \cdot 0) = 2C \quad .$$

and

$$4 = y''(0) = A \cdot 0 + B[-4\cos(2 \cdot 0)] + C[-2\sin(2 \cdot 0)] = -4B \quad .$$

So, the solution to the initial-value problem is ($\star$) with

$$C = \frac{8}{2} = 4 \quad , \quad B = \frac{4}{-4} = -1$$

and

$$A = 3 - B = 3 - (-1) = 4 \quad .$$

That is,

$$y(x) = 4 - \cos(2x) + 4\sin(2x)$$

15.5 c. Verifying that $\{y_1, y_2, y_3, y_4\}$ is a fundamental solution set: We have

$$y_1(x) = \cos(x) \quad \rightarrowtail \quad {y_1}'(x) = -\sin(x)$$

$$\hookrightarrow \quad {y_1}''(x) = -\cos(x) \quad \rightarrowtail \quad {y_1}'''(x) = \sin(x)$$

$$\hookrightarrow \quad {y_1}^{(4)}(x) = \cos(x) \quad ,$$

and

$$y_2(x) = \sin(x) \quad \rightarrowtail \quad {y_2}'(x) = \cos(x)$$

$$\hookrightarrow \quad {y_2}''(x) = -\sin(x)\,{y_2}'''(x) = -\cos(x)$$

$$\hookrightarrow \quad {y_2}^{(4)}(x) = \sin(x) \quad .$$

and

$$y_3(x) = \cosh(x) \rightarrowtail y_3{}'(x) = \sinh(x)$$

$$\hookrightarrow \quad y_3{}''(x) = \cosh(x) \rightarrowtail y_3{}'''(x) = \sinh(x)$$

$$\hookrightarrow \quad y_3{}^{(4)}(x) = \cosh(x) \quad ,$$

and

$$y_4(x) = \sinh(x) \rightarrowtail y_4{}'(x) = \cosh(x)$$

$$\hookrightarrow \quad y_4{}''(x) = \sinh(x) \rightarrowtail y_4{}'''(x) = \cosh(x)$$

$$\hookrightarrow \quad y_4{}^{(4)}(x) = \sinh(x) \quad .$$

Thus,

$$y_1{}^{(4)} - y_1 = \cos(x) - \cos(x) = 0 \quad ,$$

$$y_2{}^{(4)} - y_2 = \sin(x) - \sin(x) = 0 \quad ,$$

$$y_3{}^{(4)} - y_3 = \cosh(x) - \cosh(x) = 0 \quad ,$$

and

$$y_4{}^{(4)} - y_4 = \sinh(x) - \sinh(x) = 0 \quad ,$$

verifying that these four functions are solutions to the given differential equation. To confirm that they form a fundamental set of solutions for this fourth-order equation, we must show that they form a linearly independent set. To do that, first form the corresponding Wronskian,

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1{}' & y_2{}' & y_3{}' & y_4{}' \\ y_1{}'' & y_2{}'' & y_3{}'' & y_4{}'' \\ y_1{}''' & y_2{}''' & y_3{}''' & y_4{}''' \end{vmatrix} = \begin{vmatrix} \cos(x) & \sin(x) & \cosh(x) & \sinh(x) \\ -\sin(x) & \cos(x) & \sinh(x) & \cosh(x) \\ -\cos(x) & -\sin(x) & \cosh(x) & \sinh(x) \\ \sin(x) & -\cos(x) & \sinh(x) & \cosh(x) \end{vmatrix} \quad .$$

Plugging in a convenient value for x, say $x = 0$, we have

$$W(0) = \begin{vmatrix} \cos(0) & \sin(0) & \cosh(0) & \sinh(0) \\ -\sin(0) & \cos(0) & \sinh(0) & \cosh(0) \\ -\cos(0) & -\sin(0) & \cosh(0) & \sinh(0) \\ \sin(0) & -\cos(0) & \sinh(0) & \cosh(0) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{vmatrix}$$

$$= 1 \cdot \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 1 \end{vmatrix} = 1 \cdot 2 + 1 \cdot 2 \neq 0 \quad .$$

Since the Wronskian is nonzero at one point, Theorem 15.6 assures us that

$$\{\cos(x), \sin(x), \cosh(x), \sinh(x)\}$$

is a fundamental set of solutions for this differential equation.

Solving the initial-value problem: Set

$$y(x) = A\cos(x) + B\sin(x) + C\cosh(x) + \sinh(x) \quad . \qquad (\star)$$

Applying the initial conditions and using the above derivatives, we have

$$0 = y(0) = A\cos(0) + B\sin(0) + C\cosh(0) + \sinh(0) = A + C \quad ,$$

$$4 = y'(0) = -A\sin(0) + B\cos(0) + C\sinh(0) + D\cosh(0) = B + D \quad ,$$

$$0 = y''(0) = -A\cos(0) - B\sin(0) + C\cosh(0) + \sinh(0) = -A + C$$

and

$$0 = y'''(0) = A\sin(0) - B\cos(0) + C\sinh(0) + D\cosh(0) = -B + D \quad .$$

Solving for A and C is easy:

$$0 = A + C \qquad \text{and} \qquad 0 = -A + C$$

$$\hookrightarrow \quad C = -A \qquad \text{and} \qquad 0 = -A + C = -A - A = -2A$$

$$\hookrightarrow \quad C = -A = -\frac{0}{-2} = 0 \qquad \text{and} \qquad A = \frac{0}{-2} = 0 \quad .$$

For B and D:

$$4 = B + D \qquad \text{and} \qquad 0 = -B + D$$

$$\hookrightarrow \quad 4 = B + D \qquad \text{and} \qquad D = B$$

$$\hookrightarrow \quad 4 = B + B = 2B \qquad \text{and} \qquad D = B$$

$$\hookrightarrow \quad B = \frac{4}{2} = 2 \qquad \text{and} \qquad D = B = 2 \quad .$$

Plugging these values into $(\star)$ then gives the solution,

$$\begin{aligned} y(x) &= 0\cos(x) + 2\sin(x) + 0\cosh(x) + 2\sinh(x) \\ &= 2\sin(x) + 2\sinh(x) \quad . \end{aligned}$$

15.6 a. Setting $\quad y = e^{rx} \rightarrowtail y' = re^{rx} \rightarrowtail y'' = r^2e^{rx} \quad ,$

we have

$$0 = y'' - 4y = r^2e^{rx} - 4e^{rx} = \left[r^2 - 4\right]e^{rx} \quad .$$

Since $e^{rx} \neq 0$ for all x, it follows that

$$0 = r^2 - 4 \quad .$$

But
$$0 = r^2 - 4 \quad\rightarrowtail\quad r^2 = 4 \quad\rightarrowtail\quad r = \pm\sqrt{4} = \pm 2 \quad .$$
So e^{rx} is a solution to the differential equation if $r = 2$ or $r = -2$. That is, $\{e^{2x}, e^{-2x}\}$ is a pair of solutions to the given second-order homogeneous linear differential equation. Clearly, neither is a constant multiple of each other. So, in fact, this is a fundamental set of solutions, and
$$y(x) = c_1 e^{2x} + c_2 e^{-2x} \tag{$\star$}$$
is a general solution to the differential equation.

For the initial-value problem: We have
$$y(x) = c_1 e^{2x} + c_2 e^{-2x} \quad\rightarrowtail\quad y'(x) = 2c_1 e^{2x} - 2c_2 e^{-2x} \quad .$$
Applying the initial conditions:
$$1 = y(0) = c_1 e^{2\cdot 0} + c_2 e^{-2\cdot 0} = c_1 + c_2 \quad ,$$
and
$$0 = y'(0) = 2c_1 e^{2\cdot 0} - 2c_2 e^{-2\cdot 0} = 2c_1 - 2c_2 \quad .$$
So,
$$1 = c_1 + c_2 \qquad \text{and} \qquad 0 = 2c_1 - 2c_2$$
$$\hookrightarrow \qquad 1 = c_1 + c_2 \qquad \text{and} \qquad c_2 = c_1$$
$$\hookrightarrow \qquad 1 = c_1 + c_2 = c_1 + c_1 = 2c_1 \qquad \text{and} \qquad c_2 = c_1$$
$$\hookrightarrow \qquad c_1 = \frac{1}{2} \qquad \text{and} \qquad c_2 = c_1 = \frac{1}{2} \quad .$$
Plugging these values into ($\star$) then gives the solution,
$$y(x) = \frac{1}{2}e^{2x} + \frac{1}{2}e^{-2x} \quad .$$

15.6 c. Setting $\quad y = e^{rx} \quad\rightarrowtail\quad y' = re^{rx} \quad\rightarrowtail\quad y'' = r^2 e^{rx} \quad ,$

we have
$$0 = y'' - 10y' + 9y = r^2 e^{rx} - 10re^{rx} + 9e^{rx} = \left[r^2 - 10r + 9\right] e^{rx} \quad .$$
Dividing out e^{rx} and factoring[1] gives
$$0 = r^2 - 10r + 9 = (r-1)(r-9)$$
$$\hookrightarrow \qquad r = 1 \qquad \text{or} \qquad r = 9 \quad .$$
So e^{rx} is a solution to the differential equation if $r = 1$ or $r = 9$. That is, $\{e^{1x}, e^{9x}\}$ is a pair of solutions to the given second-order homogeneous linear differential equation.

[1] Or you can use the quadratic formula to find r.

Clearly, neither is a constant multiple of each other. So, in fact, this is a fundamental set of solutions, and

$$y(x) = c_1 e^x + c_2 e^{9x} \tag{$\star$}$$

is a general solution to the differential equation.

For the initial-value problem: We have

$$y(x) = c_1 e^x + c_2 e^{9x} \quad \rightarrowtail \quad y'(x) = c_1 e^{2x} + 9c_2 e^{9x} \quad .$$

Applying the initial conditions:

$$8 = y(0) = c_1 e^0 + c_2 e^{\cdot 0} = c_1 + c_2 \quad ,$$

and

$$-24 = y'(0) = c_1 e^0 + 9c_2 e^{9\cdot 0} = c_1 + 9c_2 \quad .$$

So,

$$8 = c_1 + c_2 \qquad \text{and} \qquad -24 = c_1 + 9c_2$$

$$\hookrightarrow \quad c_2 = 8 - c_1 \qquad \text{and} \qquad -24 = c_1 + 9[8 - c_1] = 72 - 8c_1$$

$$\hookrightarrow \quad c_2 = 8 - c_1 \qquad \text{and} \qquad c_1 = \frac{72+24}{8} = 12$$

$$\hookrightarrow \quad c_2 = 8 - 12 = -4 \qquad \text{and} \qquad c_1 = 12 \quad .$$

Plugging these values into ($\star$) then gives the solution,

$$y(x) = 12e^x - 4e^{9x} \quad .$$

15.7 a. Setting $\quad y = e^{rx} \quad \rightarrowtail \quad y' = re^{rx} \quad \rightarrowtail \quad y'' = r^2 e^{rx} \quad \rightarrowtail \quad y''' = r^3 e^{rx} \quad ,$

we have

$$0 = y''' - 9y' = r^3 e^{rx} - 9re^{rx} = \left[r^3 - 9r\right] e^{rx} \quad .$$

Dividing out e^{rx} and factoring gives

$$0 = r^3 - 9r = r\left(r^2 - 9\right) = r(r-3)(r+3)$$

$$\hookrightarrow \quad r = 0 \qquad \text{or} \qquad r = 3 \qquad \text{or} \qquad 4 = -3 \quad .$$

So e^{rx} is a solution to the differential equation if $r = 0$, $r = 3$ or $r = -3$. That is, $\{e^{0x} = 1, e^{3x}, e^{-3x}\}$ is a set of three solutions to the given third-order homogeneous linear differential equation. The corresponding Wronskian is

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1{}' & y_2{}' & y_3{}' \\ y_1{}'' & y_2{}'' & y_3{}'' \end{vmatrix} = \begin{vmatrix} 1 & e^{3x} & e^{-3x} \\ 0 & 3e^{3x} & -3e^{-3x} \\ 0 & 9e^{3x} & 9e^{-3x} \end{vmatrix} \quad .$$

Plugging in a convenient value for x, say $x = 0$ so that $e^{\pm 3x} = 1$, we have

$$W(0) = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 3 & -3 \\ 0 & 9 & 9 \end{vmatrix} = 1 \cdot [(3)(9) - (-3)(9)] = 36 \neq 0 \quad .$$

Since the Wronskian is nonzero at one point, Theorem 15.6 assures us that $\{1, e^{3x}, e^{-3x}\}$ is a fundamental set of solutions for this differential equation, and the corresponding general solution is

$$y(x) = c_1 + c_2 e^{3x} + c_3 e^{-3x} \quad .$$

15.9 a i. If $\{y_1, y_2\}$ was a linearly dependent pair on the entire real line, then there would be a single constant c such that $y_2(x) = c y_1(x)$ at every point x where $y_1(x) \neq 0$, which in turn, means that there would be a single constant c such that, whenever $y(x) \neq 0$,

$$\frac{y_2(x)}{y_1(x)} = c$$

But then

$$c = \frac{y_2(x)}{y_1(x)} = \left\{ \begin{array}{ll} \dfrac{x^2}{-x^2} & \text{if} \quad x < 0 \\ \dfrac{3x^2}{x^2} & \text{if} \quad 0 < x \end{array} \right\} = \begin{cases} -1 & \text{if} \quad x < 0 \\ 3 & \text{if} \quad 0 < x \end{cases} \quad .$$

So there isn't a single such constant c. Hence, $\{y_1, y_2\}$ is not linearly dependent on the entire real line.

15.9 a ii. If $x < 0$,

$$\begin{aligned} W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} &= \begin{vmatrix} -x^2 & x^2 \\ -2x & 2x \end{vmatrix} \\ &= (-x^2)(2x) - (x^2)(-2x) = -2x^3 + 2x^3 = 0 \quad . \end{aligned}$$

If $0 < x$,

$$\begin{aligned} W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} &= \begin{vmatrix} x^2 & 3x^2 \\ 2x & 6x \end{vmatrix} \\ &= (x^2)(6x) - (3x^2)(2x) = 6x^3 - 6x^3 = 0 \quad . \end{aligned}$$

If $x = 0$, then the derivatives would have to be computed using the basic limit definition (they do exist). However, no matter what they end up being,

$$W(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ y_1'(x) & y_2'(x) \end{vmatrix} = 0 \quad .$$

15.9 b. For that theorem to apply, $\{y_1, y_2\}$ must be a pair of solutions to some differential equation of the form

$$ay'' + by' + cy = 0$$

where a, b and c are continuous functions on $(-\infty, \infty)$ and with a never being zero on that interval. Obviously, $\{y_1, y_2\}$ is not a pair of solutions to any such differential equation.

15.9 c. From the first part of this exercise, we know

$$y_2(x) = (-1)y_1(x) \qquad \text{for} \quad x < 0 \quad ,$$

telling us that $\{y_1, y_2\}$ is linearly dependent on $(-\infty, 0)$ or any subinterval of $(-\infty, 0)$.

Also, we have

$$y_2(x) = (3)y_1(x) \qquad \text{for} \quad x < 0 \quad ,$$

telling us that $\{y_1, y_2\}$ is linearly dependent on $(0, \infty)$ or any subinterval of $(0, \infty)$.