Chapter 14: Higher-Order Linear Equation and the Reduction of Order Method

14.1 a. The equation is in the form

$$ay'' + by' + cy = g$$

where a, b, c and g are the functions of x

$$a = 1 \quad , \quad b = x^2 \quad , \quad c = -4 \quad \text{and} \quad g = x^3 \quad .$$

which is the standard form for a second-order linear differential equation. So the given equation is second order and linear. Since the forcing function g is x^3 and not 0, the equation is not homogeneous.

14.1 c. The equation can be rewritten in the form

$$ay'' + by' + cy = g$$

where a, b, c and g are the functions of x

$$a = 1 \quad , \quad b = x^2 \quad , \quad c = -4 \quad \text{and} \quad g = 0 \quad .$$

which is the standard form for a second-order linear differential equation. So the given equation is second order and linear. Since the forcing function g is x^3 and not 0, the equation is homogeneous.

14.1 e. The equation is in the form

$$ay' + by = g$$

where a, b, c and g are the functions of x

$$a = x \quad , \quad b = 3 \quad \text{and} \quad g = e^{2x} \quad .$$

which is the standard form for a first-order linear differential equation. So the given equation is first order and linear. Since the forcing function g is e^{2x} and not 0, the equation is nonhomogeneous.

14.1 g. The highest order derivative of y in the equation is y''; so the equation is second order. Because of the yy'' and $(y'')^3$ terms, the equation cannot be put in the form

$$ay'' + by' + cy = g$$

where a, b, c and g are functions of x only. So the equation is not linear.

14.1 i. The highest order derivative of y in the equation is $y^{(iv)}$; so the equation is forth order. By adding 25 to both sides of the equation we get

$$y^{(iv)} + 0y''' + 6y'' + 3y' - 83y = 25 \quad .$$

Since each term on the left is y or a derivative of y multiplied by a function of x (constants in this case), and the right side is a nonzero function of x only (again, just a constant), the equation is linear and nonhomogeneous.

14.1 k. The highest order derivative of y in the equation is y'''; so the equation is third order. Rewriting the equation as

$$y''' + 0y'' + 3y' + x^2 y = 0 \quad ,$$

we see that the equation is linear and homogeneous.

14.2 a. Letting $y = y_1 = e^{2x}$, we have

$$y = e^{2x} \quad \rightarrowtail \quad y' = 2e^{2x} \quad \rightarrowtail \quad y'' = 4e^{2x} \quad .$$

Thus,

$$\begin{aligned} y'' - 5y' + 6y &= 4e^{2x} - 5 \cdot 2e^{2x} + 6e^{2x} \\ &= [4 - 10 + 6]e^{2x} = 0 \cdot e^{2x} = 0 \quad , \end{aligned}$$

showing that $y_1 = e^{2x}$ is one solution to the differential equation.

To find the general solution, now let $y = y_1 u = e^{2x} u$ where $u = u(x)$ is a yet unknown function of x. Computing the derivatives, we have

$$y' = \left(e^{2x} u\right)' = 2e^{2x} u + e^{2x} u'$$

and

$$\begin{aligned} y'' = (y')' &= \left(2e^{2x} u + e^{2x} u'\right)' \\ &= \left(2e^{2x} u\right)' + \left(e^{2x} u'\right)' \\ &= 4e^{2x} u + 2e^{2x} u' + 2e^{2x} u' + e^{2x} u'' \\ &= 4e^{2x} u + 4e^{2x} u' + e^{2x} u'' \quad . \end{aligned}$$

So,

$$\begin{aligned} 0 &= y'' - 5y' + 6y \\ &= \left[4e^{2x} u + 4e^{2x} u' + e^{2x} u''\right] - 5\left[2e^{2x} u + e^{2x} u'\right] + 6\left[e^{2x} u\right] \\ &= e^{2x} u'' + [4 - 5 \cdot 1]\, e^{2x} u' + [4 - 5 \cdot 2 + 6]\, e^{2x} u \\ &= e^{2x}\left[u'' - u' + 0u\right] \quad , \end{aligned}$$

which reduces to $u'' - u' = 0$. Setting $v = u'$, we have

$$v' - v = 0 \quad \rightarrowtail \quad v' = v$$

a simple separable equation with constant solution $v = 0$. For the other solutions, we divide by v and continue:

$$\frac{1}{v}\frac{dv}{dx} = 1 \quad \rightarrowtail \quad \int \frac{1}{v}\frac{dv}{dx}\,dx = \int 1\,dx$$

$$\hookrightarrow \qquad \ln|v| = x + c_1 \quad \rightarrowtail \quad v = Ae^x \quad .$$

Since this last equation reduces to the constant solution $v = 0$ if $A = 0$, it describes all possible formulas for v. And since $v = u'$,

$$u = \int u'\,dx = \int v\,dx = \int Ae^x\,dx = Ae^x + B \quad .$$

Hence, the general solution to the original differential equation is

$$y = y_1 u = e^{2x}\left[Ae^x + B\right] = Ae^{3x} + Be^x \quad .$$

14.2 c. Letting $y(x) = y_1 = x^3$, we have

$$y = x^3 \quad \rightarrowtail \quad y' = 3x^2 \quad \rightarrowtail \quad y'' = 6x \quad .$$

Thus,

$$\begin{aligned} x^2y'' - 6xy' + 12y &= x^2[6x] - 6x\left[3x^2\right] + 12\left[x^3\right] \\ &= 6x^3 - 18x^3 + 12x^3 = 0x^3 = 0 \quad , \end{aligned}$$

showing that $y_1 = x^3$ is one solution to the differential equation.

To find the general solution, now let $y = y_1u = x^3u$ where $u = u(x)$ is a yet unknown function of x. Computing the derivatives, we have

$$y' = \left(x^3u\right)' = 3x^2u + x^3u'$$

and

$$\begin{aligned} y'' = (y')' &= \left(3x^2u + x^3u'\right)' \\ &= \left(3x^2u\right)' + \left(x^3u'\right)' \\ &= 6xu + 3x^2u' + 3x^2u' + x^3u'' \\ &= 6xu + 6x^2u' + x^3u'' \quad . \end{aligned}$$

So,

$$\begin{aligned} 0 &= x^2y'' - 6xy' + 12y \\ &= x^2\left[6xu + 6x^2u' + x^3u''\right] - 6x\left[3x^2u + x^3u'\right] + 12\left[x^3u\right] \\ &= 6x^3u + 6x^4u' + x^5u'' - 18x^3u - 6x^4u' + 12x^3u \\ &= x^5u'' + \left[6x^4 - 6x^4\right]u' + \left[6x^3 - 18x^3 + 12x^3\right]u \\ &= x^5u'' + 0u' + 0u \quad , \end{aligned}$$

which reduces to $u'' = 0$. Setting $v = u'$, we have

$$v' = 0 \quad \rightarrowtail \quad u' = v = c_1$$

$$\hookrightarrow \qquad u = \int u'\,dx = \int c_1\,dx = c_1x + c_2 \quad .$$

Hence, the general solution to the original differential equation is

$$y = y_1u = x^3[c_1x + c_2] = c_1x^4 + c_2x^3 \quad .$$

14.2 e. Letting $y = y_1 = \sqrt{x}$, we have

$$y = x^{1/2} \quad \rightarrowtail \quad y' = \frac{1}{2}x^{-1/2} \quad \rightarrowtail \quad y'' = -\frac{1}{4}x^{-3/2} \quad .$$

Thus,

$$4x^2y'' + y = 4x^2\left[-\frac{1}{4}x^{-3/2}\right] + x^{1/2} = -x^{1/2} + x^{1/2} = 0 \quad ,$$

showing that $y_1 = \sqrt{x}$ is one solution to the differential equation.

Now, letting the general solution be $y = y_1 u = x^{1/2} u$, we have

$$y' = \left(x^{1/2} u\right)' = \frac{1}{2}x^{-1/2} u + x^{1/2} u'$$

and

$$\begin{aligned} y'' &= \left(\frac{1}{2}x^{-1/2} u + x^{1/2} u'\right)' \\ &= -\frac{1}{4}x^{-3/2} u + \frac{1}{2}x^{-1/2} u' + \frac{1}{2}x^{-1/2} u' + x^{1/2} u'' \\ &= -\frac{1}{4}x^{-3/2} u + x^{-1/2} u' + x^{1/2} u'' \quad . \end{aligned}$$

So,

$$\begin{aligned} 0 &= 4x^2y'' + y \\ &= 4x^2\left[-\frac{1}{4}x^{-3/2} u + x^{-1/2} u' + x^{1/2} u''\right] + \left[x^{1/2} u\right] \\ &= -x^{1/2} u + 4x^{3/2} u' + 4x^{5/2} u'' + x^{1/2} u \\ &= 4x^{5/2} u'' + 4x^{3/2} u' + 0u \\ &= 4x^{3/2}\left[xu'' + u'\right] \quad , \end{aligned}$$

which reduces to $xu'' + u' = 0$. Setting $v = u'$, we have

$$xv' + v = 0 \quad .$$

A simple equation both linear and separable. For variety, let's solve this as a linear equation:

$$xv' + v = 0 \quad \rightarrowtail \quad \frac{dv}{dx} + \frac{1}{x}v = 0$$

$$\hookrightarrow \quad \mu = e^{\int (1/x)\,dx} = e^{\ln|x|} = x \qquad \text{(integrating factor on } x > 0\text{)}$$

$$\hookrightarrow \quad x\left[v' + \frac{1}{x}v = 0\right] \quad \rightarrowtail \quad \frac{d}{dx}[xv] = 0 \quad \rightarrowtail \quad xv = c_1$$

$$\hookrightarrow \quad u' = v = \frac{c_1}{x}$$

$$\hookrightarrow \quad u = \int u'\,dx = \int \frac{c_1}{x}\,dx = c_1 \ln|x| + c_2 \quad .$$

Hence, the general solution to the original differential equation is

$$y = y_1 u = \sqrt{x}\,[c_1 \ln|x| + c_2] = c_1\sqrt{x}\ln|x| + c_2\sqrt{x} \quad .$$

14.2 g. Letting $y = y_1 = e^{-x}$, we have

$$y = e^{-x} \quad \rightarrowtail \quad y' = -e^{-x} \quad \rightarrowtail \quad y'' = e^{-x} \quad .$$

Thus,

$$(x+1)y'' + xy' - y = (x+1)e^{-x} - xe^{-x} - e^{-x}$$
$$= [x+1-x-1]e^{-x} = 0e^{-x} = 0 \quad ,$$

showing that $y_1 = e^{-x}$ is one solution to the differential equation.

For the general solution, let $y = y_1 u = e^{-x} u$. Computing the derivatives, we have

$$y' = \left(e^{-x}u\right)' = -e^{-x}u + e^{-x}u'$$

and

$$\begin{aligned} y'' &= \left(-e^{-x}u + e^{-x}u'\right)' \\ &= e^{-x}u - e^{-x}u' - e^{-x}u' + e^{-x}u'' \\ &= e^{-x}u - 2e^{-x}u' + e^{-x}u'' \quad . \end{aligned}$$

So,

$$\begin{aligned} 0 &= (x+1)y'' + xy' - y \\ &= (x+1)\left[e^{-x}u - 2e^{-x}u' + e^{-x}u''\right] + x\left[-e^{-x}u + e^{-x}u'\right] - \left[e^{-x}u\right] \\ &= \Big([x+1]u'' + [-2x-2+x]\,u' + [x+1-x-1]\,u\Big)e^{-x} \\ &= \Big([x+1]u'' - [x+2]u' + 0u\Big)e^{-x} \quad . \end{aligned}$$

Setting $v = u'$, this reduces to

$$[x+1]v' - [x+2]v = 0 \quad \rightarrowtail \quad \frac{dv}{dx} = \frac{x+2}{x+1}v \quad ,$$

a separable equation with constant solution $v = 0$. For the other solutions:

$$\frac{dv}{dx} = \frac{x+2}{x+1}v \quad \rightarrowtail \quad \frac{1}{v}\frac{dv}{dx} = \frac{x+2}{x+1} = 1 + \frac{1}{x+1}$$

$$\hookrightarrow \qquad \int \frac{1}{v}\frac{dv}{dx}\,dx = \int \left[1 + \frac{1}{x+1}\right] dx$$

$$\hookrightarrow \qquad \ln|v| = x + \ln|x+1| + c_1$$

$$\hookrightarrow \qquad v = \pm e^{x+\ln|x+1|+c_1} = A(x+1)e^x \quad .$$

Since this last equation reduces to the constant solution $v = 0$ if $A = 0$, it describes all possible formulas for $v = u'$. Then, using integration by parts

$$\begin{aligned} u = \int v\,dx &= \int A(x+1)e^x\,dx \\ &= A\left[(x+1)e^x - \int e^x\,dx\right] \\ &= A\left[xe^x + e^x - e^x + c_2\right] = Axe^x + B \quad . \end{aligned}$$

Hence, the general solution to the original differential equation is

$$y = y_1 u = e^{-x}\left[Axe^x + B\right] = Ax + Be^{-x} \quad .$$

14.2 i. Letting $y = y_1 = \sin(x)$, we have

$$y = \sin(x) \quad \rightarrowtail \quad y' = \cos(x) \quad \rightarrowtail \quad y'' = -\sin(x) \quad .$$

Thus,

$$y'' + y = -\sin(x) + \sin(x) = 0 \quad ,$$

showing that $y_1 = \sin(x)$ is one solution to the differential equation.

For the general solution, let $y = y_1 u = \sin(x)u$. Computing the derivatives,

$$y' = (\sin(x)u)' = \cos(x)u + \sin(x)u'$$

and

$$\begin{aligned} y'' &= \big(\cos(x)u + \sin(x)u'\big)' \\ &= -\sin(x)u + \cos(x)u' + \cos(x)u' + \sin(x)u'' \\ &= -\sin(x)u + 2\cos(x)u' + \sin(x)u'' \quad . \end{aligned}$$

So,

$$\begin{aligned} 0 &= y'' + y \\ &= -\sin(x)u + 2\cos(x)u' + \sin(x)u'' + \sin(x)u \\ &= \sin(x)u'' + 2\cos(x)u' + 0u \quad . \end{aligned}$$

Setting $v = u'$, this reduces to

$$\sin(x)v' + 2\cos(x)v = 0 \quad \rightarrowtail \quad \frac{dv}{dx} = -2\frac{\cos(x)}{\sin(x)}v$$

a separable equation with constant solution $v = 0$. For the other solutions, we divide by v and continue:

$$\frac{1}{v}\frac{dv}{dx} = -2\frac{\cos(x)}{\sin(x)} \quad \rightarrowtail \quad \int \frac{1}{v}\frac{dv}{dx}\,dx = -2\int \frac{\cos(x)}{\sin(x)}\,dx$$

$$\hookrightarrow \qquad \ln|v| = -2\ln|\sin(x)| + c_1$$

$$\hookrightarrow \qquad u' = v = \pm e^{-2\ln|\sin(x)|+c_1} = C\sin^{-2}(x) \quad .$$

Since this last equation reduces to the constant solution $v = 0$ if $A = 0$, it describes all possible formulas for $v = u'$. Then, after recalling that $\sin^{-2}(x)$ is the derivative of $-\cot(x)$, we have

$$\begin{aligned} u &= \int C\sin^{-2}(x)\,dx \\ &= -C\cot(x) + B = A\frac{\cos(x)}{\sin(x)} + B \quad . \end{aligned}$$

Hence, the general solution to the original differential equation is

$$y = y_1 u = \sin(x)\left[A\frac{\cos(x)}{\sin(x)} + c_2\right] = A\cos(x) + B\sin(x) \quad .$$

14.2 k. Letting $y = y_1 = \sin(x)$, we have

$$y = \sin(x) \quad \rightarrowtail \quad y' = \cos(x) \quad \rightarrowtail \quad y'' = -\sin(x) \quad .$$

Thus,

$$\begin{aligned}
\sin^2(x)y'' \;-\; 2\cos(x)\sin(x)y' \;+\; \left(1+\cos^2(x)\right)y \\
&= \left[-\sin^2(x) - 2\cos^2(x) + 1 + \cos^2(x)\right]\sin(x) \\
&= \left[1 - (\sin^2(x) + \cos^2(x))\right]\sin(x) \;=\; [1-1]\sin(x) \;=\; 0 \quad ,
\end{aligned}$$

showing that $y_1 = \sin(x)$ is one solution to the differential equation.

For the general solution, let $y = y_1 u = \sin(x)u$. Computing the derivatives, we have

$$y' \;=\; (\sin(x)u)' \;=\; \cos(x)u \;+\; \sin(x)u'$$

and

$$\begin{aligned}
y'' &= \left(\cos(x)u \;+\; \sin(x)u'\right)' \\
&= -\sin(x)u \;+\; \cos(x)u' \;+\; \cos(x)u' \;+\; \sin(x)u'' \\
&= -\sin(x)u \;+\; 2\cos(x)u' \;+\; \sin(x)u'' \quad .
\end{aligned}$$

So,

$$\begin{aligned}
0 &= \sin^2(x)y'' \;-\; 2\cos(x)\sin(x)y' \;+\; \left(1+\cos^2(x)\right)y \\
&= \sin^2(x)\left[-\sin(x)u + 2\cos(x)u' \;+\; \sin(x)u''\right] \\
&\qquad -\; 2\cos(x)\sin(x)\left[\cos(x)u + \sin(x)u'\right] \;+\; \left(1+\cos^2(x)\right)[\sin(x)u] \\
&= \sin^3(x)u'' \;+\; \left[2\cos(x)\sin^2(x) - 2\cos(x)\sin^2(x)\right]u' \\
&\qquad +\; \left[-\sin^3(x) - 2\cos^2(x)\sin(x) + \left(1+\cos^2(x)\right)\sin(x)\right]u \\
&= \sin(x)\left(\sin^2(x)u'' + 0u' + \left[1 - \sin^2(x) + \cos^2(x)\right]u\right) \\
&= \sin(x)\left(\sin^2(x)u'' + 0u' + 0u\right) \quad ,
\end{aligned}$$

which reduces to $u'' = 0$. Setting $v = u'$ would be silly; just integrate:

$$u' \;=\; \int u''\,dx \;=\; \int 0\,dx \;=\; c_1$$

$$\hookrightarrow \qquad u \;=\; \int u'\,dx \;=\; \int c_1\,dx \;=\; c_1x \;+\; c_2 \quad .$$

Thus, the general solution to the original differential equation is

$$y \;=\; y_1u \;=\; \sin(x)\,[c_1x + c_2] \;=\; c_1x\sin(x) \;+\; c_2\sin(x) \quad .$$

14.2 m. Letting $y = y_1 = \sin(\ln|x|)$, we have

$$y \;=\; \sin(\ln|x|) \quad \rightarrowtail \quad y' \;=\; \cos(\ln|x|)x^{-1}$$

$$\hookrightarrow \qquad y'' \;=\; -\sin(\ln|x|)x^{-2} \;-\; \cos(\ln|x|)x^{-2} \quad .$$

Thus,

$$\begin{aligned} x^2y'' + xy' + y &= x^2\left[-\sin(\ln|x|)x^{-2} - \cos(\ln|x|)x^{-2}\right] \\ &\quad + x\left[\cos(\ln|x|)\right] + \sin(\ln|x|) \\ &= -\sin(\ln|x|) - \cos(\ln|x|) + \cos(\ln|x|) + \sin(\ln|x|) \\ &= 0 \quad , \end{aligned}$$

showing that $y_1 = \sin(\ln|x|)$ is one solution to the differential equation.

For the general solution, let $y = y_1 u = \sin(\ln|x|)u$. Computing the derivatives, we have

$$y' = (\sin(\ln|x|)u)' = \cos(\ln|x|)x^{-1}u + \sin(\ln|x|)u'$$

and

$$\begin{aligned} y'' = (y')' &= \left(\cos(\ln|x|)x^{-1}u + \sin(\ln|x|)u'\right)' \\ &= \left[-\sin(\ln|x|)x^{-2} - \cos(\ln|x|)x^{-2}\right]u \\ &\quad + \cos(\ln|x|)x^{-1}u' + \cos(\ln|x|)x^{-1}u' + \sin(\ln|x|)u'' \\ &= -x^{-2}[\sin(\ln|x|) + \cos(\ln|x|)]u \\ &\quad + 2\cos(\ln|x|)x^{-1}u' + \sin(\ln|x|)u'' \quad . \end{aligned}$$

So,

$$\begin{aligned} 0 &= x^2y'' + xy' + y \\ &= x^2\left[x^{-2}[-\sin(\ln|x|) - \cos(\ln|x|)]u + 2\cos(\ln|x|)x^{-1}u' + \sin(\ln|x|)u''\right] \\ &\quad + x\left[\cos(\ln|x|)x^{-1}u + \sin(\ln|x|)u'\right] + \sin(\ln|x|)u \\ &= x^2\sin(\ln|x|)u'' + x\left[2\cos(\ln|x|) + \sin(\ln|x|)\right]u' \\ &\quad + \left[-\sin(\ln|x|) - \cos(\ln|x|) + \cos(\ln|x|) + \sin(\ln|x|)\right]u \\ &= x\left(x\sin(\ln|x|)u'' + \left[2\cos(\ln|x|) + \sin(\ln|x|)\right]u'\right) + 0u \quad . \end{aligned}$$

Setting $v = u'$, this reduces to

$$x\sin(\ln|x|)v' + \left[2\cos(\ln|x|) + \sin(\ln|x|)\right]v = 0$$

$$\hookrightarrow \qquad \frac{dv}{dx} = \left[-2\frac{\cos(\ln|x|)}{x\sin(\ln|x|)} - \frac{1}{x}\right]v$$

a separable equation with constant solution $v = 0$. For the other solutions, we divide by v, integrate, and continue:

$$\int \frac{1}{v}\frac{dv}{dx}\,dx = \int \left[-2\frac{\cos(\ln|x|)}{x\sin(\ln|x|)} - \frac{1}{x}\right]dx$$

$$\hookrightarrow \qquad \ln|v| = -2\ln|\sin(\ln|x|)| - \ln|x| + c_1$$

$$\hookrightarrow \qquad v = Ax^{-1}\sin^{-2}(\ln|x|) \quad .$$

Since this last equation reduces to the constant solution $v = 0$ if $A = 0$, it describes all possible formulas for $v = u'$. Integrating (use a substitution $s = \ln|x|$ and then recall that $\sin^{-2}(s)$ is the derivative of $-\cot(s)$), we get

$$u = \int v\,dx = \int Ax^{-1}\sin^{-2}(\ln|x|)\,dx$$
$$= -A\cot(\ln|x|) + c_2 = C\frac{\cos(\ln|x|)}{\sin(\ln|x|)} + B \quad .$$

Hence, the general solution to the original differential equation is

$$y = y_1 u = \sin(\ln|x|)\left[C\frac{\cos(\ln|x|)}{\sin(\ln|x|)} + B\right] = C\cos(\ln|x|) + B\sin(\ln|x|) \quad .$$

14.3 a. Letting $y = y_1 = e^{3x}$, we have

$$y = e^{3x} \quad \rightarrowtail \quad y' = 3e^{3x} \quad \rightarrowtail \quad y'' = 9e^{3x} \quad .$$

Thus,

$$y'' - 4y' + 3y = 9y'' - 4\cdot 3e^{2x} + 3e^{2x}$$
$$= [9 - 12 + 3]e^{2x} = 0e^{2x} = 0 \quad ,$$

showing that $y_1 = e^{3x}$ is one solution to the corresponding homogeneous differential equation.

For the general solution, let $y = y_1 u = e^{3x}u$. Computing the derivatives, we have

$$y' = \left(e^{3x}u\right)' = 3e^{3x}u + e^{3x}u' = \left[3u + u'\right]e^{3x}$$

and

$$y'' = (y')' = \left(3e^{3x}u + e^{3x}u'\right)'$$
$$= 9e^{3x}u + 3e^{3x}u' + 3e^{3x}u' + e^{3x}u''$$
$$= \left[9u + 6u' + u''\right]e^{3x} \quad .$$

So,

$$9e^{2x} = y'' - 4y' + 3y$$
$$= \left[9u + 6u' + u''\right]e^{3x} - 4\left[3u + u'\right]e^{3x} + 3e^{3x}u$$
$$= \left(u'' + [6-4]u' + [9 - 12 + 3]u\right)e^{3x}$$
$$= \left(u'' + 2u'\right)e^{3x} \quad .$$

Setting $v = u'$ and continuing:

$$9e^{2x} = \left(v' + 2v\right)e^{3x} \quad \rightarrowtail \quad \frac{dv}{dx} + 2v = 9e^{-x}$$

$$\hookrightarrow \quad \mu = e^{\int 2\,dx} = e^{2x} \qquad \text{(integrating factor for above equation)}$$

$$\hookrightarrow \quad e^{2x}\left[\frac{dv}{dx} + 2v = 9e^{-x}\right] \quad \rightarrowtail \quad \frac{d}{dx}\left[e^{2x}v\right] = 9e^{x}$$

$$\hookrightarrow \qquad e^{2x}v = \int 9e^{x}\,dx = 9e^{x} + c_1$$

$$\hookrightarrow \qquad u' = v = 9e^{-x} + c_1 e^{-2x}$$

$$\hookrightarrow \qquad u = \int \left[9e^{-x} + c_1 e^{-2x}\right] = -9e^{-x} - \frac{c_1}{2}e^{-2x} + c_3$$

$$\hookrightarrow \qquad u = A + Be^{-2x} - 9e^{-x} \quad .$$

Hence, the general solution to the original differential equation is

$$y = y_1 u = e^{3x}\left[A + Be^{-2x} - 9e^{-x}\right] = Ae^{3x} + Be^{x} - 9e^{2x} \quad .$$

14.3 c. Letting $y = y_1 = x$, we have

$$y = x \quad \rightarrowtail \quad y' = 1 \quad \rightarrowtail \quad y'' = 0 \quad .$$

Thus,

$$x^2 y'' + xy' - y = x^2 \cdot 0 + x \cdot 1 - x = x - x = 0 \quad ,$$

showing that $y_1 = x$ is one solution to the corresponding homogeneous differential equation.

Now let $y = y_1 u = xu$. Computing the derivatives, we have

$$y' = (xu)' = u + xu'$$

and

$$y'' = \left(u + xu'\right)' = u' + u' + xu'' = 2u' + xu'' \quad .$$

So,

$$\begin{aligned}
\sqrt{x} &= x^2 y'' + xy' - y \\
&= x^2\left[2u' + xu''\right] + x\left[u + xu'\right] - xu \\
&= x^3 u'' + \left[2x^2 + x^2\right]u' + [x - x]\,u \\
&= x^2\left[xu'' + 3u'\right] \quad .
\end{aligned}$$

Setting $v = u'$ and continuing:

$$\sqrt{x} = x^2\left[xv' + 3v\right] \quad \rightarrowtail \quad \frac{dv}{dx} + \frac{3}{x}v = x^{-5/2}$$

$$\hookrightarrow \qquad \mu = e^{\int \left(3/x\right)dx} = x^3 \qquad \text{(integrating factor for above equation)}$$

$$\hookrightarrow \qquad x^3\left[\frac{dv}{dx} + \frac{3}{x}v = x^{-5/2}\right] \quad \rightarrowtail \quad \frac{d}{dx}\left[x^3 v\right] = x^{1/2}$$

$$\hookrightarrow \qquad x^3 v = \int x^{1/2}\,dx = \tfrac{2}{3}x^{3/2} + c_1$$

$$\hookrightarrow \qquad u' = v = \tfrac{2}{3}x^{-3/2} + c_1 x^{-3}$$

$$\hookrightarrow \quad u = \int \left[\tfrac{2}{3}x^{-3/2} + c_1 x^{-3}\right] dx = -\tfrac{3}{4}x^{-1/2} - \tfrac{c_1}{2}x^{-2} + c_2$$

$$\hookrightarrow \quad u = A + Bx^{-2} - \tfrac{3}{4}x^{-1/2} \quad .$$

Hence, the general solution to the original differential equation is

$$y = y_1 u = x\left[A + Bx^{-2} - \tfrac{3}{4}x^{-1/2}\right] = Ax + Bx^{-1} - \tfrac{3}{4}x^{1/2} \quad .$$

14.3 e. Letting $y = y_1 = x^{-1}$, we have

$$y = x^{-1} \quad \rightarrowtail \quad y' = -x^{-2} \quad \rightarrowtail \quad y'' = 2x^{-3} \quad .$$

Thus,

$$\begin{aligned} xy'' + (2+2x)y' + 2y &= x\left[2x^{-3}\right] + (2+2x)\left[-x^{-2}\right] + 2\left[x^{-1}\right] \\ &= 2x^{-2} - 2x^{-2} - 2x^{-1} + 2x^{-1} = 0 \quad , \end{aligned}$$

showing that $y_1 = x^{-1}$ is one solution to the corresponding homogeneous differential equation.

Now let $y = y_1 u = x^{-1}u$. Computing the derivatives, we have

$$y' = \left(x^{-1}u\right)' = -x^{-2}u + x^{-1}u'$$

and

$$\begin{aligned} y'' &= \left(-x^{-2}u + x^{-1}u'\right)' \\ &= 2x^{-3}u - x^{-2}u' - x^{-2}u' + x^{-1}u'' \\ &= 2x^{-3}u - 2x^{-2}u' + x^{-1}u'' \quad . \end{aligned}$$

So,

$$\begin{aligned} 8e^{2x} &= xy'' + (2+2x)y' + 2y \\ &= x\left[2x^{-3}u - 2x^{-2}u' + x^{-1}u''\right] \\ &\qquad + (2+2x)\left[-x^{-2}u + x^{-1}u'\right] + 2\left[x^{-1}u\right] \\ &= u'' + \left[-2x^{-1} + 2x^{-1} + 2\right]u' + \left[2x^{-2} - 2x^{-2} - 2x^{-1} + 2x^{-1}\right]u \\ &= u'' + 2u' \quad . \end{aligned}$$

Setting $v = u'$ and continuing:

$$8e^{2x} = v' + 2v \quad \rightarrowtail \quad \frac{dv}{dx} + 2v = 8e^{2x}$$

$$\hookrightarrow \quad \mu = e^{\int 2\,dx} = e2x \qquad \text{(integrating factor for above equation)}$$

$$\hookrightarrow \quad e^{2x}\left[\frac{dv}{dx} + 2v = 8e^{2x}\right] \quad \rightarrowtail \quad \frac{d}{dx}\left[e^{2x}v\right] = 8e^{4x}$$

$$\hookrightarrow \quad e^{2x}v = \int 8e^{4x}\,dx = 2e^{4x} + c_1 \quad \rightarrowtail \quad v = 2e^{2x} + c_1e^{-2x}$$

$$\hookrightarrow \quad u = \int v\,dx = \int \left[2e^{2x} + c_1e^{-2x}\right]dx = e^{2x} - \frac{c_1}{2}e^{-2x} + c_2$$

$$\hookrightarrow \quad u = e^{2x} + Be^{-2x} + A \quad .$$

Hence, the general solution to the original differential equation is

$$y = y_1u = x^{-1}\left[e^{2x} + A + Be^{-2x}\right] \quad .$$

14.5 a. Letting $y = y_1 = e^{3x}$, we have

$$y = e^{3x} \quad \rightarrowtail \quad y' = 3e^{3x} \quad \rightarrowtail \quad y'' = 9e^{3x} \quad \rightarrowtail \quad y''' = 27e^{3x} \quad .$$

Thus,

$$\begin{aligned} y''' - 9y'' + 27y' - 27y &= 27e^{3x} - 9\cdot 9e^{3x} + 27\cdot 3e^{3x} - 27e^{3x} \\ &= [27 - 81 + 81 - 27]\,e^{3x} = 0 \quad , \end{aligned}$$

showing that $y_1 = e^{3x}$ is one solution to the homogeneous differential equation.

Now let $y = y_1u = e^{3x}u$. Computing the derivatives, we have

$$y' = \left(e^{3x}u\right)' = 3e^{3x}u + e^{3x}u' = e^{3x}\left[3u + u'\right] \quad ,$$

$$\begin{aligned} y'' &= \left(3e^{3x}u + e^{3x}u'\right)' \\ &= 9e^{3x}u + 3e^{3x}u' + 3e^{3x}u' + e^{3x}u'' = e^{3x}\left[9u + 6u' + u''\right] \quad , \end{aligned}$$

and

$$\begin{aligned} y''' &= \left(e^{3x}\left[9u + 6u' + u''\right]\right)' \\ &= 3e^{3x}\left[9u + 6u' + u''\right] + e^{3x}\left[9u' + 6u'' + u'''\right] \\ &= e^{3x}\left[27u + 27u' + 9u'' + u'''\right] \quad . \end{aligned}$$

So,

$$\begin{aligned} 0 &= y''' - 9y'' + 27y' - 27y \\ &= e^{3x}\Big(\left[27u + 27u' + 9u'' + u'''\right] \\ &\qquad - 9\left[9u + 6u' + u''\right] + 27\left[3u + u'\right] - 27[u]\Big) \\ &= e^{3x}\Big(u''' + [9-9]u'' + [27 - 54 + 27]u' + [27 - 81 + 81 - 27]u\Big) \\ &= e^{3x}\Big(u''' + 0u'' + 0u' + 0u\Big) \quad . \end{aligned}$$

So $u''' = 0$, and we can find u simply by integrating:

$$u'' = \int u'''\,dx = \int 0\,dx = c_1$$

$$\hookrightarrow \quad u' = \int u''\,dx = \int c_1\,dx = c_1x + c_2$$

$$\hookrightarrow \qquad u = \int u'\,dx = \int [c_1 x + c_2]\,dx = \frac{c_1}{2}x^2 + c_2 x + c_3$$

$$\hookrightarrow \qquad u = Cx^2 + Bx + A \quad .$$

Hence, the general solution to the original differential equation is

$$y = y_1 u = e^{3x}\left[A + Bx + Cx^2\right] \quad .$$

14.5 c. Letting $y = y_1 = e^{3x}$, we have

$$y = e^{2x} \quad \rightarrowtail \quad y' = 2e^{2x} \quad \rightarrowtail \quad y'' = 4e^{3x}$$

$$\hookrightarrow \qquad y''' = 8e^{2x} \quad \rightarrowtail \quad y^{(4)} = 16e^{2x} \quad .$$

Thus,

$$\begin{aligned} y^{(4)} - 8y''' + 24y'' - 32y' + 16y & \\ &= 16e^{2x} - 8 \cdot 8e^{2x} + 24 \cdot 4e^{2x} - 32 \cdot 2e^{2x} + 16e^{2x} \\ &= [16 - 64 + 96 - 64 + 16]\,e^{2x} = [0]e^{2x} = 0 \quad , \end{aligned}$$

showing that $y_1 = e^{2x}$ is one solution to the homogeneous differential equation.

Now let $y = y_1 u = e^{2x}u$. Computing the derivatives, we have

$$y' = \left(e^{2x}u\right)' = 2e^{2x}u + e^{2x}u' = e^{2x}\left[2u + u'\right] \quad ,$$

$$\begin{aligned} y'' &= \left(2e^{2x}u + e^{2x}u'\right)' \\ &= 4e^{2x}u + 2e^{2x}u' + 2e^{2x}u' + e^{2x}u'' = e^{2x}\left[4u + 4u' + u''\right] \quad , \end{aligned}$$

$$\begin{aligned} y''' &= \left(e^{2x}\left[4u + 4u' + u''\right]\right)' \\ &= 2e^{2x}\left[4u + 4u' + u''\right] + e^{2x}\left[4u' + 4u'' + u'''\right] \\ &= e^{2x}\left[8u + 12u' + 6u'' + u'''\right] \quad , \end{aligned}$$

and

$$\begin{aligned} y^{(4)} &= \left(e^{2x}\left[8u + 12u' + 6u'' + u'''\right]\right)' \\ &= 2e^{2x}\left[8u + 12u' + 6u'' + u'''\right] + e^{2x}\left[8u' + 12u'' + 6u''' + u^{(4)}\right] \\ &= e^{2x}\left[16u + 32u' + 24u'' + 8u''' + u^{(4)}\right] \quad . \end{aligned}$$

So,

$$\begin{aligned} 0 &= y^{(4)} - 8y''' + 24y'' - 32y' + 16y \\ &= e^{2x}\Big(\left[16u + 32u' + 24u'' + 8u''' + u^{(4)}\right] - 8\left[8u + 12u' + 6u'' + u'''\right] \\ &\qquad + 24\left[4u + 4u' + u''\right] - 32\left[2u + u'\right] + 16[u]\Big) \end{aligned}$$

$$= e^{2x}\Big(u^{(4)} + [8-8]u''' + [24-48+24]u''$$
$$+ [32-96+96-32]u' + [16-64+96-64+16]u\Big)$$
$$= e^{2x}\Big(u^{(4)} + 0u''' + 0u'' + 0u' + 0u\Big) \quad .$$

So $u^{(4)} = 0$, and we can find u simply by integrating:

$$u''' = \int u^{(4)}\,dx = \int 0\,dx = c_1$$
$$\hookrightarrow \quad u'' = \int u'''\,dx = \int c_1\,dx = c_1x + c_2$$
$$\hookrightarrow \quad u' = \int u''\,dx = \int [c_1x + c_2]\,dx = \frac{c_1}{2}x^2 + c_2x + c_3$$
$$\hookrightarrow \quad u = \int u'\,dx = \int \left[\frac{c_1}{2}x^2 + c_2x + c_3\right] dx = \frac{c_1}{6}x^3 + \frac{c_2}{2}x^2 + c_3x + c_4$$
$$\hookrightarrow \quad u = Dx^3 + Cx^2 + Bx + A \quad .$$

Hence, the general solution to the original differential equation is

$$y = y_1u = e^{2x}\left[A + Bx + Cx^2 + Dx^3\right] \quad .$$