

Chapter 6: Simplifying Through Substitution

6.1 a. Let $u = 3x + 3y + 2$. Solving for y and computing y' , we then see that

$$3y = u - 3x - 2 \quad \rightsquigarrow \quad y = \frac{u}{3} - x - \frac{2}{3}$$

and

$$\frac{dy}{dx} = \frac{d}{dx} \left[\frac{u}{3} - x - \frac{2}{3} \right] = \frac{1}{3} \frac{du}{dx} - 1 \quad .$$

Substituting this into the differential equation and solving for u , we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{(3x+3y+2)^2} \quad \rightsquigarrow \quad \frac{1}{3} \frac{du}{dx} - 1 = \frac{1}{u^2} \\ \Leftrightarrow \quad \frac{1}{3} \frac{du}{dx} &= \frac{1}{u^2} + 1 = \frac{1}{u^2} + \frac{u^2}{u^2} = \frac{1+u^2}{u^2} \\ \Leftrightarrow \quad \frac{u^2}{1+u^2} \frac{du}{dx} &= 3 \quad \rightsquigarrow \quad \int \frac{u^2}{1+u^2} \frac{du}{dx} dx = \int 3 dx \\ \Leftrightarrow \quad \int \frac{u^2}{1+u^2} du &= 3x + c \quad . \end{aligned} \quad (\star)$$

Computing the last integral (and ignoring the arbitrary constant):

$$\begin{aligned} \int \frac{u^2}{1+u^2} du &= \int \frac{1+u^2-1}{1+u^2} du \\ &= \int \left[1 - \frac{1}{u^2+1} \right] du = u - \arctan(u) \quad . \end{aligned}$$

So equation (\star) becomes

$$u - \arctan(u) = 3x + c \quad ,$$

which cannot be solved for u . So we now replace each u with its formula in terms of y from the original substitution ($u = 3x + 3y + 2$), and simplify the resulting equation as much as practical:

$$\begin{aligned} 3x + 3y + 2 - \arctan(3x + 3y + 2) &= 3x + c \\ \Leftrightarrow \quad \arctan(3x + 3y + 2) &= 3y + 2 + c = 3y + C \\ \Leftrightarrow \quad 3x + 3y + 2 &= \tan(3y + C) \quad . \end{aligned}$$

6.1 c. Using the substitution $u = 4y - 8x + 3$ we have

$$y = \frac{1}{4} [u + 8x - 3] = \frac{u}{4} + 2x - \frac{3}{4} \quad , \quad \frac{dy}{dx} = \frac{1}{4} \frac{du}{dx} + 2 \quad ,$$

and

$$\begin{aligned} \cos(4y - 8x + 3) \frac{dy}{dx} &= 2 + 2 \cos(4y - 8x + 3) \\ \Leftrightarrow \quad \cos(u) \left[\frac{1}{4} \frac{du}{dx} + 2 \right] &= 2 + 2 \cos(u) \end{aligned}$$

$$\Leftrightarrow \frac{1}{4} \cos(u) \frac{du}{dx} + 2 \cos(u) = 2 + 2 \cos(u) \quad \rightsquigarrow \quad \cos(u) \frac{du}{dx} = 8$$

$$\Leftrightarrow \int \cos(u) \frac{du}{dx} dx = \int 8 dx \quad \rightsquigarrow \quad \sin(u) = 8x + c$$

$$\Leftrightarrow u = \arcsin(8x + c) \quad .$$

Plugging the last formula for u into the formula for y based on the original substitution yields the final answer,

$$\begin{aligned} y &= \frac{1}{4} [u + 8x - 3] \\ &= \frac{1}{4} [\arcsin(8x + c) + 8x - 3] = \frac{1}{4} \arcsin(8x + c) + 2x - \frac{3}{4} \quad . \end{aligned}$$

6.3 a. Rewriting the differential equation:

$$x^2 \frac{dy}{dx} - xy = y^2 \quad \rightsquigarrow \quad \frac{dy}{dx} = \frac{y}{x} + \left(\frac{y}{x}\right)^2 \quad .$$

Since the equation is homogeneous, our substitution is based on $u = \frac{y}{x}$, from which we derive

$$y = xu \quad \text{and} \quad \frac{dy}{dx} = \frac{d}{dx}[xu] = \frac{dx}{dx} \cdot u + x \frac{du}{dx} = u + x \frac{du}{dx} \quad .$$

Using this with our differential equation:

$$\frac{dy}{dx} = \frac{y}{x} + \left(\frac{y}{x}\right)^2 \quad \rightsquigarrow \quad u + x \frac{du}{dx} = u + u^2 \quad \rightsquigarrow \quad \frac{du}{dx} = \frac{u^2}{x} \quad .$$

Obviously, $u = 0$ is the only constant solution, which, in turn, also yields

$$y = xu = x \cdot 0 = 0$$

as a constant solution to the original equation. For the other solutions:

$$\begin{aligned} \frac{du}{dx} &= \frac{u^2}{x} \quad \rightsquigarrow \quad u^{-2} \frac{du}{dx} = \frac{1}{x} \quad \rightsquigarrow \quad \int u^{-2} \frac{du}{dx} dx = \int \frac{1}{x} dx \\ \Leftrightarrow \quad -u^{-1} &= \ln|x| + c \quad \rightsquigarrow \quad u = \frac{-1}{\ln|x| + c} \quad . \end{aligned}$$

This, along with our formula for y in terms of u , yields

$$y = xu = x \left[\frac{-1}{\ln|x| + c} \right] = \frac{-x}{\ln|x| + c} \quad ,$$

which, along with $y = 0$, describes all the solutions.

6.3 c. The substitution: $u = \frac{y}{x} \quad \rightsquigarrow \quad y = xu$

$$\Leftrightarrow \frac{dy}{dx} = \frac{d}{dx}[xu] = u + x \frac{du}{dx} \quad .$$

So,

$$\cos\left(\frac{y}{x}\right) \left[\frac{dy}{dx} - \frac{y}{x} \right] = 1 + \sin\left(\frac{y}{x}\right)$$

$$\hookrightarrow \cos(u) \left[\left(u + x \frac{du}{dx} \right) - u \right] = 1 + \sin(u)$$

$$\hookrightarrow \frac{du}{dx} = \frac{1 + \sin(u)}{x \cos(u)}$$

The constant solutions of this are given by those values of u where $1 + \sin(u) = 0$; that is,
 $u = \arcsin(-1) = 2n\pi - \frac{\pi}{2}$ for $n = 0, \pm 1, \pm 2, \dots$.

For the nonconstant solutions:

$$\frac{du}{dx} = \frac{1 + \sin(u)}{x \cos(u)} \quad \rightsquigarrow \quad \frac{\cos(u)}{1 + \sin(u)} \frac{du}{dx} = \frac{1}{x}$$

$$\hookrightarrow \int \frac{\cos(u)}{1 + \sin(u)} \frac{du}{dx} dx = \int \frac{1}{x} dx \quad \rightsquigarrow \quad \ln |1 + \sin(u)| = \ln |x| + c$$

$$\hookrightarrow 1 + \sin(u) = \pm e^{\ln|x|+c} = Ax \quad \rightsquigarrow \quad u = \arcsin(Ax - 1) \quad .$$

Note that the last formula for u also yields the constant solutions. So all the solutions to our original differential equation are given by

$$y = xu = x \arcsin(Ax - 1) \quad .$$

6.5 a. First, observe that $y = 0$ is clearly a constant solution.

Here, $n = 3$ (from the $3y^3$ term). So the substitution is based on $u = y^{1-3} = y^{-2}$. Hence, we also have

$$y = \pm u^{-1/2} \quad \text{and} \quad \frac{dy}{dx} = \frac{d}{dx} \left[\pm u^{-1/2} \right] = \pm \left[-\frac{1}{2} u^{-3/2} \frac{du}{dx} \right] \quad .$$

Using this substitution:

$$\frac{dy}{dx} + 3y = 3y^3 \quad \rightsquigarrow \quad -\frac{1}{2} u^{-3/2} \frac{du}{dx} + 3u^{-1/2} = 3 \left(u^{-1/2} \right)^3$$

$$\hookrightarrow \left(-2u^{3/2} \right) \left[-\frac{1}{2} u^{-3/2} \frac{du}{dx} + 3u^{-1/2} = 3u^{-3/2} \right]$$

$$\hookrightarrow \frac{du}{dx} - 6u = -6 \quad .$$

The integrating factor for the last differential equation is

$$\mu(x) = e^{\int (-6) dx} = e^{-6x}$$

Using this with the last differential equation, above:

$$e^{-6x} \left[\frac{du}{dx} - 6u = -6 \right] \quad \rightsquigarrow \quad e^{-6x} \frac{du}{dx} - 6e^{-6x} u = -6e^{-6x}$$

$$\hookrightarrow \frac{d}{dx} \left[e^{-6x} u \right] = -6e^{-6x} \quad \rightsquigarrow \quad e^{-6x} u = -\int 6e^{-6x} dx = e^{-6x} + c$$

$$\hookrightarrow u = 1 + ce^{6x} \quad .$$

This with the original substitution then gives

$$y = \pm u^{-1/2} = \pm (1 + ce^{6x})^{-1/2} .$$

So any solution to the original differential equation is given by the above formula for y or $y = 0$.

6.5 c. Again, we note that $y = 0$ is a solution.
The substitution:

$$u = y^{1-2/3} = y^{1/3} \quad \rightsquigarrow \quad y = u^3 \quad \rightsquigarrow \quad \frac{dy}{dx} = 3u^2 \frac{du}{dx} .$$

Using the substitution to find the linear equation for u :

$$\begin{aligned} \frac{dy}{dx} + 3 \cot(x)y &= 6 \cos(x)y^{2/3} \\ \Leftrightarrow 3u^2 \frac{du}{dx} + 3 \cot(x)u^3 &= 6 \cos(x)(u^3)^{2/3} \\ \Leftrightarrow \frac{du}{dx} + \cot(x)u &= 2 \cos(x) . \end{aligned}$$

The integrating factor is given by

$$\mu(x) = e^{\int \cot(x) dx} = \exp\left(\int \frac{\cos(x)}{\sin(x)} dx\right) = \exp(\ln |\sin(x)|) = |\sin(x)| .$$

As noted in chapter 5, we can simply use $\mu = \sin(x)$. Doing so:

$$\begin{aligned} \sin(x) \left[\frac{du}{dx} + \cot(x)u \right] &= 2 \cos(x) \\ \Leftrightarrow \frac{d}{dx} [\sin(x)u] &= 2 \sin(x) \cos(x) \\ \Leftrightarrow \sin(x)u &= \int 2 \sin(x) \cos(x) dx = \sin^2(x) + c \\ \Leftrightarrow u &= \sin(x) + \frac{c}{\sin(x)} \\ \Leftrightarrow y = u^3 &= \left(\sin(x) + \frac{c}{\sin(x)} \right)^3 . \end{aligned}$$

The last equation, along with $y = 0$, describes all solutions to our original differential equation.

6.7 a. This is obviously in the form

$$\frac{dy}{dx} = \text{formula of } \frac{y}{x} .$$

So it is a homogeneous equation, and we use the corresponding substitution,

$$u = \frac{y}{x} \quad , \quad y = xu \quad \text{and} \quad \frac{dy}{dx} = \frac{d}{dx} [xu] = u + x \frac{du}{dx} .$$

Doing so:

$$\frac{dy}{dx} = \frac{y}{x} + \left(\frac{x}{y}\right)^2 \quad \rightsquigarrow \quad u + x \frac{du}{dx} = u + u^{-2}$$

$$\hookrightarrow \quad u^2 \frac{du}{dx} = \frac{1}{x} \quad \rightsquigarrow \quad \int u^2 \frac{du}{dx} dx = \int \frac{1}{x} dx$$

$$\hookrightarrow \quad \frac{1}{3}u^3 = \ln|x| + c \quad \rightsquigarrow \quad u = (3 \ln|x| + 3c)^{1/3}$$

$$\hookrightarrow \quad y = xu = x(3 \ln|x| + C)^{1/3} \quad .$$

6.7 c. Since the equation is

$$\frac{dy}{dx} + \frac{2}{x}y = 4y^{1/2} \quad ,$$

it is a Bernoulli equation with $n = 1/2$. One solution is $y = 0$. The others are obtained using the substitution

$$u = y^{1-1/2} = y^{1/2} \quad .$$

Hence,

$$y = u^2 \quad , \quad \frac{dy}{dx} = 2u \frac{du}{dx} \quad ,$$

and

$$\frac{dy}{dx} + \frac{2}{x}y = 4y^{1/2} \quad \rightsquigarrow \quad 2u \frac{du}{dx} + \frac{2}{x}u^2 = 4(u^2)^{1/2}$$

$$\hookrightarrow \quad \frac{du}{dx} + \frac{1}{x}u = 2 \quad .$$

The integrating factor is $\mu(x) = e^{\int 1/x dx} = \dots = x$.

So

$$\frac{du}{dx} + \frac{1}{x}u = 2 \quad \rightsquigarrow \quad x \left[\frac{du}{dx} + \frac{1}{x}u = 2 \right]$$

$$\hookrightarrow \quad \frac{d}{dx}[xu] = 2x \quad \rightsquigarrow \quad xu = \int 2x dx = x^2 + c$$

$$\hookrightarrow \quad u = x + \frac{c}{x} \quad \rightsquigarrow \quad y = u^2 = \left(x + \frac{c}{x}\right)^2 \quad .$$

6.7 e. The $y - x$ factor suggests using the linear substitution $u = y - x$. Then

$$y = u + x \quad \text{and} \quad \frac{dy}{dx} = \frac{d}{dx}[u + x] = \frac{du}{dx} + 1 \quad .$$

Using this,

$$(y - x) \frac{dy}{dx} = 1 \quad \rightsquigarrow \quad u \left[\frac{du}{dx} + 1 \right] = 1$$

$$\hookrightarrow \quad u \frac{du}{dx} = 1 - u \quad \rightsquigarrow \quad \frac{du}{dx} = \frac{1 - u}{u} \quad .$$

Note that the derivative of u is zero if and only if $u = 1$. So $u = 1$ is the only constant solution to this last equation. The corresponding solution to the original equation is then

$$y = u + x = 1 + x \quad .$$

For the other solutions, we'll need

$$\begin{aligned}\int \frac{u}{1-u} du &= -\int \frac{1-u-1}{1-u} du \\ &= -\int \left[1 - \frac{1}{1-u}\right] du = -u - \ln|1-u| + C .\end{aligned}$$

Back to the last differential equation above:

$$\begin{aligned}\frac{du}{dx} &= \frac{1-u}{u} \quad \rightsquigarrow \quad \frac{u}{1-u} \frac{du}{dx} = 1 \\ \Leftrightarrow \int \frac{u}{1-u} \frac{du}{dx} dx &= \int 1 dx \quad \rightsquigarrow \quad -u - \ln|1-u| = x + c \\ \Leftrightarrow -(y-x) - \ln|1-(y-x)| &= x + c \\ \Leftrightarrow \ln|1-y+x| &= -c - y . \\ \Leftrightarrow 1 - y + x = \pm e^{-c-y} = Ae^{-y} &\rightsquigarrow y = 1 + x - Ae^{-y} .\end{aligned}$$

This last equation describes all solutions since it does reduce to the solution $y(x) = 1 + x$ when $A = 0$.

6.7 g.

$$\begin{aligned}(2xy + 2x^2) \frac{dy}{dx} &= x^2 + 2xy + 2y^2 \\ \Leftrightarrow \frac{dy}{dx} &= \frac{x^2 + 2xy + 2y^2}{2xy + 2x^2} = \frac{x^2 \left[1 + 2\frac{y}{x} + 2\left(\frac{y}{x}\right)^2\right]}{x^2 \left[2\frac{y}{x} + 2\right]} = \frac{1 + 2\frac{y}{x} + 2\left(\frac{y}{x}\right)^2}{2\frac{y}{x} + 2} .\end{aligned}$$

So the equation is homogeneous, and we use the substitution $u = \frac{y}{x}$.

Hence

$$y = xu \quad , \quad \frac{dy}{dx} = \frac{d}{dx} [xu] = u + x \frac{du}{dx} \quad ,$$

and

$$\begin{aligned}\frac{dy}{dx} &= \frac{1 + 2\frac{y}{x} + 2\left(\frac{y}{x}\right)^2}{2\frac{y}{x} + 2} \quad \rightsquigarrow \quad u + x \frac{du}{dx} = \frac{1 + 2u + 2u^2}{2u + 2} \\ \Leftrightarrow x \frac{du}{dx} &= \frac{1 + 2u + 2u^2}{2u + 2} - u = \frac{1 + 2u + 2u^2}{2u + 2} - \frac{2u^2 + 2u}{2u + 2} = \frac{1}{2u + 2} \\ \Leftrightarrow (2u + 2) \frac{du}{dx} &= \frac{1}{x} \quad \rightsquigarrow \quad \int (2u + 2) \frac{du}{dx} dx = \int \frac{1}{x} dx \\ \Leftrightarrow u^2 + 2u &= \ln|x| + c \\ \Leftrightarrow u^2 + 2u - (\ln|x| + c) &= 0\end{aligned}$$

$$\begin{aligned} \hookrightarrow u &= \frac{-2 \pm \sqrt{2^2 - 4[-(\ln|x| + c)]}}{2} = -1 \pm \sqrt{\ln|x| + C} \\ \hookrightarrow y = xu &= x \left[-1 \pm \sqrt{\ln|x| + C} \right] = -x \pm x\sqrt{\ln|x| + C} . \end{aligned}$$

6.7 i. Clearly, the substitution $u = 2x + y - 3$ is in order. With this,

$$y = u - 2x + 3 \quad , \quad \frac{dy}{dx} = \frac{d}{dx}[u - 2x + 3] = \frac{du}{dx} - 2 \quad ,$$

and

$$\frac{dy}{dx} = 2\sqrt{2x + y - 3} - 2 \quad \rightsquigarrow \quad \frac{du}{dx} - 2 = 2\sqrt{u} - 2$$

$$\hookrightarrow \frac{du}{dx} = 2\sqrt{u} \quad .$$

Note that this last equation has one constant solution $u = 0$. Corresponding to this is the solution to the original equation

$$y = u - 2x + 3 = 0 - 2x + 3 = 3 - 2x \quad .$$

For the other solutions, we continue the above computations:

$$\frac{du}{dx} = \sqrt{u} \quad \rightsquigarrow \quad \int u^{-1/2} \frac{du}{dx} dx = \int 2 dx$$

$$\hookrightarrow 2u^{1/2} = 2x + C \quad \rightsquigarrow \quad u = (x + c)^2$$

$$\hookrightarrow y = u - 2x + 3 = (x + c)^2 - 2x + 3 \quad .$$

So each solution is given by either the last formula or by $y = 3 - 2x$.

6.7 k.
$$x \frac{dy}{dx} - y = \sqrt{xy + x^2}$$

$$\hookrightarrow \frac{dy}{dx} = \frac{y + \sqrt{xy + x^2}}{x} \quad \rightsquigarrow \quad \frac{dy}{dx} = \frac{y}{x} + \left(\frac{y}{x} + 1\right)^{1/2} \quad .$$

So the equation is homogeneous, and we use the substitution $u = \frac{y}{x}$.

Hence

$$y = xu \quad , \quad \frac{dy}{dx} = \frac{d}{dx}[xu] = u + x \frac{du}{dx} \quad ,$$

and

$$\frac{dy}{dx} = \frac{y}{x} + \left(\frac{y}{x} + 1\right)^{1/2} \quad \rightsquigarrow \quad u + x \frac{du}{dx} = u + (u + 1)^{1/2}$$

$$\hookrightarrow \frac{du}{dx} = \frac{1}{x} \cdot (u + 1)^{1/2} \quad . \quad (\star)$$

Note that the derivative of u is zero if $u = -1$. So $u = -1$ is a constant solution, and the corresponding solution to our original equation is

$$y = xu = x(-1) = -x \quad .$$

For the rest of the solutions, we divide equation (\star) by $(u + 1)^{1/2}$ and continue the process:

$$\begin{aligned} (u + 1)^{-1/2} \frac{du}{dx} &= \frac{1}{x} \quad \rightsquigarrow \quad \int (u + 1)^{-1/2} \frac{du}{dx} dx = \int \frac{1}{x} dx \\ \Leftrightarrow \quad 2(u + 1)^{1/2} &= \ln |x| + C \quad \rightsquigarrow \quad u = \left(\frac{1}{2} \ln |x| + c \right)^2 - 1 \\ \Leftrightarrow \quad y = xu &= x \left[\left(\frac{1}{2} \ln |x| + c \right)^2 - 1 \right] = x \left(\frac{1}{2} \ln |x| + c \right)^2 - x \quad . \end{aligned}$$

So every solution is given by either the last formula for y or by $y = -x$.

6.7 m. Clearly, the linear substitution $u = x - y + 3$ is appropriate. Using this,

$$y = x - u + 3 \quad , \quad \frac{dy}{dx} = \frac{d}{dx} [x - u + 3] = 1 - \frac{du}{dx} \quad ,$$

and

$$\begin{aligned} \frac{dy}{dx} &= (x - y + 3)^2 \quad \rightsquigarrow \quad 1 - \frac{du}{dx} = u^2 \\ \Leftrightarrow \quad \frac{du}{dx} &= 1 - u^2 \quad . \end{aligned}$$

Here, the derivative of u is zero if $u = \pm 1$. So this differential equation has two constant solutions $u = 1$ and $u = -1$. Since $y = x - u + 3$, the corresponding solutions to the original equation are

$$y = x - 1 + 3 = x + 2 \quad \text{and} \quad y = x - (-1) + 3 = x + 4 \quad .$$

To find the other solutions, we divide the last differential equation above by $1 - u^2$ and proceed as usual, using partial fractions to rewrite $(1 - u^2)^{-1}$ in more convenient form for integration:

$$\begin{aligned} \frac{1}{1 - u^2} \frac{du}{dx} &= 1 \quad \rightsquigarrow \quad \int \frac{1}{1 - u^2} \frac{du}{dx} dx = \int 1 dx \\ \Leftrightarrow \quad \int \frac{1}{(1 + u)(1 - u)} dx &= x + c \quad \rightsquigarrow \quad \dots \\ \Leftrightarrow \quad \int \frac{1}{2} \left[\frac{1}{1 + u} + \frac{1}{1 - u} \right] dx &= x + c \\ \Leftrightarrow \quad \frac{1}{2} [\ln |1 + u| - \ln |1 - u|] &= x + c \\ \Leftrightarrow \quad \ln \left| \frac{1 + u}{1 - u} \right| = 2x + 2c \quad \rightsquigarrow \quad \frac{1 + u}{1 - u} &= \pm e^{2x + 2c} = Ae^{2x} \\ \Leftrightarrow \quad 1 + u = Ae^{2x} - uAe^{2x} \quad \rightsquigarrow \quad u &= \frac{Ae^{2x} - 1}{Ae^{2x} + 1} \\ \Leftrightarrow \quad y(x) = x - u + 3 = x + 3 + \frac{Ae^{2x} - 1}{Ae^{2x} + 1} \quad . \end{aligned}$$

Note that the last formula for y reduces to the constant solution $y = x + 2$ if $A = 0$, but does not reduce to the constant solution $y = x + 4$ for any value of A . So to describe all solutions, we need both

$$y = x + 3 + \frac{Ae^{2x} - 1}{Ae^{2x} + 1} \quad \text{and} \quad y = x + 4 \quad .$$

- 6.7 o.** We need a formula of y and maybe x , $f(x, y)$, such that the differential equation reduces to a 'simple' differential equation for u when we let $u = f(x, y)$. Because the given equation involves both $\sin(y)$ and $\cos(y)$, two possible choices for the substitution are obviously

$$u = \cos(y) \quad \text{and} \quad u = \sin(y) \quad .$$

Differentiating these yield, respectively,

$$\frac{du}{dx} = -\sin(y)\frac{dy}{dx} \quad \text{and} \quad \frac{du}{dx} = \cos(y)\frac{dy}{dx} \quad ,$$

the second of which matches exactly the left side of the given differential equation. So let's use the second choice,

$$u = \sin(y) \quad \text{with} \quad \frac{du}{dx} = \cos(y)\frac{dy}{dx}$$

as our substitution. Doing so, we get

$$\cos(y)\frac{dy}{dx} = e^{-x} - \sin(y) \quad \rightsquigarrow \quad \frac{du}{dx} = e^{-x} - u$$

$$\hookrightarrow \quad \frac{du}{dx} + u = e^{-x} \quad .$$

This is a simple linear equation with integrating factor $\mu(x) = e^{\int 1 dx} = e^x$.

Multiplying our differential equation for u by this integrating factor and continuing:

$$e^x \left[\frac{du}{dx} + u = e^{-x} \right] \rightsquigarrow \frac{d}{dx} [e^x u] = 1$$

$$\hookrightarrow \quad e^x u = \int 1 dx = x + c \rightsquigarrow u = (x + c)e^{-x}$$

$$\hookrightarrow \quad \sin(y) = (x + c)e^{-x} \rightsquigarrow y = \arcsin((x + c)e^{-x}) \quad .$$