

Chapter 9: Euler's Numerical Method

9.1 a. The differential equation, $\frac{dy}{dx} = \frac{y}{x}$, already is in derivative formula form. So

$$f(x, y) = \frac{y}{x} .$$

From the initial condition and given parameters:

$$x_0 = 1 \quad , \quad y_0 = -1 \quad , \quad \Delta x = \frac{1}{3} \quad , \quad N = 3$$

and

$$x_{\max} = x_0 + N\Delta x = 1 + 3 \cdot \frac{1}{3} = 2 .$$

So, for each step,

$$x_{k+1} = x_k + \Delta x = x_k + \frac{1}{3}$$

and

$$y_{k+1} = y_k + \Delta x f(x_k, y_k) = y_k + \frac{1}{3} \cdot \frac{y_k}{x_k} .$$

Repeatedly using these with $k = 0$ to $k = N - 1 = 2$:

$$x_1 = x_0 + \frac{1}{3} = 1 + \frac{1}{3} = \frac{4}{3} ,$$

$$y_1 = y_0 + \frac{1}{3} \cdot \frac{y_0}{x_0} = -1 + \frac{1}{3} \cdot \frac{-1}{1} = -\frac{4}{3}$$

and

$$x_2 = x_1 + \frac{1}{3} = \frac{4}{3} + \frac{1}{3} = \frac{5}{3} ,$$

$$y_2 = y_1 + \frac{1}{3} \cdot \frac{y_1}{x_1} = -\frac{4}{3} + \frac{1}{3} \cdot \frac{-4/3}{4/3} = -\frac{5}{3}$$

and

$$x_3 = x_2 + \frac{1}{3} = \frac{5}{3} + \frac{1}{3} = 2 \quad (= x_{\max}) ,$$

$$y_3 = y_2 + \frac{1}{3} \cdot \frac{y_2}{x_2} = -\frac{5}{3} + \frac{1}{3} \cdot \frac{-5/3}{5/3} = -2 .$$

In summary:

k	x_k	y_k
0	1	-1
1	$\frac{4}{3}$	$-\frac{4}{3}$
2	$\frac{5}{3}$	$-\frac{5}{3}$
3	2	-2

9.1 c. $4x + \frac{dy}{dx} = y^2 \rightsquigarrow \frac{dy}{dx} = y^2 - 4x \rightsquigarrow f(x, y) = y^2 - 4x .$

From the initial condition and given parameters:

$$x_0 = 0 \quad , \quad y_0 = 2 \quad , \quad \Delta x = \frac{1}{2} \quad , \quad x_{\max} = 2$$

and

$$N = \frac{x_{\max} - x_0}{\Delta x} = \frac{2 - 0}{1/2} = 4 .$$

So, for each step,

$$x_{k+1} = x_k + \Delta x = x_k + \frac{1}{2}$$

and

$$y_{k+1} = y_k + \Delta x f(x_k, y_k) = y_k + \frac{1}{2} [y_k^2 - 4x_k] .$$

Repeatedly using these with $k = 0$ to $k = N - 1 = 3$:

$$x_1 = x_0 + \frac{1}{2} = 0 + \frac{1}{2} = \frac{1}{2} ,$$

$$y_1 = y_0 + \frac{1}{2} [y_0^2 - 4x_0] = 2 + \frac{1}{2} [2^2 - 4 \cdot 0] = 4$$

and

$$x_2 = x_1 + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1 ,$$

$$y_2 = y_1 + \frac{1}{2} [y_1^2 - 4x_1] = 4 + \frac{1}{2} [4^2 - 4 \cdot \frac{1}{2}] = 11$$

and

$$x_3 = x_2 + \frac{1}{2} = 1 + \frac{1}{2} = \frac{3}{2} ,$$

$$y_3 = y_2 + \frac{1}{2} [y_2^2 - 4x_2] = 11 + \frac{1}{2} [11^2 - 4 \cdot 1] = \frac{139}{2}$$

and

$$x_4 = x_3 + \frac{1}{2} = \frac{3}{2} + \frac{1}{2} = 2 \quad (= x_{\max}) ,$$

$$y_4 = y_3 + \frac{1}{2} [y_3^2 - 4x_3] = \frac{139}{2} + \frac{1}{2} \left[\left(\frac{139}{2} \right)^2 - 4 \cdot \frac{3}{2} \right] = \frac{19853}{8} .$$

In summary:

k	x_k	y_k
0	0	2
1	$\frac{1}{2}$	4
2	1	11
3	$\frac{3}{2}$	$\frac{139}{2}$
4	2	$\frac{19853}{8}$

9.2 a. $4 \frac{dy}{dx} = \sqrt{2x + y} \rightsquigarrow f(x, y) = \sqrt{2x + y} .$

From the initial condition and given parameters:

$$x_0 = 0 , \quad y_0 = 0 , \quad \Delta x = \frac{1}{2} , \quad N = 6$$

and

$$x_{\max} = N \Delta x = 6 \cdot \frac{1}{2} = 3 .$$

So, for each step,

$$x_{k+1} = x_k + \Delta x = x_k + \frac{1}{2}$$

and

$$y_{k+1} = y_k + \Delta x f(x_k, y_k) = y_k + \frac{1}{2} \sqrt{2x_k + y_k} .$$

For brevity, we'll simply note that

$$x_1 = \frac{1}{2} , \quad x_2 = 1 , \quad x_3 = \frac{3}{2} , \quad x_4 = 2 , \quad x_5 = \frac{5}{2} \quad \text{and} \quad x_6 = 3 .$$

Repeatedly using the formula for y_k with a calculator and rounding off the long decimals:

$$y_1 = y_0 + \frac{1}{2}\sqrt{2x_0 + y_0} = 0 + \frac{1}{2}\sqrt{2 \cdot 0 + 0} = 0 \quad ,$$

$$y_2 = y_1 + \frac{1}{2}\sqrt{2x_1 + y_1} = 0 + \frac{1}{2}\sqrt{2 \cdot 0.5 + 0} = \frac{1}{2} \quad ,$$

$$y_3 = y_2 + \frac{1}{2}\sqrt{2x_2 + y_2} = \frac{1}{2} + \frac{1}{2}\sqrt{2 \cdot 1 + \frac{1}{2}} = 1.29056941 \dots$$

$$\rightsquigarrow \text{ We'll use: } y_3 = 1.2906 \quad (\text{rounding off}) \quad ,$$

$$y_4 = y_3 + \frac{1}{2}\sqrt{2x_3 + y_3}$$

$$= 1.2906 + \frac{1}{2}\sqrt{2 \cdot \frac{3}{2} + 1.2906} \dots = 2.326288177 \dots$$

$$\rightsquigarrow \text{ We'll use: } y_4 = 2.3263 \quad (\text{rounding off}) \quad ,$$

$$y_5 = y_4 + \frac{1}{2}\sqrt{2x_4 + y_4}$$

$$= 2.3263 + \frac{1}{2}\sqrt{2 \cdot 2 + 2.3263} = 3.58390686 \dots$$

$$\rightsquigarrow \text{ We'll use: } y_5 = 3.5839 \quad (\text{rounding off}) \quad ,$$

and

$$y_6 = y_5 + \frac{1}{2}\sqrt{2x_5 + y_5}$$

$$= 3.5839 + \frac{1}{2}\sqrt{2 \cdot \frac{5}{2} + 3.5839} = 5.04881467 \dots$$

$$\rightsquigarrow \text{ We'll use: } y_6 = 5.0488 \quad (\text{rounding off}) \quad .$$

In summary:

k	x_k	y_k
0	0	0.0000
1	.5	0.0000
2	1.0	0.5000
3	1.5	1.2906
4	2.0	2.3263
5	2.5	3.5839
6	3.0	5.0488

(Note: The values obtained for y_3 to y_6 will depend on the number of decimal places kept when rounding off.)

9.2 c. $\frac{dy}{dx} = y^x \rightsquigarrow f(x, y) = y^x \quad .$

From the initial condition and given parameters:

$$x_0 = 1 \quad , \quad y_0 = 2 \quad , \quad \Delta x = 0.1 \quad , \quad x_{\max} = 1.5$$

and

$$N = \frac{x_{\max} - x_0}{\Delta x} = \frac{1.5 - 1.0}{0.1} = 5 \quad .$$

So, for each step,

$$x_{k+1} = x_k + \Delta x = x_k + 0.1$$

and

$$y_{k+1} = y_k + \Delta x f(x_k, y_k) = y_k + \frac{1}{10}(y_k)^{x_k} \quad .$$

For brevity, we'll simply note that

$$x_1 = 1.1 \quad , \quad x_2 = 1.2 \quad , \quad x_3 = 1.3 \quad , \quad x_4 = 1.4 \quad \text{and} \quad x_5 = 1.5 \quad .$$

Repeatedly using the formula for y_k with a calculator and rounding off the long decimals:

$$y_1 = y_0 + \frac{1}{10}(y_0)^{x_0} = 2 + \frac{1}{10}(2)^1 = 2.2 \quad ,$$

$$y_2 = y_1 + \frac{1}{10}(y_1)^{x_1} = 2.2 + \frac{1}{10}(2.2)^{1.1} = 2.43804822\dots$$

$$\rightsquigarrow \text{ We'll use: } y_2 = 2.4380 \quad (\text{rounding off}) \quad ,$$

$$y_3 = y_2 + \frac{1}{10}(y_2)^{x_2} = 2.4380 + \frac{1}{10}(2.4380)^{1.2} = 2.729367053\dots$$

$$\rightsquigarrow \text{ We'll use: } y_3 = 2.7294 \quad (\text{rounding off}) \quad ,$$

$$y_4 = y_3 + \frac{1}{10}(y_3)^{x_3} = 2.7294 + \frac{1}{10}(2.7294)^{1.3} = 3.098281898\dots$$

$$\rightsquigarrow \text{ We'll use: } y_4 = 3.0984 \quad (\text{rounding off}) \quad ,$$

and

$$y_5 = y_4 + \frac{1}{10}(y_4)^{x_4} = 3.0984 + \frac{1}{10}(3.0984)^{1.4} = 3.585471230\dots$$

$$\rightsquigarrow \text{ We'll use: } y_5 = 3.5855 \quad (\text{rounding off}) \quad .$$

In summary:

k	x_k	y_k
0	1.0	2.0000
1	1.1	2.2000
2	1.2	2.4380
3	1.3	2.7294
4	1.4	3.0984
5	1.5	3.5855

(Note: The values obtained for y_2 to y_5 will depend on the number of decimal places kept when rounding off.)