Chapter 4: Separable First-Order Equations

4.3 a. Factoring out y^2, we get $\frac{dy}{dx} = (3 - \sin(x))\, y^2$,
which is $\frac{dy}{ds} = f(x)g(y)$,
with $f(x) = 3 - \sin(x)$ and $g(y) = y^2$.
So the equation is separable.

4.3 c. $x\frac{dy}{dx} = (x-y)^2 \;\rightarrowtail\; \frac{dy}{dx} = \frac{(x-y)^2}{x} \neq f(x)g(y)$ for any choice of f and g .
So the differential equation is not separable.

4.3 e. $\frac{dy}{dx} + 4y = 8 \;\rightarrowtail\; \frac{dy}{dx} = 8 - y = 4(2-y) = f(x)g(y)$
with $f(x) = 4$ and $g(y) = 2 - y$. So the differential equation is separable.

4.3 g. $\frac{dy}{dx} + 4y = x^2 \;\rightarrowtail\; \frac{dy}{ds} = x^2 - 4y \neq f(x)g(y)$ for any choice of f and g .
So the differential equation is not separable.

4.4 a.

$$\frac{dy}{dx} = \frac{x}{y} \;\rightarrowtail\; y\frac{dy}{dx} = x \;\rightarrowtail\; \int y\frac{dy}{dx}\,dx = \int x\,dx$$

$$\hookrightarrow \int y\,dy = \int x\,dx \;\rightarrowtail\; \tfrac{1}{2}y^2 = \tfrac{1}{2}x^2 + C \;\rightarrowtail\; y^2 = x^2 + \underbrace{2C}_{c}$$

$$\hookrightarrow \qquad y = \pm\sqrt{x^2 + c} \quad .$$

4.4 c.

$$\left[xy\frac{dy}{dx} = y^2 + 9\right]\left[\frac{1}{x\left(y^2+9\right)}\right] \;\rightarrowtail\; \frac{y}{y^2+9}\frac{dy}{dx} = \frac{1}{x}$$

$$\hookrightarrow \int \frac{y}{y^2+9}\frac{dy}{dx}\,dx = \int \frac{1}{x}\,dx \;\rightarrowtail\; \frac{1}{2}\int \frac{2y}{y^2+9}\,dy = \int \frac{1}{x}\,dx$$

$$\hookrightarrow \tfrac{1}{2}\ln\left|y^2+9\right| = \ln|x| + C \;\rightarrowtail\; \ln\left|y^2+9\right| = 2\ln|x| + 2C$$

$$\hookrightarrow y^2 + 9 = \pm e^{2\ln|x|+2C} = \pm e^{2\ln|x|}e^{2C} = \pm e^{2C}e^{\ln x^2} = Ax^2$$

$$\hookrightarrow \qquad y^2 = Ax^2 - 9 \;\rightarrowtail\; y = \pm\sqrt{Ax^2 - 9} \quad .$$

4.4 e.

$$\int \cos(y)\,\frac{dy}{dx}\,dx = \int \sin(x)\,dx \;\rightarrowtail\; \int \cos(y)\,dy = \int \sin(x)\,dx$$

$$\hookrightarrow \qquad \sin(y) = -\cos(x) + c \;\rightarrowtail\; y = \arcsin(c - \cos(x)) \quad .$$

4.5 a. The general solution (from the solution to exercise 4.4 a) is

$$y = \pm\sqrt{x^2 + c} \quad .$$

Applying the initial condition, we have

$$3 = y(1) = \pm\sqrt{1^2 + c} = \pm\sqrt{1 + c}$$

Since 3 is positive, we must take the positive square root. For c, we then have

$$3 = \sqrt{1 + c} \quad \rightarrowtail \quad 3^2 = 1 + c \quad \rightarrowtail \quad c = 9 - 1 = 8 \quad .$$

So the solution is $y = \sqrt{x^2 + 8}$.

4.5 c. Finding the general solution to the differential equation:

$$y\frac{dy}{dx} = xy^2 + x = x\left(y^2 + 1\right) \quad \rightarrowtail \quad \frac{y}{y^2 + 1}\frac{dy}{dx} = x$$

$$\hookrightarrow \quad \int \frac{y}{y^2 + 1}\frac{dy}{dx}\,dx = \int x\,dx \quad \rightarrowtail \quad \frac{1}{2}\int \frac{2y}{y^2 + 1}\,dy = \int x\,dx$$

$$\hookrightarrow \quad \frac{1}{2}\ln\left(y^2 + 1\right) = \frac{1}{2}x^2 + C \quad \rightarrowtail \quad \ln\left(y^2 + 1\right) = x^2 + c$$

$$\hookrightarrow \quad y^2 + 1 = e^{x^2 + c} = e^{x^2}e^c = e^{x^2}A$$

$$\hookrightarrow \quad y^2 = Ae^{x^2} - 1 \quad \rightarrowtail \quad y = \pm\sqrt{Ae^{x^2} - 1} \quad .$$

Applying the initial condition:

$$-2 = y(0) = \pm\sqrt{Ae^{0^2} - 1} = \pm\sqrt{A - 1} \quad .$$

So we take the negative square root, and then solve for A :

$$-2 = -\sqrt{A - 1} \quad \rightarrowtail \quad 4 = A - 1 \quad \rightarrowtail \quad A = 5 \quad .$$

So the solution is $y = -\sqrt{5e^{x^2} - 1}$.

4.6 a. $0 = \dfrac{dy}{dx} = xy - 4x = x(y - 4) \quad \rightarrowtail \quad 0 = y - 4 \quad \rightarrowtail \quad y = 4 \quad .$

4.6 c.

$$y\frac{dy}{dx} = xy^2 - 9x \quad \rightarrowtail \quad \frac{dy}{dx} = \frac{xy^2 - 9x}{y} = x \cdot \frac{y^2 - 9}{y}$$

$$\hookrightarrow \quad 0 = y^2 - 9 \quad \rightarrowtail \quad y^2 = 9 \quad \rightarrowtail \quad y = \pm\sqrt{9} = \pm 3 \quad .$$

So the two constant solutions are $y = 3$ and $y = -3$.

4.6 e. $0 = \dfrac{dy}{dx} = e^{x + y^2} = e^x e^{y^2} \quad .$

But there are no values of y such that $e^{y^2} = 0$. So there are no constant solutions.

4.7 a. From the answer to exercise 4.6 a, we know $y = 4$ is the only constant solution. To find the nonconstant solutions:

$$\frac{dy}{dx} = xy - 4x \quad \rightarrowtail \quad \frac{dy}{dx} = x(y-4) \quad \rightarrowtail \quad \frac{1}{y-4}\frac{dy}{dx} = x$$

$$\hookrightarrow \quad \int \frac{1}{y-4}\frac{dy}{dx}dx = \int x\,dx \quad \rightarrowtail \quad \ln|y-4| = \frac{1}{2}x^2 + c$$

$$\hookrightarrow \quad |y-4| = \exp\left(\frac{1}{2}x^2 + c\right) = e^c \exp\left(\frac{1}{2}x^2\right)$$

$$\hookrightarrow \quad y - 4 = \pm e^c \exp\left(\frac{1}{2}x^2\right) = A\exp\left(\frac{1}{2}x^2\right)$$

$$\hookrightarrow \quad y = 4 + A\exp\left(\frac{1}{2}x^2\right) \qquad (\text{with } A = \pm e^c \neq 0) \quad .$$

Since, the last equation reduces to the constant solution $y = 4$ when $A = 0$, that last equation without restrictions on A can serve as the general solution.

4.7 c. $\frac{dy}{dx} = 3y^2 - y^2\sin(x) = y^2(3 - \sin(x))$.
Constant solutions:

$$0 = \frac{dy}{dx} = y^2(3-\sin(x)) \quad \rightarrowtail \quad y = 0 \text{ is the constant solution} \quad .$$

Other solutions: $\frac{dy}{dx} = y^2(3-\sin(x)) \quad \rightarrowtail \quad y^{-2}\frac{dy}{dx} = 3 - \sin(x)$

$$\hookrightarrow \quad \int y^{-2}\frac{dy}{dx}\,dx = \int (3-\sin(x))\,dx \quad \rightarrowtail \quad -y^{-1} = 3x + \cos(x) + C$$

$$\hookrightarrow \quad \frac{1}{y} = -3x - \cos(x) + c \quad \rightarrowtail \quad y = \frac{1}{c - 3x - \cos(x)} \quad .$$

In this case, no value of c in the last line yields the constant solution $y = 0$. So for the general solution we need both

$$y = \frac{1}{c - 3x - \cos(x)} \qquad \text{and} \qquad y = 0 \quad .$$

4.7 e. Constant solutions: $0 = \frac{dy}{dx} = \frac{y}{x} \quad \rightarrowtail \quad y = 0$ is the constant solution .
Other solutions:

$$\frac{dy}{dx} = \frac{y}{x} \quad \rightarrowtail \quad \frac{1}{y}\frac{dy}{dx} = \frac{1}{x} \quad \rightarrowtail \quad \int \frac{1}{y}\frac{dy}{dx}\,dx = \int \frac{1}{x}\,dx$$

$$\hookrightarrow \quad \ln|y| = \ln|x| + C \quad \rightarrowtail \quad y = \pm e^{\ln|x|+C} = \pm e^C e^{\ln|x|} = Ax \quad .$$

Since the last equation becomes the constant solution $y = 0$ when $A = 0$, we can use that equation. $y(x) = Ax$, for the general solution.

4.7 g. $\left(x^2+1\right)\frac{dy}{dx} = y^2 + 1 \quad\rightarrowtail\quad \frac{dy}{dx} = \frac{y^2+1}{x^2+1} > 0 \quad\rightarrowtail\quad$ no constant solution.
Nonconstant solutions:

$$\frac{dy}{dx} = \frac{y^2+1}{x^2+1} \quad\rightarrowtail\quad \frac{1}{y^2+1}\frac{dy}{dx} = \frac{1}{x^2+1}$$

$$\hookrightarrow \quad \int \frac{1}{y^2+1}\frac{dy}{dx}\,dx = \int \frac{1}{x^2}\,dx \quad\rightarrowtail\quad \arctan(y) = \arctan(x) + c \quad .$$

Solving for y then yields the general solution $y = \tan(\arctan(x) + c)$.

4.7 i. There are no constant solutions since $e^{-y} > 0$ for all y. So all solutions are nonconstant:

$$\frac{dy}{dx} = e^{-y} \quad\rightarrowtail\quad e^{y}\frac{dy}{dx} = 1 \quad\rightarrowtail\quad \int e^{y}\frac{dy}{dx}\,dx = \int 1\,dx$$

$$\hookrightarrow \quad e^{y} = x + c \quad\rightarrowtail\quad y = \ln(x+c) \quad .$$

4.7 k. $0 = \frac{dy}{dx} = 3xy^3 \quad\rightarrowtail\quad y = 0$ is the constant solution.
Nonconstant solutions:

$$\frac{dy}{dx} = 3xy^3 \quad\rightarrowtail\quad y^{-3}\frac{dy}{dx} = 3x \quad\rightarrowtail\quad \int y^{-3}\frac{dy}{dx}\,dx = \int 3x\,dx$$

$$\hookrightarrow \quad -\frac{1}{2}y^{-2} = \frac{3}{2}x^2 + C \quad\rightarrowtail\quad y^{-2} = -3x^2 + c$$

$$\hookrightarrow \quad y = \pm\left(c - 3x^2\right)^{-1/2} \quad .$$

This last equation does not reduce to the constant solution $y = 0$ for any choice of c. So, to describe the general solution we need both

$$y = \pm\left(c - 3x^2\right)^{-1/2} \qquad \text{and} \qquad y = 0 \quad .$$

4.7 m. $\frac{dy}{dx} - 3x^2y^2 = -3x^2 \quad\rightarrowtail\quad \frac{dy}{dx} = 3xy^2 - 3x^2 = 3x^2\left(y^2-1\right) \quad .$
Constant solutions:

$$0 = \frac{dy}{dx} = 3x^2\left(y^2-1\right) \quad\rightarrowtail\quad y^2 - 1 = 0 \quad\rightarrowtail\quad y = \pm 1$$

$$\hookrightarrow \quad y = 1 \quad \text{and} \quad y = -1 \quad \text{are the two constant solutions} \quad .$$

Nonconstant solutions:

$$\frac{dy}{dx} = 3x^2\left(y^2-1\right) \quad\rightarrowtail\quad \frac{1}{y^2-1}\frac{dy}{dx} = 3x^2$$

$$\hookrightarrow \quad \int \frac{1}{y^2-1}\frac{dy}{dx}\,dx = \int 3x^2\,dx = x^3 + C \quad . \qquad (\star)$$

For the remaining integral, you can use partial fractions. Begin by noting that

$$\frac{1}{y^2-1} = \frac{1}{(y+1)(y-1)} = \frac{A}{y+1} + \frac{B}{y-1}$$
$$= \frac{A(y-1) + B(y+1)}{(y+1)(y-1)} = \frac{A(y-1) + B(y+1)}{y^2-1} \quad .$$

So A and B are numbers such that

$$1 = A(y-1) + B(y+1) \qquad \text{for all} \quad y \quad .$$

Now solve for A and B, possibly by first setting $y = 1$,

$$1 = A(1-1) + B(1+1) \quad \rightarrowtail \quad B = \frac{1}{2} \quad ,$$

and then setting $y = -1$,

$$1 = A(-1-1) + B(-1+1) \quad \rightarrowtail \quad A = -\frac{1}{2} \quad .$$

So,

$$\frac{1}{y^2-1} = \frac{-1/2}{y+1} + \frac{1/2}{y-1} \quad ,$$

and (ignoring the arbitrary constant)

$$\int \frac{1}{y^2-1}\frac{dy}{dx}\,dx = \int \left[\frac{1}{2}\cdot\frac{1}{y-1} - \frac{1}{2}\cdot\frac{1}{y+1}\right] dx$$
$$= \frac{1}{2}\left[\ln|y-1| - \ln|y+1|\right] = \frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| \quad .$$

Combining this with equation (⋆):

$$\frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| = x^3 + C \quad \rightarrowtail \quad \frac{y-1}{y+1} = \pm e^{2x^3+2C} = Ae^{2x^3}$$

$$\hookrightarrow \quad y - 1 = yAe^{2x^3} + Ae^{2x^3} \quad \rightarrowtail \quad y - yAe^{2x^3} = 1 + Ae^{2x^3}$$

$$\hookrightarrow \quad y\left(1 - Ae^{2x^3}\right) = 1 + Ae^{2x^3} \quad \rightarrowtail \quad y = \frac{1 + Ae^{2x^3}}{1 - Ae^{2x^3}} \quad .$$

If $A = 0$, the last equation reduces to $y = 1$, but the equation does not reduce to $y = -1$ for any choice of A. So all the solutions are given by using both

$$y(x) = \frac{1 + Ae^{2x^3}}{1 - Ae^{2x^3}} \qquad \text{and} \qquad y = -1 \quad .$$

4.7 o. $\frac{dy}{dx} = 200y - 2y^2 = 2(100-y)y \quad .$
Constant solutions: $0 = 2(100-y)y \quad \rightarrowtail \quad y = 0 \quad \text{and} \quad y = 100 \quad .$
Nonconstant solutions:

$$\frac{dy}{dx} = 2(100-y)y \quad \rightarrowtail \quad \int \frac{1}{(100-y)y}\frac{dy}{dx}\,dx = \int 2\,dx$$

$$\hookrightarrow \quad \int \frac{1}{(100-y)y}\,dy = 2x + c \quad . \qquad (\star)$$

Using partial fractions (see the solution to exercise 4.7 m),

$$\int \frac{1}{(100-y)y}\frac{dy}{dx}\,dx = \cdots$$

$$= \frac{1}{100}\int \left[\frac{1}{y} + \frac{1}{100-y}\right] dy$$

$$= \frac{1}{100}\left[\ln|y| - \ln|100-y|\right] = \frac{1}{100}\ln\left|\frac{y}{100-y}\right| \quad .$$

Combined with equation (⋆), this gives

$$\frac{1}{100}\ln\left|\frac{y}{100-y}\right| = 2x + c \quad \rightarrowtail \quad \frac{y}{100-y} = \pm e^{100(2x+c)} = Ae^{200x}$$

$$\hookrightarrow \quad y = 100Ae^{200x} - yAe^{200x} \quad \rightarrowtail \quad y\left(1 + Ae^{200x}\right) = 100Ae^{200x}$$

$$\hookrightarrow \quad y = \frac{100Ae^{200x}}{1 + Ae^{200x}} \quad .$$

The last reduces to $y = 0$ if $A = 0$, but does not reduce to $y = 100$ for any choice of A. So all the solutions are given by using both

$$y = \frac{100Ae^{200x}}{1 + Ae^{200x}} \qquad \text{and} \qquad y = 100$$

4.8 a. $\frac{dy}{dx} - 2y = -10 \quad \rightarrowtail \quad \frac{dy}{dx} = 2y - 10 = 2(y-5)$

Clearly, the only constant solution is $y = 5$ which does not satisfy the initial condition $y(0) = 8$. So we must find the nonconstant solutions:

$$\frac{dy}{dx} = 2(y-5) \quad \rightarrowtail \quad \frac{1}{y-5}\frac{dy}{dx} = 2$$

$$\hookrightarrow \quad \int \frac{1}{y-5}\frac{dy}{dx}\,dx = \int 2\,dx \quad \rightarrowtail \quad \ln|y-5| = 2x + c$$

$$\hookrightarrow \quad y - 5 = \pm e^{2s+c} = Ae^{2x} \quad \rightarrowtail \quad y = 5 + Ae^{2x} \quad .$$

Applying the initial condition:

$$8 = y(0) = 5 + Ae^{2\cdot 0} = 5 + A \quad \rightarrowtail \quad A = 8 - 5 = 3 \quad .$$

So the solution to the initial-value problem is $y = 5 + 3e^{2x}$.

4.8 c.

$$\frac{dy}{dx} = 2x - 1 + 2xy - y \quad \rightarrowtail \quad \frac{dy}{dx} = (2x-1) + (2x-1)y$$

$$\hookrightarrow \quad \frac{dy}{dx} = (2x-1)(1+y) \quad .$$

In this case, the derivative is zero if $y = -1$. So $y = -1$ is the constant solution. Moreover, this constant solution satisfies the initial condition $y(0) = -1$. So the constant solution $y = -1$ is the solution to the initial-value problem.

4.8 e. $x\frac{dy}{dx} = y^2 - y \quad \rightarrowtail \quad \frac{dy}{dx} = \frac{y(y-1)}{x}$.

Clearly, the only constant solutions are $y = 0$ and $y = 1$, neither of which satisfies the initial condition $y(1) = 2$. So we must find the nonconstant solutions:

$$\frac{dy}{dx} = \frac{y(y-1)}{x} \quad \rightarrowtail \quad \frac{1}{y(y-1)}\frac{dy}{dx} = \frac{1}{x}$$

$$\hookrightarrow \quad \int \frac{1}{y(y-1)}\frac{dy}{dx}\,dx = \int \frac{1}{x}\,dx = \ln|x| + c \quad . \qquad (\star)$$

Using partial fractions (see the solution to exercise 4.7 m), we find that

$$\frac{1}{y(y-1)} = \cdots = \frac{1}{y-1} - \frac{1}{y} \quad .$$

So, continuing from equation ($\star$), we have

$$\ln|y-1| - \ln|y| = \ln|x| + c$$

$$\hookrightarrow \quad \ln\left|\frac{y-1}{y}\right| = \ln|x| + c \quad \rightarrowtail \quad \frac{y-1}{y} = \pm e^{\ln|x|+c} = Ax$$

$$\hookrightarrow \quad 1 - \frac{1}{y} = Ax \quad \rightarrowtail \quad \frac{1}{y} = 1 - Ax \quad \rightarrowtail \quad y = \frac{1}{1-Ax} \quad .$$

Applying the initial condition:

$$2 = y(1) = \frac{1}{1 - A\cdot 1} \quad \rightarrowtail \quad 1 - A = \frac{1}{2} \quad \rightarrowtail \quad A = \frac{1}{2} \quad .$$

So,
$$y = \frac{1}{1 - \frac{1}{2}x} = \frac{2}{2-x} \quad .$$

4.8 g. $\left(y^2 - 1\right)\frac{dy}{dx} = 4xy \quad \rightarrowtail \quad \frac{dy}{dx} = 4x \cdot \frac{y}{y^2-1}$.

The only constant solution is $y = 0$, which does not satisfy the initial condition $y(0) = 1$. So we must find the nonconstant solutions:

$$\left(y^2 - 1\right)\frac{dy}{dx} = 4xy \quad \rightarrowtail \quad \frac{y^2-1}{y}\frac{dy}{dx} = 4x$$

$$\hookrightarrow \quad \int \frac{y^2-1}{y}\frac{dy}{dx}\,dx = \int 4x\,dx \quad \rightarrowtail \quad \int \left[y - \frac{1}{y}\right] dy = \int 4x\,dx$$

$$\hookrightarrow \quad \frac{1}{2}y^2 - \ln|y| = 2x^2 + c \quad \rightarrowtail \quad y^2 - 2\ln|y| = 4x^2 + 2c \quad .$$

Getting an explicit solution here is not practical. So we will apply the initial condition $y(0) = 1$ to the last equation:

$$1^2 - 2\ln|1| = 4\cdot 2^2 + 2c \quad \rightarrowtail \quad 1 - 0 = 16 + 2c \quad \rightarrowtail \quad 2c = -15 \quad .$$

So the (implicit) solution is $y^2 - 2\ln|y| = 4x^2 - 15$.

4.10 a. From the answer to exercise 4.8 a we know the solution is $y(x) = 5 + 3e^{2x}$ which is valid for all values of x. So the interval is $(-\infty, \infty)$.

4.10 c. From the answer to exercise 4.8 e, we know the solution is

$$y = \frac{2}{2-x} \quad ,$$

which is continuous everywhere except at $x = 2$ where it 'blows up'. And since the initial condition is given at $x = 1 < 2$, the interval over which this solution is valid is $(-\infty, 2)$.

4.10 e. From the answer to exercise 4.7 k, we know all the solutions to the differential equation are given by

$$y(x) = \pm\left(c - 3x^2\right)^{-1/2} \qquad \text{and} \qquad y = 0 \quad .$$

Obviously, $y = 0$ does not satisfy the initial condition. So we apply the initial condition with the nonconstant solution formula:

$$\frac{1}{2} = y(0) = \pm\left(c - 3 \cdot 0^2\right)^{-1/2} = \pm\frac{1}{\sqrt{c}} \quad \rightarrowtail \quad c = 4 \quad ,$$

and

$$y = +\left(4 - 3x^2\right)^{-1/2} = \frac{1}{\sqrt{4-3x^2}} \quad .$$

This is valid over the largest interval containing 0 and with $4 - 3x^2 > 0$. But

$$4 - 3x^2 > 0 \quad \rightarrowtail \quad x^2 < \frac{4}{3} \quad \rightarrowtail \quad -\sqrt{\frac{4}{3}} < x < +\frac{2}{\sqrt{3}} \quad .$$

So the interval is $\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$.