Chapter 12: Numerical Methods II: Beyond the Euler Method

12.1 a. The differential equation, $\frac{dy}{dx} = \frac{y}{x}$, already is in derivative formula form. So

$$f(x, y) = \frac{y}{x} \quad .$$

From the initial condition and given parameters:

$$x_0 = 1 \quad , \quad y_0 = -1 \quad , \quad \Delta x = \frac{1}{3} \quad , \quad N = 3$$

and

$$x_{\max} = x_0 + N\Delta x = 1 + 3 \cdot \frac{1}{3} = 2 \quad .$$

Since we are doing these computations "by hand", let us do a little "pre-computing" to save time and space later:

$$x_{k+1} = x_k + \Delta x = x_k + \frac{1}{3} \quad ,$$

$$p = y_k + \Delta x \cdot f(x_k, y_k)$$

$$= y_k + \frac{1}{3} \cdot \frac{y_k}{x_k} = y_k + \frac{y_k}{3x_k}$$

and

$$y_{k+1} = y_k + \Delta x \cdot \frac{1}{2}\left[f(x_k, y_k) + f(x_{k+1}, p)\right]$$

$$= y_k + \frac{1}{6}\left[\frac{y_k}{x_k} + \frac{p}{x_{k+1}}\right] \quad .$$

Repeatedly using these with $k = 0, 1, \ldots$:
With $k = 0$:

$$x_1 = x_0 + \frac{1}{3} = 1 + \frac{1}{3} = \frac{4}{3} \quad ,$$

$$p = y_0 + \frac{y_0}{3x_0} = -1 + \frac{-1}{3 \cdot 1} = -\frac{4}{3}$$

and

$$y_1 = y_k + \frac{1}{6}\left[\frac{y_0}{x_0} + \frac{p}{x_{0+1}}\right] = -1 + \frac{1}{6}\left[\frac{-1}{1} + \frac{-4/3}{4/3}\right] = -\frac{4}{3} \quad .$$

With $k = 1$:

$$x_2 = x_1 + \frac{1}{3} = \frac{4}{3} + \frac{1}{3} = \frac{5}{3} \quad ,$$

$$p = y_1 + \frac{y_1}{3x_1} = -\frac{4}{3} + \frac{-4/3}{3 \cdot 4/3} = -\frac{5}{3}$$

and

$$y_2 = y_1 + \frac{1}{6}\left[\frac{y_1}{x_1} + \frac{p}{x_2}\right] = -\frac{4}{3} + \frac{1}{6}\left[\frac{-4/3}{4/3} + \frac{-5/3}{5/3}\right] = -\frac{5}{3} \quad .$$

With $k = 1$:

$$x_3 = x_2 + \frac{1}{3} = \frac{5}{3} + \frac{1}{3} = 2 \quad (= x_{\max}) \quad ,$$

$$p = y_2 + \frac{y_2}{3x_2} = -\frac{5}{3} + \frac{-5/3}{3 \cdot 5/3} = -2 \quad .$$

and

$$y_3 = y_2 + \frac{1}{6}\left[\frac{y_2}{x_2} + \frac{p}{x_3}\right] = -\frac{5}{3} + \frac{1}{6}\left[\frac{-5/3}{5/3} + \frac{-2}{2}\right] = -2 \quad .$$

In summary:

k	x_k	y_k
0	1	-1
1	$4/3$	$-4/3$
2	$5/3$	$-5/3$
3	2	-2

12.1 c. $2x + 5\frac{dy}{dx} = y \quad \rightarrowtail \quad \frac{dy}{dx} = \frac{1}{5}[y - 2x] \quad \rightarrowtail \quad f(x,y) = \frac{1}{5}[y-2x] \quad .$

From the initial condition and given parameters:

$$x_0 = 0 \quad , \quad y_0 = 2 \quad , \quad \Delta x = \frac{1}{2} \quad , \quad x_{max} = 2$$

and

$$N = \frac{x_{max} - x_0}{\Delta x} = \frac{2 - 0}{1/2} = 4 \quad .$$

Since we are doing these computations "by hand", let us do a little "pre-computing" to save time and space later:

$$x_{k+1} = x_k + \Delta x = x_k + \frac{1}{2} \quad ,$$

$$\begin{aligned} p &= y_k + \Delta x \cdot f(x_k, y_k) \\ &= y_k + \frac{1}{2} \cdot \frac{1}{5}[y_k - 2x_k] = y_k + \frac{1}{10}[y_k - 2x_k] \end{aligned}$$

and

$$\begin{aligned} y_{k+1} &= y_k + \Delta x \cdot \frac{1}{2}\left[f(x_k, y_k) + f(x_{k+1}, p)\right] \\ &= y_k + \frac{1}{2} \cdot \frac{1}{2}\left[\frac{1}{5}[y_k - 2x_k] + \frac{1}{5}\left[p - 2\left(x_k + \frac{1}{2}\right)\right]\right] \\ &= y_k + \frac{1}{20}[y_k + p - 4x_k - 1] \quad . \end{aligned}$$

Repeatedly using these with $k = 0, 1, \ldots$:

With $k = 0$:

$$x_1 = 0 + \frac{1}{2} = \frac{1}{2} \quad ,$$

$$p = y_k + \frac{1}{10}[y_k - 2x_k] = 2 + \frac{1}{10}[2 - 2 \cdot 0] = \frac{11}{5}$$

and

$$\begin{aligned} y_1 &= y_0 + \frac{1}{20}[y_0 + p - 4x_0 - 1] \\ &= 2 + \frac{1}{20}\left[2 + \frac{11}{5} - 4 \cdot 0 - 1\right] = \frac{54}{25} \quad . \end{aligned}$$

With $k = 1$:

$$x_2 = x_1 + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1 \quad ,$$

$$p = y_1 + \frac{1}{10}[y_1 - 2x_1] = \frac{54}{25} + \frac{1}{10}\left[\frac{54}{25} - 2 \cdot \frac{1}{2}\right] = \frac{569}{250}$$

and

$$\begin{aligned} y_2 &= y_1 + \frac{1}{20}[y_1 + p - 4x_k - 1] \\ &= \frac{54}{25} + \frac{1}{20}\left[\frac{54}{25} + \frac{569}{250} - 4 \cdot \frac{1}{2} - 1\right] = \frac{11{,}159}{5{,}000} \quad . \end{aligned}$$

With $k = 2$:

$$x_3 = x_2 + \frac{1}{2} = 1 + \frac{1}{2} = \frac{3}{2} \quad ,$$

$$p = y_2 + \frac{1}{10}[y_2 - 2x_2] = \frac{11{,}159}{5{,}000} + \frac{1}{10}\left[\frac{11{,}159}{5{,}000} - 2 \cdot 1\right] = \frac{112{,}749}{50{,}000}$$

and

$$y_3 = y_2 + \frac{1}{20}[y_2 + p - 4x_2 - 1]$$

$$= \frac{11{,}159}{5{,}000} + \frac{1}{20}\left[\frac{11{,}159}{5{,}000} + \frac{112{,}749}{50{,}000} - 4 \cdot 1 - 1\right] = \frac{2{,}206{,}139}{1{,}000{,}000} \quad .$$

With $k = 3$:

$$x_4 = x_3 + \frac{1}{2} = \frac{3}{2} + \frac{1}{2} = 2 \quad (= x_{\max}) \quad ,$$

$$p = y_3 + \frac{1}{10}[y_3 - 2x_3] = \frac{2{,}206{,}139}{1{,}000{,}000} + \frac{1}{10}\left[\frac{2{,}206{,}139}{1{,}000{,}000} - 2 \cdot \frac{3}{2}\right] = \frac{21{,}267{,}529}{10{,}000{,}000}$$

and

$$y_4 = y_3 + \frac{1}{20}[y_3 + p - 4x_3 - 1]$$

$$= \frac{2{,}206{,}139}{1{,}000{,}000} + \frac{1}{20}\left[\frac{2{,}206{,}139}{1{,}000{,}000} + \frac{21{,}267{,}529}{10{,}000{,}000} - 4 \cdot \frac{3}{2} - 1\right] = \frac{414{,}556{,}719}{200{,}000{,}000} \quad .$$

In summary:

k	x_k	y_k
0	0	2
1	$^1/_2$	$^{54}/_{25}$
2	1	$^{11,159}/_{5,000}$
3	$^3/_2$	$^{2,206,139}/_{1,000,000}$
4	2	$^{414,556,719}/_{200,000,000}$

12.6 a. $\dfrac{dy}{dx} = -8xy \quad \rightarrowtail \quad f(x, y) = -8xy \quad .$

From the initial condition and given parameters:

$$x_0 = 0 \quad , \quad y_0 = 10 \quad , \quad x_{\max} = 1 \quad , \quad N = 2$$

and, since $x_{\max} = x_0 + N\Delta x$,

$$\Delta x = \frac{x_{\max} - x_0}{N} = \frac{1 - 0}{2} = \frac{1}{2} \quad .$$

Repeatedly using

$$x_{k+1} = x_k + \Delta x \quad ,$$

$$s_1 = f(x_k, y_k) \quad ,$$

$$s_2 = f\left(x_k + \frac{1}{2}\Delta x \, , \, y_k + \frac{1}{2}\Delta x \cdot s_1\right) \quad ,$$

$$s_3 = f\left(x_k + \frac{1}{2}\Delta x \, , \, y_k + \frac{1}{2}\Delta x \cdot s_2\right) \quad ,$$

$$s_4 = f(x_{k+1} \, , \, y_k + \Delta x \cdot s_3)$$

and

$$y_{k+1} = y_k + \frac{\Delta x}{6}[s_1 + 2s_2 + 2s_3 + s_4]$$

with $k = 0, 1, \ldots$, we have:

With $k = 0$:

$$x_1 = x_0 + \Delta x = 0 + \frac{1}{2} = \frac{1}{2} \quad ,$$

$$\begin{aligned} s_1 &= f(x_0, y_0) \\ &= f(0, 10) = -8 \cdot 0 \cdot 10 = 0 \quad , \end{aligned}$$

$$\begin{aligned} s_2 &= f\left(x_0 + \tfrac{1}{2}\Delta x\,,\, y_0 + \tfrac{1}{2}\Delta x \cdot s_1\right) \\ &= f\left(0 + \tfrac{1}{2} \cdot \tfrac{1}{2}\,,\, 10 + \tfrac{1}{2} \cdot \tfrac{1}{2} \cdot 0\right) = f\left(\tfrac{1}{4}, 10\right) = -8 \cdot \tfrac{1}{4} \cdot 10 = -20 \quad , \end{aligned}$$

$$\begin{aligned} s_3 &= f\left(x_0 + \tfrac{1}{2}\Delta x\,,\, y_0 + \tfrac{1}{2}\Delta x \cdot s_2\right) \\ &= f\left(0 + \tfrac{1}{2} \cdot \tfrac{1}{2}\,,\, 10 + \tfrac{1}{2} \cdot \tfrac{1}{2} \cdot (-20)\right) = f\left(\tfrac{1}{4}, 5\right) = -8 \cdot \tfrac{1}{4} \cdot 5 = -10 \quad , \end{aligned}$$

$$\begin{aligned} s_4 &= f(x_{0+1}\,,\, y_0 + \Delta x \cdot s_3) \\ &= f\left(\tfrac{1}{2}, 10 + \tfrac{1}{2} \cdot (-10)\right) = f\left(\tfrac{1}{2}, 5\right) = -8 \cdot \tfrac{1}{2} \cdot (5) = -20 \end{aligned}$$

and

$$\begin{aligned} y_1 &= y_0 + \frac{\Delta x}{6}\left[s_1 + 2s_2 + 2s_3 + s_4\right] \\ &= 10 + \frac{1}{2} \cdot \frac{1}{6}\left[0 + 2 \cdot (-20) + 2 \cdot (-10) + (-20)\right] = \frac{10}{3} \quad . \end{aligned}$$

With $k = 1$:

$$x_2 = x_1 + \Delta x = \frac{1}{2} + \frac{1}{2} = 1 \quad ,$$

$$\begin{aligned} s_1 &= f(x_1, y_1) \\ &= f\left(\tfrac{1}{2}, \tfrac{10}{3}\right) = -8 \cdot \tfrac{1}{2} \cdot \tfrac{10}{3} = -\tfrac{40}{3} \quad , \end{aligned}$$

$$\begin{aligned} s_2 &= f\left(x_1 + \tfrac{1}{2}\Delta x\,,\, y_1 + \tfrac{1}{2}\Delta x \cdot s_1\right) \\ &= f\left(\tfrac{1}{2} + \tfrac{1}{2} \cdot \tfrac{1}{2}\,,\, \tfrac{10}{3} + \tfrac{1}{2} \cdot \tfrac{1}{2} \cdot (-\tfrac{40}{3})\right) = f\left(\tfrac{3}{4}, 0\right) = -8 \cdot \tfrac{3}{4} \cdot 0 = 0 \quad , \end{aligned}$$

$$\begin{aligned} s_3 &= f\left(x_1 + \tfrac{1}{2}\Delta x\,,\, y_1 + \tfrac{1}{2}\Delta x \cdot s_2\right) \\ &= f\left(\tfrac{1}{2} + \tfrac{1}{2} \cdot \tfrac{1}{2}\,,\, \tfrac{10}{3} + \tfrac{1}{2} \cdot \tfrac{1}{2} \cdot 0\right) = f\left(\tfrac{3}{4}, \tfrac{10}{3}\right) = -8 \cdot \tfrac{3}{4} \cdot \tfrac{10}{3} = -20 \quad , \end{aligned}$$

$$\begin{aligned} s_4 &= f(x_{1+1}\,,\, y_1 + \Delta x \cdot s_3) \\ &= f\left(1, \tfrac{10}{3} + \tfrac{1}{2} \cdot (-20)\right) = f\left(1, -\tfrac{20}{3}\right) = -8 \cdot 1 \cdot (-\tfrac{20}{3}) = \tfrac{160}{3} \end{aligned}$$

and

$$\begin{aligned} y_2 &= y_1 + \frac{\Delta x}{6}\left[s_1 + 2s_2 + 2s_3 + s_4\right] \\ &= \frac{10}{3} + \frac{1}{2} \cdot \frac{1}{6}\left[-\frac{40}{3} + 2 \cdot 0 + 2 \cdot (-20) + \frac{160}{3}\right] = \frac{10}{3} \quad . \end{aligned}$$

In summary:

k	x_k	y_k
0	0	10
1	$^1/_2$	$^{10}/_3$
2	1	$^{10}/_3$

12.6 c. $\dfrac{dy}{dx} + \dfrac{y}{x} = 4 \quad\rightarrowtail\quad \dfrac{dy}{dx} = 4 - \dfrac{y}{x} \quad\rightarrowtail\quad f(x,y) = 4 - \dfrac{y}{x} \quad .$

From the initial condition and given parameters:

$$x_0 = 1 \quad , \quad y_0 = 8 \quad , \quad \Delta x = \tfrac{1}{2} \quad , \quad N = 2$$

and

$$x_{\max} = x_0 + N\Delta x = 2 \quad .$$

Repeatedly using

$$x_{k+1} = x_k + \Delta x \quad ,$$

$$s_1 = f(x_k, y_k) \quad ,$$

$$s_2 = f\Big(x_k + \tfrac{1}{2}\Delta x \,,\, y_k + \tfrac{1}{2}\Delta x \cdot s_1\Big) \quad ,$$

$$s_3 = f\Big(x_k + \tfrac{1}{2}\Delta x \,,\, y_k + \tfrac{1}{2}\Delta x \cdot s_2\Big) \quad ,$$

$$s_4 = f(x_{k+1} \,,\, y_k + \Delta x \cdot s_3)$$

and

$$y_{k+1} = y_k + \frac{\Delta x}{6}\left[s_1 + 2s_2 + 2s_3 + s_4\right]$$

with $k = 0, 1, \ldots$, we have:

With $k = 0$:

$$x_1 = x_0 + \Delta x = 1 + \tfrac{1}{2} = \tfrac{3}{2} \quad ,$$

$$s_1 = f(x_0, y_0) \qquad \left(= 4 - \tfrac{y}{x}\right)$$

$$= f(1, 8) = 4 - \tfrac{8}{1} = -4 \quad ,$$

$$s_2 = f\Big(x_0 + \tfrac{1}{2}\Delta x \,,\, y_0 + \tfrac{1}{2}\Delta x \cdot s_1\Big)$$

$$= f\Big(1 + \tfrac{1}{2}\cdot\tfrac{1}{2} \,,\, 8 + \tfrac{1}{2}\cdot\tfrac{1}{2}\cdot(-4)\Big) = f\Big(\tfrac{5}{4}, 7\Big) = 4 - \frac{7}{^5/_4} = -\tfrac{8}{5} \quad ,$$

$$s_3 = f\Big(x_0 + \tfrac{1}{2}\Delta x \,,\, y_0 + \tfrac{1}{2}\Delta x \cdot s_2\Big)$$

$$= f\Big(1 + \tfrac{1}{2}\cdot\tfrac{1}{2} \,,\, 8 + \tfrac{1}{2}\cdot\tfrac{1}{2}\cdot\Big(-\tfrac{8}{5}\Big)\Big) = f\Big(\tfrac{5}{4}, \tfrac{38}{5}\Big) = 4 - \frac{^{38}/_5}{^5/_4} = -\tfrac{52}{25} \quad ,$$

$$s_4 = f(x_{0+1} \,,\, y_0 + \Delta x \cdot s_3)$$

$$= f\Big(\tfrac{3}{2}, 8 + \tfrac{1}{2}\cdot\Big(-\tfrac{52}{25}\Big)\Big) = f\Big(\tfrac{3}{2}, \tfrac{174}{25}\Big) = 4 - \frac{^{174}/_{25}}{^3/_2} = -\tfrac{16}{25}$$

and

$$\begin{aligned} y_1 &= y_0 + \frac{\Delta x}{6}[s_1 + 2s_2 + 2s_3 + s_4] \\ &= 10 + \frac{1}{2}\cdot\frac{1}{6}\left[-4 + 2\cdot\left(-\frac{8}{5}\right) + 2\cdot\left(-\frac{52}{25}\right) - \frac{16}{25}\right] = 7 \quad . \end{aligned}$$

With $k = 1$:

$$x_2 = x_1 + \Delta x = \frac{3}{2} + \frac{1}{2} = 2 \quad ,$$

$$\begin{aligned} s_1 &= f(x_1, y_1) \\ &= f\left(\frac{3}{2}, 7\right) = 4 - \frac{7}{3/2} = -\frac{2}{3} \quad , \end{aligned}$$

$$\begin{aligned} s_2 &= f\left(x_1 + \frac{1}{2}\Delta x\, ,\, y_1 + \frac{1}{2}\Delta x \cdot s_1\right) \\ &= f\left(\frac{3}{2} + \frac{1}{2}\cdot\frac{1}{2}\, , 7 + \frac{1}{2}\cdot\frac{1}{2}\cdot\left(-\frac{2}{3}\right)\right) = f\left(\frac{7}{4}, \frac{41}{6}\right) = 4 - \frac{41/6}{7/4} = \frac{2}{21} \quad , \end{aligned}$$

$$\begin{aligned} s_3 &= f\left(x_1 + \frac{1}{2}\Delta x\, ,\, y_1 + \frac{1}{2}\Delta x \cdot s_2\right) \\ &= f\left(\frac{3}{2} + \frac{1}{2}\cdot\frac{1}{2}\, , 7 + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{2}{21}\right) = f\left(\frac{7}{4}, \frac{295}{42}\right) = 4 - \frac{295/42}{7/4} = -\frac{2}{147} \quad , \end{aligned}$$

$$\begin{aligned} s_4 &= f(x_{1+1}\, ,\, y_1 + \Delta x \cdot s_3) \\ &= f\left(2, 7 + \frac{1}{2}\cdot\left(-\frac{2}{147}\right)\right) = f\left(2, -\frac{1{,}028}{147}\right) = 4 - \frac{1{,}028/147}{2} = \frac{74}{147} \end{aligned}$$

and

$$\begin{aligned} y_2 &= y_1 + \frac{\Delta x}{6}[s_1 + 2s_2 + 2s_3 + s_4] \\ &= 7 + \frac{1}{2}\cdot\frac{1}{6}\left[-\frac{2}{3} + 2\cdot\frac{2}{21} + 2\cdot\left(-\frac{2}{147}\right) + \frac{74}{147}\right] = 7 \quad . \end{aligned}$$

In summary:

k	x_k	y_k
0	1	8
1	$3/2$	7
2	2	7