

**Chapter 7: The Exact Form and General Integrating Factors**

**7.1 a.** By definition  $\phi = 3xy$  is potential function for

$$M + N \frac{dy}{dx} = 0$$

if

$$M = \frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}[3xy] = 3y \quad \text{and} \quad N = \frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}[3xy] = 3x \quad .$$

So,  $\phi = 3xy$  is potential function for

$$3y + 3x \frac{dy}{dx} = 0 \quad .$$

An implicit solution is given by  $\phi = c$ ; that is,

$$3xy = c \quad .$$

Solving this for  $y$  gives the explicit solution  $y = \frac{c}{3x} = \frac{C}{x}$  .

**7.1 c.** By definition  $\phi = x^2y - xy^3$  is potential function for

$$M + N \frac{dy}{dx} = 0$$

if

$$M = \frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x} [x^2y - xy^3] = 2xy - y^3$$

and

$$N = \frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y} [x^2y - xy^3] = x^2 - 3xy^2 \quad .$$

So,  $\phi = x^2y - xy^3$  is potential function for

$$2xy - y^3 + [x^2 - 3xy^2] \frac{dy}{dx} = 0 \quad .$$

An implicit solution is given by  $\phi = c$ ; that is,

$$x^2y - xy^3 = c \quad .$$

Solving this for  $y$  to obtain an explicit solution is, alas, not practical.

**7.2 a.** By subtracting the left side of the equation from both sides, and then multiplying both sides by  $2xy$ ,

$$\frac{dy}{dx} = \frac{1}{y} - \frac{y}{2x} \quad \rightsquigarrow \quad \frac{y}{2x} - \frac{1}{y} + \frac{dy}{dx} = 0$$

$$\Leftrightarrow \quad \underbrace{[y^2 - 2x]}_M + \underbrace{2xy \frac{dy}{dx}}_N = 0 \quad .$$

To show this equation is exact with potential function  $\phi = xy^2 - x^2$ , we only need to show that  $\frac{\partial \phi}{\partial x} = M$  and  $\frac{\partial \phi}{\partial y} = N$  where  $M$  and  $N$  are as indicated above. Checking this, we see that

$$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x} [xy^2 - x^2] = y^2 - 2x = M$$

and

$$\frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y} [xy^2 - x^2] = 2xy = N \quad ,$$

confirming the claim.

**7.2 b.** Starting with the implicit solution  $\phi = c$ ,

$$xy^2 - x^2 = c \quad \rightsquigarrow \quad y^2 = \frac{x^2 + c}{x} = x + \frac{c}{x} \quad \rightsquigarrow \quad y = \pm \sqrt{x + \frac{c}{x}} \quad .$$

**7.2 c.** By definition, the equation  $M + N^{dy/dx} = 0$   $\psi$  is in exact form and has potential function  $\psi$  if

$$\frac{\partial \psi}{\partial x} = e^{xy^2 - x^2} [y^2 - 2x] \quad \text{and} \quad \frac{\partial \psi}{\partial y} = e^{xy^2 - x^2} 2xy \quad .$$

Computing the derivatives, we get that, in fact,

$$\frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x} e^{xy^2 - x^2} = e^{xy^2 - x^2} [y^2 - 2x]$$

and

$$\frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y} e^{xy^2 - x^2} = e^{xy^2 - x^2} 2xy \quad ,$$

confirming the claim.

**7.4 a.** Here, the potential function  $\phi(x, y)$  must satisfy the pair of equations

$$\frac{\partial \phi}{\partial x} = M(x, y) = 2xy + y^2 \quad \text{and} \quad \frac{\partial \phi}{\partial y} = N(x, y) = 2xy + x^2 \quad .$$

Integrating the first equation with respect to  $x$  yields

$$\phi(x, y) = \int \frac{\partial \phi}{\partial x} dx = \int [2xy + y^2] dx = x^2y + xy^2 + h(y) \quad (\star)$$

where  $h(y)$  is a yet undetermined function of  $y$ . To determine this function, plug the last formula for  $\phi$  into the second of the initial pair of equations:

$$\frac{\partial \phi}{\partial y} = 2xy + x^2$$

$$\hookrightarrow \quad \frac{\partial}{\partial y} [x^2y + xy^2 + h(y)] = 2xy + x^2$$

$$\hookrightarrow \quad x^2 + 2xy + h'(y) = 2xy + x^2$$

$$\hookrightarrow \quad h'(y) = 0 \quad .$$

Hence, since  $h(y)$  is a function of  $y$  only,

$$h(y) = \int h'(y) dy = \int 0 dy = c_1 \quad ,$$

and formula (★) for  $\phi$  becomes

$$\phi(x, y) = x^2y + xy^2 + h(y) = x^2y + xy^2 + c_1 .$$

An implicit solution to the differential equation is then given by  $\phi = c_2$ ,

$$x^2y + xy^2 + c_1 = c_2 ,$$

which can be rewritten as

$$xy^2 + x^2y + C = 0 \quad (\text{with } c = c_1 - c_2) .$$

Solving for  $y$  via the quadratic formula then yields the explicit solution

$$y(x) = \frac{-x^2 \pm \sqrt{(x^2)^2 - 4xc}}{2x} = -\frac{x}{2} \pm \frac{1}{2}\sqrt{x^2 + \frac{C}{x}} .$$

**7.4 c.** Here, the potential function  $\phi(x, y)$  must satisfy

$$\frac{\partial\phi}{\partial x} = M(x, y) = 2 - 2x \quad \text{and} \quad \frac{\partial\phi}{\partial y} = N(x, y) = 3y^2 .$$

Integrating the first equation with respect to  $x$  yields

$$\phi(x, y) = \int \frac{\partial\phi}{\partial x} dx = \int [2 - 2x] dx = 2x - x^2 + h(y) . \quad (\star)$$

Combining this formula for  $\phi$  with the second of the initial pair of equations:

$$3y^2 = \frac{\partial\phi}{\partial y} = \frac{\partial}{\partial y} [2x - x^2 + h(y)] = h'(y) .$$

Hence,

$$h(y) = \int h'(y) dy = \int 3y^2 dy = y^3 + c_1 ,$$

and formula (★) for  $\phi$  becomes

$$\phi(x, y) = 2x - x^2 + h(y) = 2x - x^2 + y^3 + c_1 .$$

An implicit solution to the differential equation is then given by  $\phi = c_2$ ,

$$2x - x^2 + y^3 + c_1 = c_2 .$$

Solving for  $y$  then yields the explicit solution

$$y(x) = (x^2 - 2x + C)^{1/3} .$$

**7.4 e.** Here, the potential function  $\phi(x, y)$  must satisfy

$$\frac{\partial\phi}{\partial x} = M(x, y) = 4x^3y \quad \text{and} \quad \frac{\partial\phi}{\partial y} = N(x, y) = x^4 - y^4 .$$

Integrating the first equation with respect to  $x$  yields

$$\phi(x, y) = \int \frac{\partial\phi}{\partial x} dx = \int 4x^3y dx = x^4y + h(y) . \quad (\star)$$

Combining this formula for  $\phi$  with the second of the initial pair of equations:

$$x^4 - y^4 = \frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y} [x^4 y + h(y)] = x^4 + h'(y) \quad .$$

Hence,  $h'(y) = -y^4$ ,

$$h(y) = \int h'(y) dy = - \int y^4 dy = -\frac{1}{5}y^5 + c_1 \quad ,$$

and formula (★) for  $\phi$  becomes

$$\phi(x, y) = x^4 y + h(y) = x^4 y - \frac{1}{5}y^5 + c_1 \quad .$$

An implicit solution to the differential equation is then given by  $\phi = c_2$ , which, after combining arbitrary constants, is

$$x^4 y - \frac{1}{5}y^5 = c \quad .$$

Solving this for an explicit solution  $y$  is not practical.

**7.4 g.** Here, the potential function  $\phi(x, y)$  must satisfy

$$\frac{\partial \phi}{\partial x} = M(x, y) = 1 + e^y \quad \text{and} \quad \frac{\partial \phi}{\partial y} = N(x, y) = xe^y \quad .$$

Integrating the first equation with respect to  $x$  yields

$$\phi(x, y) = \int \frac{\partial \phi}{\partial x} dx = \int [1 + e^y] dx = x + xe^y + h(y) \quad . \quad (\star)$$

Combining this formula for  $\phi$  with the second of the initial pair of equations:

$$xe^y = \frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y} [x + xe^y + h(y)] = xe^y + h'(y) \quad .$$

Hence,  $h'(y) = 0$ ,

$$h(y) = \int h'(y) dy = \int 0 dy = c_1 \quad ,$$

and formula (★) for  $\phi$  becomes

$$\phi(x, y) = x + xe^y + h(y) = x + xe^y + c_1 \quad .$$

An implicit solution to the differential equation is then given by  $\phi = c_2$ , which, after combining arbitrary constants, is

$$x + xe^y = c \quad .$$

Solving for  $y$  then yields the explicit solution  $y(x) = \ln \left| \frac{c-x}{x} \right| \quad .$

**7.5 a.** For part i: For  $M + N^{dy/dx} = 0$  to be in exact form, we must have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad .$$

Here

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} [1 + y^4] = 4y^3 \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x} [xy^3] = y^3 .$$

So  $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ , telling us the equation is not in exact form.

For part ii: We need to find a function  $\mu = \mu(x, y)$  such that

$$\frac{\partial}{\partial y} [\mu M] = \frac{\partial}{\partial x} [\mu N] .$$

For this problem, this equation is

$$\frac{\partial}{\partial y} [\mu [1 + y^4]] = \frac{\partial}{\partial x} [\mu [xy^3]] .$$

Let us see if the integrating factor  $\mu$  can be a function of  $x$  only,  $\mu = \mu(x)$ :

$$\frac{\partial}{\partial y} [\mu(x) [1 + y^4]] = \frac{\partial}{\partial x} [\mu(x) [xy^3]]$$

$$\hookrightarrow 4\mu(x)y^3 = \frac{d\mu}{dx}xy^3 + \mu(x)y^3$$

$$\hookrightarrow \frac{d\mu}{dx}xy^3 = 3\mu(x)y^3$$

$$\hookrightarrow \frac{d\mu}{dx} = \frac{3\mu(x)}{x} .$$

The last equation is a differential equation for  $\mu(x)$  not involving the variable  $y$ . A solution (easily found by treating this as a separable or linear differential equation) is

$$\mu(x) = x^3 .$$

This is what we can use as an integrating factor.

For part iii: Multiplying the given differential equation by the integrating factor  $\mu$  just found:

$$x^3 \left[ 1 + y^4 + xy^3 \frac{dy}{dx} \right] = x^3 \cdot 0$$

$$\hookrightarrow x^3 [1 + y^4] + x^4 y^3 \frac{dy}{dx} = 0 .$$

Any potential function  $\phi$  of this equation must satisfy

$$\frac{\partial \phi}{\partial x} = x^3 [1 + y^4] \quad \text{and} \quad \frac{\partial \phi}{\partial y} = x^4 y^3 . \quad (\star)$$

Integrating the first equation with respect to  $x$ :

$$\phi(x, y) = \int \frac{\partial \phi}{\partial x} dx = \int x^3 [1 + y^4] dx = \frac{1}{4} x^4 [1 + y^4] + h(y) .$$

Combining this result with the second equation in  $(\star)$ :

$$x^4 y^3 = \frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y} \left[ \frac{1}{4} x^4 [1 + y^4] + h(y) \right] = x^4 y^3 + h'(y)$$

$$\hookrightarrow h'(y) = 0 \quad \rightsquigarrow \quad h(y) = \int 0 dy = c_1 .$$

So the potential function is

$$\phi(x, y) = \frac{1}{4}x^4 [1 + y^4] + h(y) = \frac{1}{4}x^4 [1 + y^4] + c_1 \quad ,$$

and an implicit solution to the given differential equation is

$$\frac{1}{4}x^4 [1 + y^4] + c_1 = c_2 \quad ,$$

which we rewrite as

$$x^4 [1 + y^4] = C \quad .$$

Solving this for  $y$  then yields  $y(x) = \pm (Cx^4 - 1)^{1/4}$  .

**7.5 c.** For part i: Here

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} [2x^{-1}y] = 2x^{-1}$$

and

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x} [4x^2y - 3] = 8xy \quad .$$

So  $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ , telling us the equation is not in exact form.

For part ii: We need to find a function  $\mu = \mu(x, y)$  such that

$$\frac{\partial}{\partial y} [2\mu x^{-1}y] = \frac{\partial}{\partial y} [\mu M] = \frac{\partial}{\partial x} [\mu N] = \frac{\partial}{\partial x} [\mu [4x^2y - 3]] \quad .$$

Let us see if an integrating factor  $\mu$  can be a function of  $x$  only,  $\mu = \mu(x)$ :

$$\frac{\partial}{\partial y} [\mu(x) [2x^{-1}y]] = \frac{\partial}{\partial x} [\mu(x) [4x^2y - 3]]$$

$$\hookrightarrow \mu(x) [2x^{-1}] = \frac{d\mu}{dx} [4x^2y - 3] + \mu(x) [8xy]$$

$$\hookrightarrow \frac{d\mu}{dx} [4x^2y - 3] = -\mu(x) [8xy - 2x^{-1}]$$

$$\hookrightarrow \frac{d\mu}{dx} [4x^2y - 3] = -2x^{-1}\mu(x) [4x^2y - 1] \quad .$$

In this case, we clearly cannot cancel out the  $y$  variable, and cannot obtain a differential equation for  $\mu = \mu(x)$  in terms of  $x$  and  $\mu$  only. So there is not an integrating factor of  $x$  alone.

Attempting to find an integrating factor of  $y$  only,  $\mu = \mu(y)$ :

$$\frac{\partial}{\partial y} [\mu(y) [2x^{-1}y]] = \frac{\partial}{\partial x} [\mu(y) [4x^2y - 3]]$$

$$\hookrightarrow \frac{d\mu}{dy} [2x^{-1}y] + \mu(y) [2(y)x^{-1}] = \mu(y) [8xy]$$

$$\hookrightarrow \frac{d\mu}{dy} [2x^{-1}y] = \mu(y) [8xy - 2x^{-1}]$$

$$\hookrightarrow \frac{d\mu}{dy} = \mu(y) [4x^2 - y^{-1}] \quad .$$

Since we clearly cannot obtain a differential equation for  $\mu = \mu(y)$  in terms of  $y$  and  $\mu$  only, there is not an integrating factor of  $y$  alone.

Finally, let us attempt to find an integrating factor of the form  $\mu = \mu(x, y) = x^\alpha y^\beta$ :

$$\begin{aligned} \frac{\partial}{\partial y} [\mu [2x^{-1}y]] &= \frac{\partial}{\partial x} [\mu [4x^2y - 3]] \\ \Leftrightarrow \frac{\partial}{\partial y} [x^\alpha y^\beta [2x^{-1}y]] &= \frac{\partial}{\partial x} [x^\alpha y^\beta [4x^2y - 3]] \\ \Leftrightarrow \frac{\partial}{\partial y} [2x^{\alpha-1}y^{\beta+1}] &= \frac{\partial}{\partial x} [4x^{\alpha+2}y^{\beta+1} - 3x^\alpha y^\beta] \\ \Leftrightarrow 2(\beta+1)x^{\alpha-1}y^\beta &= 4(\alpha+2)x^{\alpha+1}y^{\beta+1} - 3\alpha x^{\alpha-1}y^\beta \end{aligned}$$

Combining like terms yields

$$[2(\beta+1) + 3\alpha]x^{\alpha-1}y^\beta - 4(\alpha+2)x^{\alpha+1}y^{\beta+1} = 0 \quad ,$$

which requires that

$$2(\beta+1) + 3\alpha = 0 \quad \text{and} \quad -4(\alpha+2) = 0 \quad ,$$

and from which it follows that

$$\alpha = -2 \quad \text{and} \quad \beta = -1 - \frac{3}{2}\alpha = -1 - \frac{3}{2}(-2) = 2 \quad .$$

So,

$$\mu = \mu(x, y) = x^\alpha y^\beta = x^{-2}y^2 \quad .$$

*For part iii:* Multiplying the given differential equation by the integrating factor  $\mu$  just found:

$$\begin{aligned} x^{-2}y^2 \left[ 2x^{-1}y + [4x^2y - 3] \frac{dy}{dx} \right] &= x^{-2}y^2 \cdot 0 \\ \Leftrightarrow 2x^{-3}y^3 + [4y^3 - 3x^{-2}y^2] \frac{dy}{dx} &= 0 \quad . \end{aligned}$$

Any potential function  $\phi$  of this equation must satisfy

$$\frac{\partial \phi}{\partial x} = 2x^{-3}y^3 \quad \text{and} \quad \frac{\partial \phi}{\partial y} = 4y^3 - 3x^{-2}y^2 \quad . \quad (\star)$$

Integrating the first with respect to  $x$ :

$$\phi(x, y) = \int \frac{\partial \phi}{\partial x} dx = \int 2x^{-3}y^3 dx = -x^{-2}y^3 + h(y) \quad .$$

Combining this result with the second equation in  $(\star)$ :

$$\begin{aligned} 4y^3 - 3x^{-2}y^2 &= \frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y} [-x^{-2}y^3 + h(y)] \\ \Leftrightarrow 4y^3 - 3x^{-2}y^2 &= -3x^{-2}y^2 + h'(y) \\ \Leftrightarrow h'(y) = 4y^3 &\rightsquigarrow h(y) = \int 4y^3 dy = y^4 + c_1 \quad . \end{aligned}$$

So the potential function is

$$\phi(x, y) = -x^{-2}y^3 + h(y) = -x^{-2}y^3 + h(y) = -x^{-2}y^3 + y^4 + c_1 ,$$

and an implicit solution to the given differential equation is

$$-x^{-2}y^3 + y^4 + c_1 = c_2 ,$$

which we rewrite as

$$y^4 - x^{-2}y^3 = c ,$$

but will not attempt to solve for  $y$ .

**7.5 e.** For part i: Here

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} [3y + 3y^2] = 3 + 6y$$

and

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x} [2x + 4xy] = 2 + 4y .$$

So  $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ , telling us the equation is not in exact form.

For part ii: We need to find a function  $\mu = \mu(x, y)$  such that

$$\frac{\partial}{\partial y} [\mu [3y + 3y^2]] = \frac{\partial}{\partial y} [\mu M] = \frac{\partial}{\partial x} [\mu N] = \frac{\partial}{\partial x} [\mu [2x + 4xy]] .$$

Let us see if an integrating factor  $\mu$  can be a function of  $x$  only,  $\mu = \mu(x)$ :

$$\frac{\partial}{\partial y} [\mu(x) [3y + 3y^2]] = \frac{\partial}{\partial x} [\mu(x) [2x + 4xy]]$$

$$\hookrightarrow \mu(x) [3 + 6y] = \frac{d\mu}{dx} [2x + 4xy] + \mu(x) [2 + 4y]$$

$$\hookrightarrow \frac{d\mu}{dx} [2x + 4xy] = \mu(x) [3 + 6y - 2 - 4y]$$

$$\hookrightarrow \frac{d\mu}{dx} 2x [1 + 2y] = \mu(x) [1 + 2y]$$

$$\hookrightarrow \frac{d\mu}{dx} = \frac{\mu(x)}{2x} .$$

The last equation is a differential equation for  $\mu(x)$  not involving the variable  $y$ . A solution (easily found by treating this as a separable or linear differential equation) is

$$\mu(x) = x^{1/2} .$$

This is what we can use as an integrating factor.

For part iii: Multiplying the given differential equation by the integrating factor  $\mu$  just found:

$$x^{1/2} [3y + 3y^2 + [2x + 4xy] \frac{dy}{dx}] = x^{1/2} \cdot 0$$

$$\hookrightarrow 3x^{1/2} [y + y^2] + 2x^{3/2} [1 + 2y] \frac{dy}{dx} = 0 .$$



Any potential function  $\phi$  of this equation must satisfy

$$\frac{\partial \phi}{\partial x} = 3x^{1/2} [y + y^2] \quad \text{and} \quad \frac{\partial \phi}{\partial y} = 2x^{3/2} [1 + 2y] \quad . \quad (\star)$$

Integrating the first equation with respect to  $x$  :

$$\phi(x, y) = \int \frac{\partial \phi}{\partial x} dx = \int 3x^{1/2} [y + y^2] dx = 2x^{3/2} [y + y^2] + h(y) \quad .$$

Combining this result with the second equation in  $(\star)$ :

$$2x^{3/2} [1 + 2y] = \frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y} [2x^{3/2} [y + y^2] + h(y)]$$

$$\hookrightarrow \quad 2x^{3/2} [1 + 2y] = 2x^{3/2} [1 + 2y] + h'(y)$$

$$\hookrightarrow \quad h'(y) = 0 \quad \rightsquigarrow \quad h(y) = \int 0 dy = c_1 \quad .$$

So the potential function is

$$\phi(x, y) = 2x^{3/2} [y + y^2] + h(y) = 2x^{3/2} [y + y^2] + c_1 \quad ,$$

and an implicit solution to the given differential equation is

$$-2x^{3/2} [y + y^2] + c_1 = c_2 \quad ,$$

which we rewrite as

$$2y^2 + 2y + cx^{-3/2} = 0$$

so that we can solve for  $y$  using the quadratic formula,

$$y(x) = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 2cx^{-3/2}}}{2 \cdot 2} = -\frac{1}{2} \pm \frac{1}{2} \sqrt{1 + Cx^{-3/2}} \quad .$$

**7.5 g.** For part i:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} [2y^3] = 6y^2$$

and

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x} [4x^3y^3 - 3xy^2] = 12x^2y^3 - 3y^2 \quad .$$

So  $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ , telling us the equation is not in exact form.

For part ii: We need to find a function  $\mu = \mu(x, y)$  such that

$$\frac{\partial}{\partial y} [2\mu y^3] = \frac{\partial}{\partial y} [\mu M] = \frac{\partial}{\partial x} [\mu N] = \frac{\partial}{\partial x} [\mu [4x^3y^3 - 3xy^2]] \quad .$$

Let us see if an integrating factor  $\mu$  can be a function of  $x$  only,  $\mu = \mu(x)$  :

$$\frac{\partial}{\partial y} [2\mu(x)y^3] = \frac{\partial}{\partial x} [\mu(x) [4x^3y^3 - 3xy^2]]$$

$$\begin{aligned} \hookrightarrow \quad 6\mu(x)y^2 &= \frac{d\mu}{dx} [4x^3y^3 - 3xy^2] \\ &\quad + \mu(x) [12x^2y^3 - 3y^2] \end{aligned}$$

$$\begin{aligned} \hookrightarrow \quad \frac{d\mu}{dx} [4x^3y^3 - 3xy^2] &= -\mu(x) [4x^2y^3 - 3y^2 - 6y^2] \\ \hookrightarrow \quad \frac{d\mu}{dx} [4x^3y^3 - 3xy^2] &= -3\mu(x) [4x^2y^3 - 3y^2] \\ \hookrightarrow \quad \frac{d\mu}{dx} &= -3\mu(x) \quad . \end{aligned}$$

The last equation does not contain the variable  $y$ , and is a simple differential equation for  $\mu(x)$ . A solution (easily found by treating this as a separable or linear differential equation) is

$$\mu(x) = x^{-3} \quad .$$

This is what we can use as an integrating factor.

*For part iii:* Multiplying the given differential equation by the integrating factor  $\mu$  just found:

$$\begin{aligned} x^{-3} \left[ 2y^3 + [4x^3y^3 - 3xy^2] \frac{dy}{dx} \right] &= x^{-3} \cdot 0 \\ \hookrightarrow \quad 2x^{-3}y^3 + [4y^3 - 3x^{-2}y^2] \frac{dy}{dx} &= 0 \quad . \end{aligned}$$

Any potential function  $\phi$  of this equation must satisfy

$$\frac{\partial \phi}{\partial x} = 2x^{-3}y^3 \quad \text{and} \quad \frac{\partial \phi}{\partial y} = 4y^3 - 3x^{-2}y^2 \quad . \quad (\star)$$

Integrating the first equation with respect to  $x$ :

$$\phi(x, y) = \int \frac{\partial \phi}{\partial x} dx = \int 2x^{-3}y^3 dx = -x^{-2}y^3 + h(y) \quad .$$

Combining this result with the second equation in  $(\star)$ :

$$\begin{aligned} 4y^3 - 3x^{-2}y^2 &= \frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y} [-x^{-2}y^3 + h(y)] \\ \hookrightarrow \quad 4y^3 - 3x^{-2}y^2 &= -3x^{-2}y^2 + h'(y) \\ \hookrightarrow \quad h'(y) = 4y^3 &\rightsquigarrow h(y) = \int 4y^3 dy = y^4 + c_1 \quad . \end{aligned}$$

So the potential function is

$$\phi(x, y) = -x^{-2}y^3 + h(y) = -x^{-2}y^3 + y^4 + c_1 \quad ,$$

and an implicit solution to the given differential equation is

$$-x^{-2}y^3 + y^4 + c_1 = c_2 \quad ,$$

which we rewrite as

$$y^4 - x^{-2}y^3 = c \quad ,$$

but will not attempt to solve for  $y$ .

7.5i. For part i: Here

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} [6 + 12x^2y^2] = 24x^2y$$

and

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x} [7x^3y + xy^{-1}] = 21x^2y + y^{-1} .$$

So  $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ , telling us the equation is not in exact form.

For part ii: We need to find a function  $\mu = \mu(x, y)$  such that

$$\frac{\partial}{\partial y} [\mu [6 + 12x^2y^2]] = \frac{\partial}{\partial y} [\mu M] = \frac{\partial}{\partial x} [\mu N] = \frac{\partial}{\partial x} [\mu [7x^3y + xy^{-1}]] .$$

Let us see if an integrating factor  $\mu$  can be a function of  $x$  only,  $\mu = \mu(x)$ :

$$\begin{aligned} \frac{\partial}{\partial y} [\mu(x) [6 + 12x^2y^2]] &= \frac{\partial}{\partial x} [\mu(x) [7x^3y + xy^{-1}]] \\ \Leftrightarrow \mu(x) [24x^2y] &= \frac{d\mu}{dx} [7x^3y + xy^{-1}] \\ &\quad + \mu(x) [21x^2y + y^{-1}] \\ \Leftrightarrow \frac{d\mu}{dx} [7x^3y + xy^{-1}] &= \mu(x) [3x^2y - xy^2] . \end{aligned}$$

In this case, we clearly cannot cancel out the  $y$  variable, and cannot obtain a differential equation for  $\mu = \mu(x)$  in terms of  $x$  and  $\mu$  only. So there is not an integrating factor of  $x$  alone.

Attempting to find an integrating factor of  $y$  only,  $\mu = \mu(y)$ :

$$\begin{aligned} \frac{\partial}{\partial y} [\mu(y) [6 + 12x^2y^2]] &= \frac{\partial}{\partial x} [\mu(y) [7x^3y + xy^{-1}]] \\ \Leftrightarrow \frac{d\mu}{dy} [6 + 12x^2y^2] + \mu(y) [24x^2y] &= \mu(y) [21x^2y + y^{-1}] \\ \Leftrightarrow \frac{d\mu}{dy} [6 + 12x^2y^2] &= \mu(y) [y^{-1} - 3x^2y] . \end{aligned}$$

Since we clearly cannot obtain a differential equation for  $\mu = \mu(y)$  in terms of  $y$  and  $\mu$  only, there is not an integrating factor of  $y$  alone.

Finally, let's attempt to find an integrating factor of the form  $\mu = x^\alpha y^\beta$ :

$$\begin{aligned} \frac{\partial}{\partial y} [\mu [6 + 12x^2y^2]] &= \frac{\partial}{\partial x} [\mu [7x^3y + xy^{-1}]] \\ \Leftrightarrow \frac{\partial}{\partial y} [x^\alpha y^\beta [6 + 12x^2y^2]] &= \frac{\partial}{\partial x} [x^\alpha y^\beta [7x^3y + xy^{-1}]] \\ \Leftrightarrow \frac{\partial}{\partial y} [6x^\alpha y^\beta + 12x^{\alpha+2}y^{\beta+2}] &= \frac{\partial}{\partial x} [7x^{\alpha+3}y^{\beta+1} + x^{\alpha+1}y^{\beta-1}] \\ \Leftrightarrow 6\beta x^\alpha y^{\beta-1} + 12(\beta + 2)x^{\alpha+2}y^{\beta+1} &= 7(\alpha + 3)x^{\alpha+2}y^{\beta+1} + (\alpha + 1)x^\alpha y^{\beta-1} . \end{aligned}$$

Combining like terms yields

$$[\alpha - 6\beta + 1]x^\alpha y^{\beta-1} + [7\alpha - 12\beta - 3]x^{\alpha+2}y^{\beta+1} = 0 \quad ,$$

which requires that

$$\alpha - 6\beta + 1 = 0 \quad \text{and} \quad 7\alpha - 12\beta - 3 = 0 \quad .$$

Solving for  $\alpha$  and  $\beta$ :

$$\alpha - 6\beta + 1 = 0 \quad \text{and} \quad 7\alpha - 12\beta - 3 = 0$$

$$\hookrightarrow \quad \alpha = 6\beta - 1 \quad \text{and} \quad 7(6\beta - 1) - 12\beta = 3$$

$$\hookrightarrow \quad \alpha = 6\beta - 1 \quad \text{and} \quad \beta = \frac{3+7}{7 \cdot 6 - 12} = \frac{1}{3}$$

$$\hookrightarrow \quad \alpha = 6 \cdot \frac{1}{3} - 1 = 1 \quad \text{and} \quad \beta = \frac{1}{3} \quad .$$

So,

$$\mu = x^\alpha y^\beta = xy^{1/3} \quad .$$

For part iii: Multiplying the given differential equation by the integrating factor  $\mu$  just found:

$$xy^{1/3} \left[ 6 + 12x^2y^2 + \left[ 7x^3y + \frac{x}{y} \right] \frac{dy}{dx} \right] = xy^{1/3} \cdot 0$$

$$\hookrightarrow \quad 6xy^{1/3} + 12x^3y^{7/3} + \left[ 7x^4y^{4/3} + x^2y^{-2/3} \right] \frac{dy}{dx} = 0 \quad .$$

Any potential function  $\phi$  of this equation must satisfy

$$\frac{\partial \phi}{\partial x} = 6xy^{1/3} + 12x^3y^{7/3} \quad \text{and} \quad \frac{\partial \phi}{\partial y} = 7x^4y^{4/3} + x^2y^{-2/3} \quad . \quad (\star)$$

Integrating the first equation with respect to  $x$ :

$$\begin{aligned} \phi(x, y) &= \int \frac{\partial \phi}{\partial x} dx \\ &= \int \left[ 6xy^{1/3} + 12x^3y^{7/3} \right] dx = 3x^2y^{1/3} + 3x^4y^{7/3} + h(y) \quad . \end{aligned}$$

Combining this result with the second equation in  $(\star)$ :

$$7x^4y^{4/3} + x^2y^{-2/3} = \frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y} \left[ 3x^2y^{1/3} + 3x^4y^{7/3} + h(y) \right]$$

$$\hookrightarrow \quad 7x^4y^{4/3} + x^2y^{-2/3} = x^2y^{-2/3} + 7x^4y^{4/3} + h'(y)$$

$$\hookrightarrow \quad h'(y) = 0 \quad \rightsquigarrow \quad h(y) = \int 0 dy = c_1 \quad .$$

So the potential function is

$$\phi(x, y) = 3x^2y^{1/3} + 3x^4y^{7/3} + h(y) = 3x^2y^{1/3} + 3x^4y^{7/3} + c_1 \quad ,$$

and an implicit solution to the given differential equation is

$$3x^2y^{1/3} + 3x^4y^{7/3} + c_1 = c_2 \quad ,$$

which we rewrite as

$$x^2y^{1/3} + x^4y^{7/3} = c \quad ,$$

but will not attempt to solve for  $y$ .

**7.6 a.** If

$$\mu = \mu(x) = e^{\int P(x) dx} \quad \text{with} \quad P = P(x) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \quad ,$$

then

$$\frac{\partial}{\partial y} [\mu M] = \frac{\partial}{\partial y} \left[ e^{\int P(x) dx} M(x, y) \right] = e^{\int P(x) dx} \frac{\partial M}{\partial y} = \mu \frac{\partial M}{\partial y}$$

and

$$\begin{aligned} \frac{\partial}{\partial x} [\mu N] &= \frac{\partial}{\partial x} \left[ e^{\int P(x) dx} N(x, y) \right] \\ &= \left[ e^{\int P(x) dx} P(x) \right] N(x, y) + e^{\int P(x) dx} \frac{\partial N}{\partial x} \\ &= e^{\int P(x) dx} \left[ PN + \frac{\partial N}{\partial x} \right] \\ &= \mu \left[ \left( \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \right) N + \frac{\partial N}{\partial x} \right] \\ &= \mu \left[ \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) + \frac{\partial N}{\partial x} \right] = \mu \frac{\partial M}{\partial y} \quad . \end{aligned}$$

Thus,

$$\frac{\partial}{\partial y} [\mu M] = \frac{\partial}{\partial x} [\mu N]$$

which, according to the complete test for exactness (theorem 7.4 on page 139 of the text), ensures that the given  $\mu$  is an integrating factor for the given equation.