



Chapter 12: Numerical Methods II: Beyond the Euler Method

12.1 a. The differential equation, $\frac{dy}{dx} = \frac{y}{x}$, already is in derivative formula form. So

$$f(x, y) = \frac{y}{x} .$$

From the initial condition and given parameters:

$$x_0 = 1 , \quad y_0 = -1 , \quad \Delta x = \frac{1}{3} , \quad N = 3$$

and

$$x_{\max} = x_0 + N \Delta x = 1 + 3 \cdot \frac{1}{3} = 2 .$$

Since we are doing these computations “by hand”, let us do a little “pre-computing” to save time and space later:

$$x_{k+1} = x_k + \Delta x = x_k + \frac{1}{3} ,$$

$$\begin{aligned} p &= y_k + \Delta x \cdot f(x_k, y_k) \\ &= y_k + \frac{1}{3} \cdot \frac{y_k}{x_k} = y_k + \frac{y_k}{3x_k} \end{aligned}$$

and

$$\begin{aligned} y_{k+1} &= y_k + \Delta x \cdot \frac{1}{2} [f(x_k, y_k) + f(x_{k+1}, p)] \\ &= y_k + \frac{1}{6} \left[\frac{y_k}{x_k} + \frac{p}{x_{k+1}} \right] . \end{aligned}$$

Repeatedly using these with $k = 0, 1, \dots$:

With $k = 0$:

$$x_1 = x_0 + \frac{1}{3} = 1 + \frac{1}{3} = \frac{4}{3} ,$$

$$p = y_0 + \frac{y_0}{3x_0} = -1 + \frac{-1}{3 \cdot 1} = -\frac{4}{3}$$

and

$$y_1 = y_0 + \frac{1}{6} \left[\frac{y_0}{x_0} + \frac{p}{x_1} \right] = -1 + \frac{1}{6} \left[\frac{-1}{1} + \frac{-4/3}{4/3} \right] = -\frac{4}{3} .$$

With $k = 1$:

$$x_2 = x_1 + \frac{1}{3} = \frac{4}{3} + \frac{1}{3} = \frac{5}{3} ,$$

$$p = y_1 + \frac{y_1}{3x_1} = -\frac{4}{3} + \frac{-4/3}{3 \cdot 4/3} = -\frac{5}{3}$$

and

$$y_2 = y_1 + \frac{1}{6} \left[\frac{y_1}{x_1} + \frac{p}{x_2} \right] = -\frac{4}{3} + \frac{1}{6} \left[\frac{-4/3}{4/3} + \frac{-5/3}{5/3} \right] = -\frac{5}{3} .$$

With $k = 2$:

$$x_3 = x_2 + \frac{1}{3} = \frac{5}{3} + \frac{1}{3} = 2 (= x_{\max}) ,$$

$$p = y_2 + \frac{y_2}{3x_2} = -\frac{5}{3} + \frac{-5/3}{3 \cdot 5/3} = -2 .$$

and

$$y_3 = y_2 + \frac{1}{6} \left[\frac{y_2}{x_2} + \frac{p}{x_3} \right] = -\frac{5}{3} + \frac{1}{6} \left[\frac{-5/3}{5/3} + \frac{-2}{2} \right] = -2 .$$



In summary:

| k | x_k | y_k |
|-----|-------|-------|
| 0 | 1 | -1 |
| 1 | 4/3 | -4/3 |
| 2 | 5/3 | -5/3 |
| 3 | 2 | -2 |

$$\mathbf{12.1 c.} \quad 2x + 5\frac{dy}{dx} = y \implies \frac{dy}{dx} = \frac{1}{5}[y - 2x] \implies f(x, y) = \frac{1}{5}[y - 2x].$$

From the initial condition and given parameters:

$$x_0 = 0, \quad y_0 = 2, \quad \Delta x = \frac{1}{2}, \quad x_{\max} = 2$$

and

$$N = \frac{x_{\max} - x_0}{\Delta x} = \frac{2 - 0}{1/2} = 4.$$

Since we are doing these computations “by hand”, let us do a little “pre-computing” to save time and space later:

$$x_{k+1} = x_k + \Delta x = x_k + \frac{1}{2},$$

$$\begin{aligned} p &= y_k + \Delta x \cdot f(x_k, y_k) \\ &= y_k + \frac{1}{2} \cdot \frac{1}{5}[y_k - 2x_k] = y_k + \frac{1}{10}[y_k - 2x_k] \end{aligned}$$

and

$$\begin{aligned} y_{k+1} &= y_k + \Delta x \cdot \frac{1}{2}[f(x_k, y_k) + f(x_{k+1}, p)] \\ &= y_k + \frac{1}{2} \cdot \frac{1}{2} \left[\frac{1}{5}[y_k - 2x_k] + \frac{1}{5} \left[p - 2 \left(x_k + \frac{1}{2} \right) \right] \right] \\ &= y_k + \frac{1}{20}[y_k + p - 4x_k - 1]. \end{aligned}$$

Repeatedly using these with $k = 0, 1, \dots$:

With $k = 0$:

$$x_1 = 0 + \frac{1}{2} = \frac{1}{2},$$

$$p = y_0 + \frac{1}{10}[y_0 - 2x_0] = 2 + \frac{1}{10}[2 - 2 \cdot 0] = \frac{11}{5}$$

and

$$\begin{aligned} y_1 &= y_0 + \frac{1}{20}[y_0 + p - 4x_0 - 1] \\ &= 2 + \frac{1}{20} \left[2 + \frac{11}{5} - 4 \cdot 0 - 1 \right] = \frac{54}{25}. \end{aligned}$$

With $k = 1$:

$$x_2 = x_1 + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1,$$

$$p = y_1 + \frac{1}{10}[y_1 - 2x_1] = \frac{54}{25} + \frac{1}{10} \left[\frac{54}{25} - 2 \cdot \frac{1}{2} \right] = \frac{569}{250}$$

and

$$\begin{aligned} y_2 &= y_1 + \frac{1}{20}[y_1 + p - 4x_1 - 1] \\ &= \frac{54}{25} + \frac{1}{20} \left[\frac{54}{25} + \frac{569}{250} - 4 \cdot \frac{1}{2} - 1 \right] = \frac{11,159}{5,000}. \end{aligned}$$

With $k = 2$:

$$x_3 = x_2 + \frac{1}{2} = 1 + \frac{1}{2} = \frac{3}{2},$$

$$p = y_2 + \frac{1}{10}[y_2 - 2x_2] = \frac{11,159}{5,000} + \frac{1}{10}\left[\frac{11,159}{5,000} - 2 \cdot 1\right] = \frac{112,749}{50,000}$$

and

$$\begin{aligned} y_3 &= y_2 + \frac{1}{20}[y_2 + p - 4x_2 - 1] \\ &= \frac{11,159}{5,000} + \frac{1}{20}\left[\frac{11,159}{5,000} + \frac{112,749}{50,000} - 4 \cdot 1 - 1\right] = \frac{2,206,139}{1,000,000}. \end{aligned}$$

With $k = 3$:

$$x_4 = x_3 + \frac{1}{2} = \frac{3}{2} + \frac{1}{2} = 2 \quad (=x_{\max}),$$

$$p = y_3 + \frac{1}{10}[y_3 - 2x_3] = \frac{2,206,139}{1,000,000} + \frac{1}{10}\left[\frac{2,206,139}{1,000,000} - 2 \cdot \frac{3}{2}\right] = \frac{21,267,529}{10,000,000}$$

and

$$\begin{aligned} y_4 &= y_3 + \frac{1}{20}[y_3 + p - 4x_3 - 1] \\ &= \frac{2,206,139}{1,000,000} + \frac{1}{20}\left[\frac{2,206,139}{1,000,000} + \frac{21,267,529}{10,000,000} - 4 \cdot \frac{3}{2} - 1\right] = \frac{414,556,719}{200,000,000}. \end{aligned}$$

In summary:

| k | x_k | y_k |
|-----|---------------|-----------------------------------|
| 0 | 0 | 2 |
| 1 | $\frac{1}{2}$ | $\frac{54}{25}$ |
| 2 | 1 | $\frac{11,159}{5,000}$ |
| 3 | $\frac{3}{2}$ | $\frac{2,206,139}{1,000,000}$ |
| 4 | 2 | $\frac{414,556,719}{200,000,000}$ |

12.6 a. $\frac{dy}{dx} = -8xy \rightarrow f(x, y) = -8xy$.

From the initial condition and given parameters:

$$x_0 = 0, \quad y_0 = 10, \quad x_{\max} = 1, \quad N = 2$$

and, since $x_{\max} = x_0 + N\Delta x$,

$$\Delta x = \frac{x_{\max} - x_0}{N} = \frac{1 - 0}{2} = \frac{1}{2}.$$

Repeatedly using

$$x_{k+1} = x_k + \Delta x,$$

$$s_1 = f(x_k, y_k),$$

$$s_2 = f\left(x_k + \frac{1}{2}\Delta x, y_k + \frac{1}{2}\Delta x \cdot s_1\right),$$

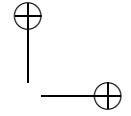
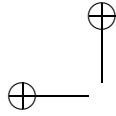
$$s_3 = f\left(x_k + \frac{1}{2}\Delta x, y_k + \frac{1}{2}\Delta x \cdot s_2\right),$$

$$s_4 = f(x_{k+1}, y_k + \Delta x \cdot s_3)$$

and

$$y_{k+1} = y_k + \frac{\Delta x}{6}[s_1 + 2s_2 + 2s_3 + s_4]$$

with $k = 0, 1, \dots$, we have:



With $k = 0$:

$$x_1 = x_0 + \Delta x = 0 + \frac{1}{2} = \frac{1}{2},$$

$$\begin{aligned}s_1 &= f(x_0, y_0) \\&= f(0, 10) = -8 \cdot 0 \cdot 10 = 0,\end{aligned}$$

$$\begin{aligned}s_2 &= f\left(x_0 + \frac{1}{2}\Delta x, y_0 + \frac{1}{2}\Delta x \cdot s_1\right) \\&= f\left(0 + \frac{1}{2} \cdot \frac{1}{2}, 10 + \frac{1}{2} \cdot \frac{1}{2} \cdot 0\right) = f\left(\frac{1}{4}, 10\right) = -8 \cdot \frac{1}{4} \cdot 10 = -20,\end{aligned}$$

$$\begin{aligned}s_3 &= f\left(x_0 + \frac{1}{2}\Delta x, y_0 + \frac{1}{2}\Delta x \cdot s_2\right) \\&= f\left(0 + \frac{1}{2} \cdot \frac{1}{2}, 10 + \frac{1}{2} \cdot \frac{1}{2} \cdot (-20)\right) = f\left(\frac{1}{4}, 5\right) = -8 \cdot \frac{1}{4} \cdot 5 = -10,\end{aligned}$$

$$\begin{aligned}s_4 &= f(x_{0+1}, y_0 + \Delta x \cdot s_3) \\&= f\left(\frac{1}{2}, 10 + \frac{1}{2} \cdot (-10)\right) = f\left(\frac{1}{2}, 5\right) = -8 \cdot \frac{1}{2} \cdot (5) = -20\end{aligned}$$

and

$$\begin{aligned}y_1 &= y_0 + \frac{\Delta x}{6} [s_1 + 2s_2 + 2s_3 + s_4] \\&= 10 + \frac{1}{2} \cdot \frac{1}{6} [0 + 2 \cdot (-20) + 2 \cdot (-10) + (-20)] = \frac{10}{3}.\end{aligned}$$

With $k = 1$:

$$x_2 = x_1 + \Delta x = \frac{1}{2} + \frac{1}{2} = 1,$$

$$\begin{aligned}s_1 &= f(x_1, y_1) \\&= f\left(\frac{1}{2}, \frac{10}{3}\right) = -8 \cdot \frac{1}{2} \cdot \frac{10}{3} = -\frac{40}{3},\end{aligned}$$

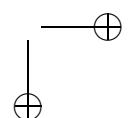
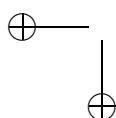
$$\begin{aligned}s_2 &= f\left(x_1 + \frac{1}{2}\Delta x, y_1 + \frac{1}{2}\Delta x \cdot s_1\right) \\&= f\left(\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}, \frac{10}{3} + \frac{1}{2} \cdot \frac{1}{2} \cdot \left(-\frac{40}{3}\right)\right) = f\left(\frac{3}{4}, 0\right) = -8 \cdot \frac{3}{4} \cdot 0 = 0,\end{aligned}$$

$$\begin{aligned}s_3 &= f\left(x_1 + \frac{1}{2}\Delta x, y_1 + \frac{1}{2}\Delta x \cdot s_2\right) \\&= f\left(\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}, \frac{10}{3} + \frac{1}{2} \cdot \frac{1}{2} \cdot 0\right) = f\left(\frac{3}{4}, \frac{10}{3}\right) = -8 \cdot \frac{3}{4} \cdot \frac{10}{3} = -20,\end{aligned}$$

$$\begin{aligned}s_4 &= f(x_{1+1}, y_1 + \Delta x \cdot s_3) \\&= f\left(1, \frac{10}{3} + \frac{1}{2} \cdot (-20)\right) = f\left(1, -\frac{20}{3}\right) = -8 \cdot 1 \cdot \left(-\frac{20}{3}\right) = \frac{160}{3}\end{aligned}$$

and

$$\begin{aligned}y_2 &= y_1 + \frac{\Delta x}{6} [s_1 + 2s_2 + 2s_3 + s_4] \\&= \frac{10}{3} + \frac{1}{2} \cdot \frac{1}{6} \left[-\frac{40}{3} + 2 \cdot 0 + 2 \cdot (-20) + \frac{160}{3} \right] = \frac{10}{3}.\end{aligned}$$





In summary:

| k | x_k | y_k |
|-----|---------------|----------------|
| 0 | 0 | 10 |
| 1 | $\frac{1}{2}$ | $\frac{10}{3}$ |
| 2 | 1 | $\frac{10}{3}$ |

$$\mathbf{12.6 \text{ c.}} \quad \frac{dy}{dx} + \frac{y}{x} = 4 \implies \frac{dy}{dx} = 4 - \frac{y}{x} \implies f(x, y) = 4 - \frac{y}{x} .$$

From the initial condition and given parameters:

$$x_0 = 1 , \quad y_0 = 8 , \quad \Delta x = \frac{1}{2} , \quad N = 2$$

and

$$x_{\max} = x_0 + N\Delta x = 2 .$$

Repeatedly using

$$x_{k+1} = x_k + \Delta x ,$$

$$s_1 = f(x_k, y_k) ,$$

$$s_2 = f\left(x_k + \frac{1}{2}\Delta x, y_k + \frac{1}{2}\Delta x \cdot s_1\right) ,$$

$$s_3 = f\left(x_k + \frac{1}{2}\Delta x, y_k + \frac{1}{2}\Delta x \cdot s_2\right) ,$$

$$s_4 = f(x_{k+1}, y_k + \Delta x \cdot s_3)$$

and

$$y_{k+1} = y_k + \frac{\Delta x}{6} [s_1 + 2s_2 + 2s_3 + s_4]$$

with $k = 0, 1, \dots$, we have:

With $k = 0$:

$$x_1 = x_0 + \Delta x = 1 + \frac{1}{2} = \frac{3}{2} ,$$

$$s_1 = f(x_0, y_0) \quad (= 4 - \frac{y}{x})$$

$$= f(1, 8) = 4 - \frac{8}{1} = -4 ,$$

$$s_2 = f\left(x_0 + \frac{1}{2}\Delta x, y_0 + \frac{1}{2}\Delta x \cdot s_1\right)$$

$$= f\left(1 + \frac{1}{2} \cdot \frac{1}{2}, 8 + \frac{1}{2} \cdot \frac{1}{2} \cdot (-4)\right) = f\left(\frac{5}{4}, 7\right) = 4 - \frac{7}{5/4} = -\frac{8}{5} ,$$

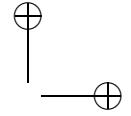
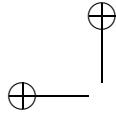
$$s_3 = f\left(x_0 + \frac{1}{2}\Delta x, y_0 + \frac{1}{2}\Delta x \cdot s_2\right)$$

$$= f\left(1 + \frac{1}{2} \cdot \frac{1}{2}, 8 + \frac{1}{2} \cdot \frac{1}{2} \cdot \left(-\frac{8}{5}\right)\right) = f\left(\frac{5}{4}, \frac{38}{5}\right) = 4 - \frac{38/5}{5/4} = -\frac{52}{25} ,$$

$$s_4 = f(x_{0+1}, y_0 + \Delta x \cdot s_3)$$

$$= f\left(\frac{3}{2}, 8 + \frac{1}{2} \cdot \left(-\frac{52}{25}\right)\right) = f\left(\frac{3}{2}, \frac{174}{25}\right) = 4 - \frac{174/25}{3/2} = -\frac{16}{25}$$





and

$$\begin{aligned} y_1 &= y_0 + \frac{\Delta x}{6} [s_1 + 2s_2 + 2s_3 + s_4] \\ &= 10 + \frac{1}{2} \cdot \frac{1}{6} \left[-4 + 2 \cdot \left(-\frac{8}{5} \right) + 2 \cdot \left(-\frac{52}{25} \right) - \frac{16}{25} \right] = 7 . \end{aligned}$$

With $k = 1$:

$$x_2 = x_1 + \Delta x = \frac{3}{2} + \frac{1}{2} = 2 ,$$

$$\begin{aligned} s_1 &= f(x_1, y_1) \\ &= f\left(\frac{3}{2}, 7\right) = 4 - \frac{7}{3/2} = -\frac{2}{3} , \end{aligned}$$

$$\begin{aligned} s_2 &= f\left(x_1 + \frac{1}{2}\Delta x, y_1 + \frac{1}{2}\Delta x \cdot s_1\right) \\ &= f\left(\frac{3}{2} + \frac{1}{2} \cdot \frac{1}{2}, 7 + \frac{1}{2} \cdot \frac{1}{2} \cdot \left(-\frac{2}{3}\right)\right) = f\left(\frac{7}{4}, \frac{41}{6}\right) = 4 - \frac{41/6}{7/4} = \frac{2}{21} , \end{aligned}$$

$$\begin{aligned} s_3 &= f\left(x_1 + \frac{1}{2}\Delta x, y_1 + \frac{1}{2}\Delta x \cdot s_2\right) \\ &= f\left(\frac{3}{2} + \frac{1}{2} \cdot \frac{1}{2}, 7 + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{2}{21}\right) = f\left(\frac{7}{4}, \frac{295}{42}\right) = 4 - \frac{295/42}{7/4} = -\frac{2}{147} , \end{aligned}$$

$$\begin{aligned} s_4 &= f(x_{1+1}, y_1 + \Delta x \cdot s_3) \\ &= f\left(2, 7 + \frac{1}{2} \cdot \left(-\frac{2}{147}\right)\right) = f\left(2, -\frac{1,028}{147}\right) = 4 - \frac{1,028/147}{2} = \frac{74}{147} \end{aligned}$$

and

$$\begin{aligned} y_2 &= y_1 + \frac{\Delta x}{6} [s_1 + 2s_2 + 2s_3 + s_4] \\ &= 7 + \frac{1}{2} \cdot \frac{1}{6} \left[-\frac{2}{3} + 2 \cdot \frac{2}{21} + 2 \cdot \left(-\frac{2}{147}\right) + \frac{74}{147} \right] = 7 . \end{aligned}$$

In summary:

| k | x_k | y_k |
|-----|---------------|-------|
| 0 | 1 | 8 |
| 1 | $\frac{3}{2}$ | 7 |
| 2 | 2 | 7 |

