

Chapter 5: Linear First-Order Equations

- 5.1 a.** Dividing through by x^2 we get $\frac{dy}{dx} + 3y = x^{-2} \sin(x)$,
 which is the standard form $\frac{dy}{dx} + p(x)y = q(x)$
 with $p(x) = 3$ and $q(x) = x^{-2} \sin(x)$.
 So the equation is linear.

- 5.1 c.** Because of the y^2 , this cannot be put into the standard form

$$\frac{dy}{dx} + p(x)y = q(x) .$$

So the differential equation is not linear.

- 5.1 e.** $\frac{dy}{dx} = 1 + xy + 3y = 1 + (x+3)y \rightsquigarrow \frac{dy}{dx} - (x+3)y = 1$.
 The last equation is in standard form for a first-order linear differential equation (with $p(x) = -(x+3)$ and $q(x) = 1$). So the equation is linear.

- 5.1 g.** $\frac{dy}{dx} - e^{2x} = 0 \rightsquigarrow \frac{dy}{dx} + 0 \cdot y = e^{2x}$.
 This equation is linear since it can be rewritten in standard form

$$\frac{dy}{dx} + p(x)y = q(x) \quad \text{with } p(x) = 0 \quad \text{and } q(x) = e^{2x} .$$

- 5.1 i.** This equation cannot be put into standard form because of the y^3 term. So it is not linear.

- 5.2 a.** The equation is already in standard form with $p(x) = 2$.
 The corresponding integrating factor is

$$\mu(x) = e^{\int p(x) dx} = e^{\int 2 dx} = e^{2x} .$$

Multiplying the differential equation by the integrating factor and proceeding with the procedure:

$$\begin{aligned} \mu \left[\frac{dy}{dx} + py = q \right] &\rightsquigarrow e^{2x} \left[\frac{dy}{dx} + 2y = 6 \right] \\ \Leftrightarrow \underbrace{e^{2x} \frac{dy}{dx} + 2e^{2x} y}_{\frac{d}{dx} [e^{2x} y]} = 6e^{2x} &\rightsquigarrow \frac{d}{dx} [e^{2x} y] = 6e^{2x} \\ \Leftrightarrow e^{2x} y = \int \frac{d}{dx} [e^{2x} y] dx = \int 6e^{2x} dx = 3e^{2x} + c & \\ \Leftrightarrow y = e^{-2x} [3e^{2x} + c] = 3 + ce^{-2x} . & \end{aligned}$$

5.2 c. Get the equation into standard form:

$$\frac{dy}{dx} = 4y + 16x \quad \rightsquigarrow \quad \frac{dy}{dx} - 4y = 16x \quad .$$

Find the corresponding integrating factor:

$$\mu(x) = e^{\int p(x) dx} = e^{\int (-4) dx} = e^{-4x} \quad .$$

Multiply the differential equation in standard form by μ and proceed:

$$\begin{aligned} e^{-4x} \left[\frac{dy}{dx} - 4y = 16x \right] &\rightsquigarrow e^{-4x} \frac{dy}{dx} - 4e^{-4x} y = 16xe^{-4x} \\ \Leftrightarrow \frac{d}{dx} [e^{-4x} y] &= 16xe^{-4x} \\ \Leftrightarrow e^{-4x} y &= \int \frac{d}{dx} [e^{-4x} y] dx = \int 16xe^{-4x} dx \quad . \end{aligned} \quad (\star)$$

For the last integral, we use integration by parts ($\int u dv = uv - \int v du$ with $u = x$ and $dv = 16e^{-4x}$):

$$\int 16xe^{-4x} dx = x(-4e^{-4x}) - \int (-4e^{-4x}) dx = -4xe^{-4x} - e^{-4x} + c \quad .$$

Plugging this back into equation (\star) and solving for y :

$$\begin{aligned} e^{-4x} y &= -4xe^{-4x} - e^{-4x} + c \\ \Leftrightarrow y &= e^{4x} [-4xe^{-4x} - e^{-4x} + c] = -4x - 1 + ce^{4x} \quad . \end{aligned}$$

5.2 e. Standard form: $x \frac{dy}{dx} + 3y - 10x^2 = 0$

$$\Leftrightarrow x \frac{dy}{dx} + 3y = 10x^2 \quad \rightsquigarrow \quad \frac{dy}{dx} + \frac{3}{x}y = 10x \quad .$$

Integrating factor: $\mu(x) = e^{\int 3/x dx} = e^{3 \ln|x|} = |x|^3 \quad .$

However, as noted on page 101 of the text, we can simply continue with $\mu(x) = x^3$ as the integrating factor:

$$\begin{aligned} x^3 \left[\frac{dy}{dx} + \frac{3}{x}y = 10x \right] \\ \Leftrightarrow x^3 \frac{dy}{dx} + 3x^2 = 10x^4 \quad \rightsquigarrow \quad \frac{d}{dx} [x^3 y] = 10x^4 \\ \Leftrightarrow x^3 y = \int \frac{d}{dx} [x^3 y] dx = \int 10x^4 dx = 2x^5 + c \\ \Leftrightarrow y = x^{-3} [2x^5 + c] = 2x^2 + cx^{-3} \quad . \end{aligned}$$

5.2 g. Standard form: $x \frac{dy}{dx} = \sqrt{x} + 3y \quad \rightsquigarrow \quad \frac{dy}{dx} - \frac{3}{x}y = x^{-1/2} \quad .$

Integrating factor: $\mu(x) = e^{\int (-3/x) dx} = e^{-3 \ln|x|} = |x|^{-3} \quad .$

Continuing with $\mu(x) = x^{-3}$ as the integrating factor:

$$x^{-3} \left[\frac{dy}{dx} - \frac{3}{x}y = x^{-1/2} \right]$$

$$\Leftrightarrow x^{-3} \frac{dy}{dx} - 3x^{-4}y = x^{-7/2} \quad \rightsquigarrow \quad \frac{d}{dx} [x^{-3}y] = x^{-7/2}$$

$$\Leftrightarrow x^{-3}y = \int \frac{d}{dx} [x^{-3}y] dx = \int x^{-7/2} dx = -\frac{2}{5}x^{-5/2} + c$$

$$\Leftrightarrow y = x^3 \left[c - \frac{2}{5}x^{-5/2} \right] = cx^3 - \frac{2}{5}x^{1/2} .$$

5.2 i. Standard form: $x \frac{dy}{dx} + (5x + 2)y = \frac{20}{x} \quad \rightsquigarrow \quad \frac{dy}{dx} + \left(5 + \frac{2}{x}\right)y = \frac{20}{x^2} .$

Integrating factor: $\mu(x) = e^{\int [5 + 2/x] dx} = e^{5x + 2 \ln x} = e^{5x} x^2 .$

Continuing with the procedure:

$$e^{5x} x^2 \left[\frac{dy}{dx} + \left(5 + \frac{2}{x}\right)y = \frac{20}{x^2} \right] \quad \rightsquigarrow \quad \frac{d}{dx} [e^{5x} x^2 y] = 20e^{5x}$$

$$\Leftrightarrow e^{5x} x^2 y = \int \frac{d}{dx} [e^{5x} x^2 y] dx = \int 20e^{5x} dx = 4e^{5x} + c$$

$$\Leftrightarrow y = e^{-5x} x^{-2} [4e^{5x} + c] = x^{-2} [4 + ce^{-5x}] .$$

5.3 a. First, find the general solution:

$$\mu(x) = e^{\int (-3) dx} = e^{-3x}$$

$$\Leftrightarrow e^{-3x} \left[\frac{dy}{dx} - 3y = 6 \right] \quad \rightsquigarrow \quad \frac{d}{dx} [e^{-3x} y] = 6e^{-3x}$$

$$\Leftrightarrow e^{-3x} y = \int 6e^{-3x} dx = -2e^{-3x} + c \quad \rightsquigarrow \quad y = ce^{3x} - 2 .$$

Then apply the initial condition:

$$5 = y(0) = ce^{3 \cdot 0} - 2 = c - 2 \quad \rightsquigarrow \quad c = 5 + 2 = 7 .$$

So the solution is $y(x) = 7e^{3x} - 2$.

5.3 c. General solution:

$$\mu(x) = e^{\int 5 dx} = e^{5x}$$

$$\Leftrightarrow e^{5x} \left[\frac{dy}{dx} + 5y = e^{-3x} \right] \quad \rightsquigarrow \quad \frac{d}{dx} [e^{5x} y] = e^{2x}$$

$$\Leftrightarrow e^{5x} y = \int e^{2x} dx = \frac{1}{2}e^{2x} + c \quad \rightsquigarrow \quad y = \frac{1}{2}e^{-3x} + ce^{-5x} .$$

Apply the initial condition:

$$0 = y(0) = \frac{1}{2}e^{-3 \cdot 0} + ce^{-5 \cdot 0} = \frac{1}{2} + c \quad \rightsquigarrow \quad c = -\frac{1}{2} .$$

So the solution is $y(x) = \frac{1}{2}e^{-3x} - \frac{1}{2}e^{-5x}$.

5.3 e. General solution:

$$\begin{aligned}
 x \frac{dy}{dx} &= y + x^2 \cos(x) \quad \rightsquigarrow \quad \frac{dy}{dx} - \frac{1}{x}y = x \cos(x) \\
 \hookrightarrow \quad \mu(x) &= e^{\int (-1/x) dx} = e^{-\ln|x|} = |x|^{-1} \quad (\text{use } \mu(x) = x^{-1}) \\
 \hookrightarrow \quad x^{-1} \left[\frac{dy}{dx} - \frac{1}{x}y = x \cos(x) \right] &\rightsquigarrow \quad \frac{d}{dx} [x^{-1}y] = \cos(x) \\
 \hookrightarrow \quad x^{-1}y &= \int \cos(x) dx = \sin(x) + c \quad \rightsquigarrow \quad y = x \sin(x) + cx \quad .
 \end{aligned}$$

Apply the initial condition:

$$0 = y\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \sin\left(\frac{\pi}{2}\right) + c \cdot \frac{\pi}{2} = \frac{\pi}{2} \cdot 1 + c \cdot \frac{\pi}{2} \quad \rightsquigarrow \quad c = -1 \quad .$$

Hence, $y(x) = x \sin(x) - x$.

5.4 a.

$$\begin{aligned}
 \mu(x) &= e^{\int 6x dx} = e^{3x^2} \\
 \hookrightarrow \quad e^{3x^2} \left[\frac{dy}{dx} + 6xy = \sin(x) \right] &\rightsquigarrow \quad \frac{d}{dx} [e^{3x^2} y] = e^{3x^2} \sin(x) \\
 \hookrightarrow \quad \int_0^x \frac{d}{ds} [e^{3s^2} y] ds &= \int_0^x e^{3s^2} \sin(s) ds \\
 \hookrightarrow \quad e^{3x^2} y(x) - \underbrace{e^{3 \cdot 0^2} y(0)}_{=1 \cdot 4} &= \int_0^x e^{3s^2} \sin(s) ds \\
 \hookrightarrow \quad e^{3x^2} y(x) &= 4 + \int_0^x e^{3s^2} \sin(s) ds \\
 \hookrightarrow \quad y &= e^{-3x^2} \left[4 + \int_0^x e^{3s^2} \sin(s) ds \right] \quad .
 \end{aligned}$$

5.4 c.

$$\begin{aligned}
 x \frac{dy}{dx} - y &= x^2 e^{-x^2} \quad \rightsquigarrow \quad \frac{dy}{dx} - \frac{1}{x}y = x e^{-x^2} \\
 \hookrightarrow \quad \mu(x) &= e^{\int (-1/x) dx} = \dots = \frac{1}{x} \\
 \hookrightarrow \quad \frac{1}{x} \left[\frac{dy}{dx} - \frac{1}{x}y = x e^{-x^2} \right] &\rightsquigarrow \quad \frac{d}{dx} \left[\frac{y}{x} \right] = e^{-x^2} \\
 \hookrightarrow \quad \int_3^x \frac{d}{ds} \left[\frac{y}{s} \right] ds &= \int_3^x e^{-s^2} ds \\
 \hookrightarrow \quad \frac{y(x)}{x} - \frac{y(3)}{3} &= \int_3^x e^{-s^2} ds \quad \rightsquigarrow \quad \frac{y(x)}{x} - \frac{8}{3} = \int_3^x e^{-s^2} ds \\
 \hookrightarrow \quad y(x) &= x \left[\frac{8}{3} + \int_3^x e^{-s^2} ds \right] \quad .
 \end{aligned}$$