Chapter 10: Numerical Methods I: The Euler Method

10.1 a. The differential equation, $\frac{dy}{dx} = \frac{y}{x}$, already is in derivative formula form. So

$$f(x,y) = \frac{y}{x} \quad .$$

From the initial condition and given parameters:

$$x_0 = 1 \quad , \quad y_0 = -1 \quad , \quad \Delta x = \frac{1}{3} \quad , \quad N = 3$$

and

$$x_{\max} = x_0 + N\Delta x = 1 + 3 \cdot \frac{1}{3} = 2 \quad .$$

Since we are doing these computations "by hand", let us do a little "pre-computing" to save time and space later:

$$x_{k+1} = x_k + \Delta x = x_k + \frac{1}{3}$$

and

$$y_{k+1} = y_k + \Delta x f(x_k, y_k) = y_k + \frac{1}{3} \cdot \frac{y_k}{x_k} = y_k + \frac{y_k}{3x_k} \quad .$$

Repeatedly using these with $k = 0, 1, \ldots$:
With $k = 0$:

$$x_1 = x_0 + \frac{1}{3} = 1 + \frac{1}{3} = \frac{4}{3}$$

and

$$y_1 = y_0 + \frac{y_0}{3x_0} = -1 + \frac{-1}{3 \cdot 1} = -\frac{4}{3}$$

With $k = 1$

$$x_2 = x_1 + \frac{1}{3} = \frac{4}{3} + \frac{1}{3} = \frac{5}{3}$$

and

$$y_2 = y_1 + \frac{y_1}{3x_1} = -\frac{4}{3} + \frac{-4/3}{3 \cdot 4/3} = -\frac{5}{3}$$

With $k = 2$

$$x_3 = x_2 + \frac{1}{3} = \frac{5}{3} + \frac{1}{3} = 2 \quad (= x_{\max})$$

and

$$y_3 = y_2 + \cdot\frac{y_2}{3x_2} = -\frac{5}{3} + \frac{-5/3}{3 \cdot 5/3} = -2 \quad .$$

In summary:

k	x_k	y_k
0	1	-1
1	$4/3$	$-4/3$
2	$5/3$	$-5/3$
3	2	-2

10.1 c. $2x + 5\frac{dy}{dx} = y \quad \longmapsto \quad \frac{dy}{dx} = \frac{1}{5}[y - 2x] \quad \longmapsto \quad f(x,y) = \frac{1}{5}[y - 2x] \quad .$

From the initial condition and given parameters:

$$x_0 = 0 \quad , \quad y_0 = 2 \quad , \quad \Delta x = \frac{1}{2} \quad , \quad x_{\max} = 2$$

and

$$N = \frac{x_{\max} - x_0}{\Delta x} = \frac{2 - 0}{1/2} = 4 \quad .$$

Since we are doing these computations "by hand", let us do a little "pre-computing" to save time and space later:

$$x_{k+1} = x_k + \Delta x = x_k + \frac{1}{2}$$

and

$$\begin{aligned} y_{k+1} &= y_k + \Delta x \cdot f(x_k, y_k) \\ &= y_k + \frac{1}{2} \cdot \frac{1}{5}[y_k - 2x_k] = y_k + \frac{1}{10}[y_k - 2x_k] \quad . \end{aligned}$$

Repeatedly using these with $k = 0, 1, \ldots$:
With $k = 0$:

$$x_1 = 0 + \frac{1}{2} = \frac{1}{2} \quad ,$$

and

$$y_1 = y_k + \frac{1}{10}[y_k - 2x_k] = 2 + \frac{1}{10}[2 - 2 \cdot 0] = \frac{11}{5} \quad .$$

With $k = 1$:

$$x_2 = x_1 + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1$$

and

$$y_2 = y_1 + \frac{1}{10}[y_1 - 2x_1] = \frac{11}{5} + \frac{1}{10}\left[\frac{11}{5} - 2 \cdot \frac{1}{2}\right] = \frac{58}{25} \quad .$$

With $k = 2$:

$$x_3 = x_2 + \frac{1}{2} = 1 + \frac{1}{2} = \frac{3}{2}$$

and

$$y_3 = y_2 + \frac{1}{10}[y_2 - 2x_2] = \frac{58}{25} + \frac{1}{10}\left[\frac{58}{25} - 2 \cdot 1\right] = \frac{294}{125} \quad .$$

With $k = 3$:

$$x_4 = x_3 + \frac{1}{2} = \frac{3}{2} + \frac{1}{2} = 2 \quad (= x_{\max})$$

and

$$y_4 = y_3 + \frac{1}{10}[y_3 - 2x_3] = \frac{294}{125} + \frac{1}{10}\left[\frac{294}{125} - 2 \cdot \frac{3}{2}\right] = \frac{2{,}859}{1{,}250} \quad .$$

In summary:

k	x_k	y_k
0	0	2
1	$1/2$	$11/5$
2	1	$58/25$
3	$3/2$	$294/125$
4	2	$2{,}859/1{,}250$

10.2 a. $4\dfrac{dy}{dx} = \sqrt{2x + y} \quad \rightarrowtail \quad f(x, y) = \sqrt{2x + y} \quad .$

From the initial condition and given parameters:

$$x_0 = 0 \quad , \quad y_0 = 0 \quad , \quad \Delta x = \frac{1}{2} \quad , \quad N = 6$$

and

$$x_{\max} = N\Delta x = 6 \cdot \frac{1}{2} = 3 \quad .$$

So, for each step,

$$x_{k+1} = x_k + \Delta x = x_k + \frac{1}{2}$$

and

$$y_{k+1} = y_k + \Delta x f(x_k, y_k) = y_k + \frac{1}{2}\sqrt{2x_k + y_k} \quad .$$

For brevity, we'll simply note that

$$x_1 = \frac{1}{2} \quad , \quad x_2 = 1 \quad , \quad x_3 = \frac{3}{2} \quad , \quad x_4 = 2 \quad , \quad x_5 = \frac{5}{2} \quad \text{and} \quad x_6 = 3 \quad .$$

Repeatedly using the formula for y_k with a calculator and rounding off the long decimals:

$$y_1 = y_0 + \frac{1}{2}\sqrt{2x_0 + y_0} = 0 + \frac{1}{2}\sqrt{2 \cdot 0 + 0} = 0 \quad ,$$

$$y_2 = y_1 + \frac{1}{2}\sqrt{2x_1 + y_1} = 0 + \frac{1}{2}\sqrt{2 \cdot 0.5 + 0} = \frac{1}{2} \quad ,$$

$$y_3 = y_2 + \frac{1}{2}\sqrt{2x_2 + y_2} = \frac{1}{2} + \frac{1}{2}\sqrt{2 \cdot 1 + \frac{1}{2}} = 1.29056941\ldots$$

$$\rightarrowtail \quad \text{We'll use: } y_3 = 1.2906 \quad \text{(rounding off)} \quad ,$$

$$y_4 = y_3 + \frac{1}{2}\sqrt{2x_3 + y_3}$$

$$= 1.2906 + \frac{1}{2}\sqrt{2 \cdot \frac{3}{2} + 1.2906\ldots} = 2.326288177\ldots$$

$$\rightarrowtail \quad \text{We'll use: } y_4 = 2.3263 \quad \text{(rounding off)} \quad ,$$

$$y_5 = y_4 + \frac{1}{2}\sqrt{2x_4 + y_4}$$

$$= 2.3263 + \frac{1}{2}\sqrt{2 \cdot 2 + 2.3263} = 3.58390686\ldots$$

$$\rightarrowtail \quad \text{We'll use: } y_5 = 3.5839 \quad \text{(rounding off)} \quad ,$$

and

$$y_6 = y_5 + \frac{1}{2}\sqrt{2x_5 + y_5}$$

$$= 3.5839 + \frac{1}{2}\sqrt{2 \cdot \frac{5}{2} + 3.5839} = 5.04881467\ldots$$

$$\rightarrowtail \quad \text{We'll use: } y_6 = 5.0488 \quad \text{(rounding off)} \quad .$$

In summary:

k	x_k	y_k
0	0	0.0000
1	.5	0.0000
2	1.0	0.5000
3	1.5	1.2906
4	2.0	2.3263
5	2.5	3.5839
6	3.0	5.0488

(Note: The values obtained for y_3 to y_6 will depend on the number of decimal places kept when rounding off.)

10.2 c. $\dfrac{dy}{dx} = y^x \quad \rightarrowtail \quad f(x,y) = y^x \quad .$

From the initial condition and given parameters:

$$x_0 = 1 \quad , \quad y_0 = 2 \quad , \quad \Delta x = 0.1 \quad , \quad x_{\max} = 1.5$$

and

$$N = \frac{x_{\max} - x_0}{\Delta x} = \frac{1.5 - 1.0}{0.1} = 5 \quad .$$

So, for each step,

$$x_{k+1} = x_k + \Delta x = x_k + 0.1$$

and

$$y_{k+1} = y_k + \Delta x f(x_k, y_k) = y_k + \frac{1}{10}(y_k)^{x_k} \quad .$$

For brevity, we'll simply note that

$$x_1 = 1.1 \quad , \quad x_2 = 1.2 \quad , \quad x_3 = 1.3 \quad , \quad x_4 = 1.4 \quad \text{and} \quad x_5 = 1.5 \quad .$$

Repeatedly using the formula for y_k with a calculator and rounding off the long decimals:

$$y_1 = y_0 + \tfrac{1}{10}(y_0)^{x_0} = 2 + \tfrac{1}{10}(2)^1 = 2.2 \quad ,$$

$$y_2 = y_1 + \tfrac{1}{10}(y_1)^{x_1} = 2.2 + \tfrac{1}{10}(2.2)^{1.1} = 2.43804822\ldots$$

$$\rightarrowtail \quad \text{We'll use: } y_2 = 2.4380 \quad \text{(rounding off)} \quad ,$$

$$y_3 = y_2 + \tfrac{1}{10}(y_2)^{x_2} = 2.4380 + \tfrac{1}{10}(2.4380)^{1.2} = 2.729367053\ldots$$

$$\rightarrowtail \quad \text{We'll use: } y_3 = 2.7294 \quad \text{(rounding off)} \quad ,$$

$$y_4 = y_3 + \tfrac{1}{10}(y_3)^{x_3} = 2.7294 + \tfrac{1}{10}(2.7294)^{1.3} = 3.098281898\ldots$$

$$\rightarrowtail \quad \text{We'll use: } y_4 = 3.0984 \quad \text{(rounding off)} \quad ,$$

and

$$y_5 = y_4 + \tfrac{1}{10}(y_4)^{x_4} = 3.0984 + \tfrac{1}{10}(3.0984)^{1.4} = 3.585471230\ldots$$

$$\rightarrowtail \quad \text{We'll use: } y_5 = 3.5855 \quad \text{(rounding off)} \quad .$$

In summary:

k	x_k	y_k
0	1.0	2.0000
1	1.1	2.2000
2	1.2	2.4380
3	1.3	2.7294
4	1.4	3.0984
5	1.5	3.5855

(Note: The values obtained for y_2 to y_5 will depend on the number of decimal places kept when rounding off.)