Chapter 36: Critical Points, Direction Fields and Trajectories

36.1 a. Each constant/equilibrium solution $(x, y) = (x_0, y_0)$ is a solution to the algebraic system

$$\begin{aligned} 0 &= 2x - 5y \\ 0 &= 3x - 7y \end{aligned} \quad , \qquad (\star)$$

which is obtained from the original system of differential equations by simply setting $x' = 0$ and $y' = 0$.

From the first equation in ($\star$), we have $y = \frac{2}{5}x$. With the second equation in ($\star$), this then gives

$$0 = 3x - \frac{2}{5}x = \frac{13}{5}x \quad \rightarrowtail \quad x = 0 \quad .$$

Plugging $x = 0$ back into $y = \frac{2}{5}x$ then yields $y = 0$.

Hence, there is just one constant/equilibrium solution; namely, $(x, y) = (0, 0)$.

36.1 c. Each constant/equilibrium solution $(x, y) = (x_0, y_0)$ is a solution to the algebraic system

$$\begin{aligned} 0 &= 3x + y \\ 0 &= 6x + 2y \end{aligned} \quad , \qquad (\star)$$

which is obtained from the original system of differential equations by simply setting $x' = 0$ and $y' = 0$.

From the first equation in ($\star$), we have $y = -3x$. This, with the second equation in ($\star$), then gives

$$0 = 6x + 2(-3x) = 0 \quad .$$

In other words, any pair of constants (x_0, y_0) with $y_0 = -3x_0$ satisfies algebraic system ($\star$). So every $(x, y) = (x_0, y_0)$ with $y_0 = -3x_0$ is a constant/equilibrium solution to the original system of differential equations.

36.1 e. The algebraic system for the equilibrium solutions is

$$\begin{aligned} 0 &= x^2 - y^2 \\ 0 &= x^2 - 6x + 8 \end{aligned} \quad . \qquad (\star)$$

From the second equation in ($\star$):

$$0 = \underbrace{x^2 - 6x + 8}_{(x-2)(x-4)} \quad \rightarrowtail \quad x = 2 \quad \text{or} \quad x = 4 \quad .$$

If $x = 2$, the first equation in system ($\star$) becomes

$$0 = x^2 - y^2 = 2^2 - y^2 \quad ,$$

from which it follows that $y = 2$ or $y = -2$. So, the two equilibrium solutions with $x = 2$ are $(2, 2)$ and $(2, -2)$.

On the other hand, if the second equation in system ($\star$) holds because $x = 4$, then the first equation in that system becomes

$$0 = x^2 - y^2 = 4^2 - y^2 \quad ,$$

from which it follows that $y = 4$ or $y = -4$. Hence, the two equilibrium solutions with $x = 4$ are $(4, 4)$ and $(4, -4)$.

Hence, the constant/equilibrium solutions to this system of differential equations are

$$(2,2) \quad , \quad (2,-2) \quad , \quad (4,4) \quad \text{and} \quad (4,-4) \quad .$$

36.1 g. The algebraic system for the equilibrium solutions is

$$\begin{aligned} 0 &= 4x - xy \\ 0 &= x^2 y + y^3 - x^2 - y^2 \end{aligned} \quad . \tag{$\star$}$$

From the first equation in ($\star$):

$$0 = 4x - xy = x(4-y) \quad \rightarrowtail \quad x = 0 \quad \text{or} \quad y = 4 \quad .$$

If the first equation in ($\star$) holds because $x = 0$, then the other equation in that system becomes

$$0 = 0 + y^3 - 0^2 y^3 - y^2 = y^2(y-1) \quad ,$$

from which it follows that $y = 0$ or $y = 1$. So, the two equilibrium solutions with $x = 0$ are $(0, 0)$ and $(0, 1)$.

On the other hand, if the first equation in system ($\star$) holds because $y = 4$, then the second equation in that system becomes

$$0 = 4x^2 + 4^3 - x^2 - 4^2 = 3x^2 + 3 \cdot 16$$

$$\hookrightarrow \qquad x^2 = -16 = \pm\sqrt{-16} = \pm 4i \quad .$$

But $4i$ is not a real number, as required. So there is no constant/equilibrium solution with $y = 4$.

Hence, the constant/equilibrium solutions to this system of differential equations are just those with $x = 0$,

$$(0,0) \quad \text{and} \quad (0,1) \quad .$$

36.2 a. At $(x, y) = (0, 0)$,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -x + 2y \\ 2x - y \end{bmatrix} = \begin{bmatrix} -0 + 2 \cdot 0 \\ 2 \cdot 0 - 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad .$$

So the direction arrow at $(x, y) = (0, 0)$ points 0 units in the X-direction and 0 units in the Y-direction. In other words, $(0, 0)$ is a critical point and we just plot a dot (not an arrow) there.

At $(x, y) = (2, 0)$,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -x + 2y \\ 2x - y \end{bmatrix} = \begin{bmatrix} -2 + 2 \cdot 0 \\ 2 \cdot 2 - 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} = 2\begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad .$$

So the direction arrow at $(x, y) = (2, 0)$ points -1 unit in the X-direction and $+2$ units in the Y-direction.

At $(x, y) = (0, 2)$,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -x + 2y \\ 2x - y \end{bmatrix} = \begin{bmatrix} -0 + 2 \cdot 2 \\ 2 \cdot 0 - 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad .$$

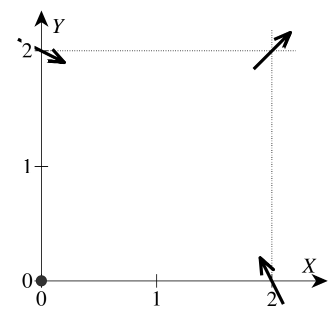
So the direction arrow at $(x, y) = (0, 2)$ points $+2$ units in the X-direction and -1 unit in the Y-direction.

At $(x, y) = (2, 2)$,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -x + 2y \\ 2x - y \end{bmatrix} = \begin{bmatrix} -2 + 2 \cdot 2 \\ 2 \cdot 2 - 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad .$$

So the direction arrow at $(x, y) = (2, 2)$ points $+1$ unit in the X-direction and $+1$ unit in the Y-direction.

Sketching these arrows at the given points then yields the direction field sketched to the right



36.3. At $(x, y) = (0, 0)$,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} (1 - 2x)(y + 1) \\ x - y \end{bmatrix} = \begin{bmatrix} (1 - 2 \cdot 0)(0 + 1) \\ 0 - 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad .$$

So the direction arrow at $(x, y) = (0, 0)$ points $+1$ unit in the X-direction and 0 units in the Y-direction.

At $(x, y) = \left(\frac{1}{2}, 0\right)$,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} (1 - 2x)(y + 1) \\ x - y \end{bmatrix} = \begin{bmatrix} \left(1 - 2 \cdot \frac{1}{2}\right)(0 + 1) \\ \frac{1}{2} - 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad .$$

So the direction arrow at $(x, y) = \left(\frac{1}{2}, 0\right)$ points 0 units in the X-direction and $+1$ unit in the Y-direction.

At $(x, y) = (1, 0)$,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} (1 - 2x)(y + 1) \\ x - y \end{bmatrix} = \begin{bmatrix} (1 - 2 \cdot 1)(0 + 1) \\ 1 - 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad .$$

So the direction arrow at $(x, y) = (1, 0)$ points -1 unit in the X-direction and $+1$ unit in the Y-direction.

At $(x, y) = \left(0, \frac{1}{2}\right)$,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} (1 - 2x)(y + 1) \\ x - y \end{bmatrix} = \begin{bmatrix} (1 - 2 \cdot 0)\left(\frac{1}{2} + 1\right) \\ 0 - \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3 \\ -2 \end{bmatrix} \quad .$$

So the direction arrow at $(x, y) = \left(0, \frac{1}{2}\right)$ points 3 units in the X-direction and -2 units in the Y-direction.

At $(x, y) = \left(\frac{1}{2}, \frac{1}{2}\right)$,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} (1-2x)(y+1) \\ x-y \end{bmatrix} = \begin{bmatrix} \left(1 - 2\cdot\frac{1}{2}\right)\left(\frac{1}{2}+1\right) \\ \frac{1}{2} - \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

So the direction arrow at $(x, y) = \left(\frac{1}{2}, \frac{1}{2}\right)$ points 0 units in the X-direction and 0 units in the Y-direction. In other words, $(0, 0)$ is a critical point and we just plot a dot (not an arrow) there.

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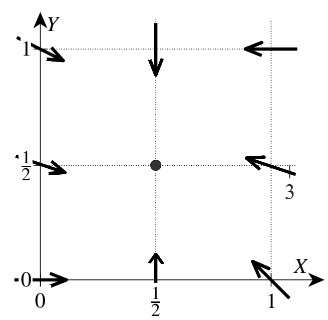
At $(x, y) = (1, 1)$,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} (1-2x)(y+1) \\ x-y \end{bmatrix} = \begin{bmatrix} (1-2\cdot 1)\,(1+1) \\ 1-1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} -1 \\ 0 \end{bmatrix} .$$

So the direction arrow at $(x, y) = (1, 1)$ points -1 unit in the X-direction and 0 units in the Y-direction.

Sketching these arrows at the given points then yields the direction field sketched to the right.

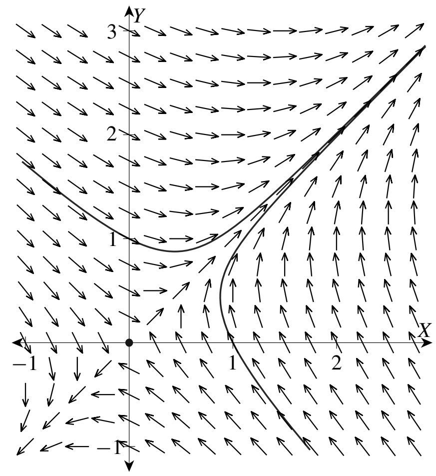


36.4 a. For the critical points, we solve the algebraic system

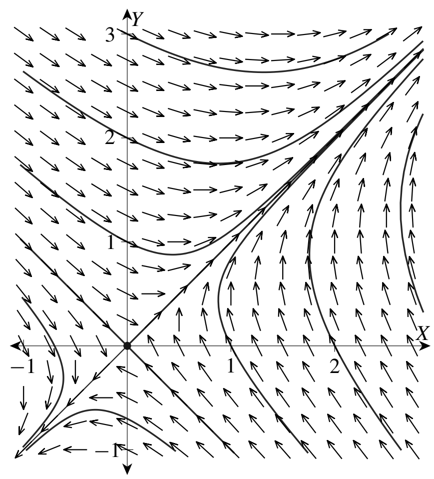
$$\begin{aligned} 0 &= -x + 2y \\ 0 &= 2x - y \end{aligned} .$$

This is easily solved and yields the single critical point $(0, 0)$. Plot this point on the direction field.

36.4 b. Starting at point $(1,0)$ sketch the curve that best "follows" the arrows. Then do the same thing starting at point $(0,1)$. The result should be similar to the sketch at the right.



36.4 c. Repeat what was done in the previous part using a number of different points. Try to get curves sketched throughout the region. The result should be similar to the sketch at the right.



36.4 d. Following the trajectory through point $(1,0)$, sketched above, it should be clear that $x(t)$ and $y(t)$ become large and nearly equal as $t \to \infty$.

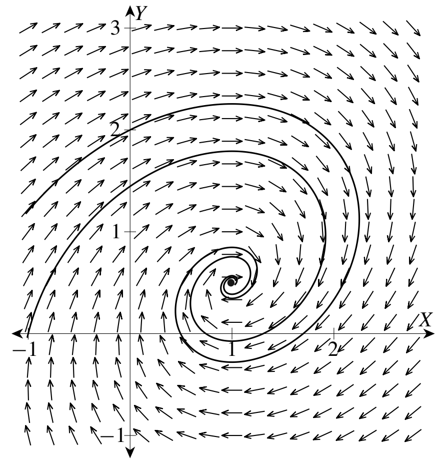
36.4 e. Since there are arrows near critical point $(0,0)$ pointing away from $(0,0)$, it would appear that this critical point is unstable.

36.5 a. For the critical points, we solve the algebraic system

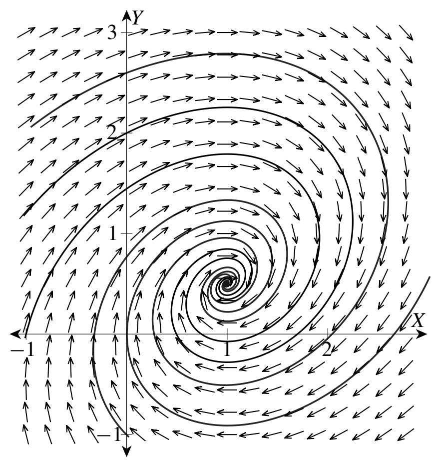
$$\begin{aligned} 0 &= -x + 2y \\ 0 &= -2x + 2 \end{aligned} \quad .$$

From the second equation we immediately get $x = 1$. This and the first equation yield $y = \frac{1}{2} \cdot 1 = \frac{1}{2}$. So there is one critical point, $\left(1, \frac{1}{2}\right)$.

36.5 b. Starting at point $(-1, 0)$ sketch the curve that best "follows" the arrows. Then do the same thing starting at point $(0, 2)$. The result should be similar to the sketch at the right.



36.5 c. Repeat what was done in the previous part using a number of different points. Try to get curves sketched throughout the region. The result should be similar to the sketch at the right.



36.5 d. Following the trajectory through point $(-1, 0)$, sketched above, it should be clear that $(x(t), y(t))$ spirals in towards the critical point as $t \to \infty$. So, as t gets large, $x(t) \to 1$ and $y(t) \to \frac{1}{2}$.

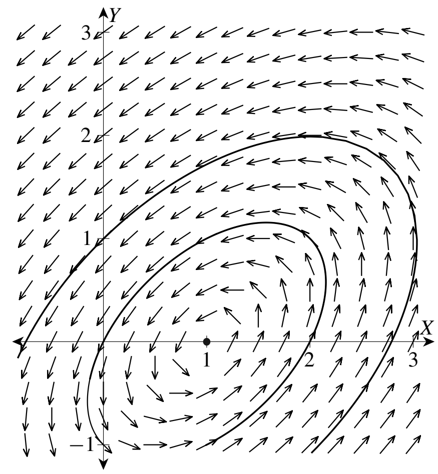
36.5 e. Since the trajectories appear to be spiralling in towards the critical point, it would appear that this critical point is asymptotically stable.

36.7 a. For the critical points, we solve the algebraic system

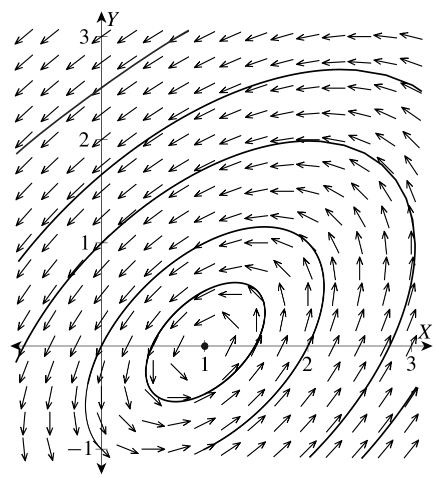
$$\begin{aligned} 0 &= x - 2y - 1 \\ 0 &= 2x - y - 2 \end{aligned} .$$

This is easily solved and yields the single critical point $(1, 0)$. Plot this point on the direction field.

36.7 b. Starting at point $(0, 0)$ sketch the curve that best "follows" the arrows. Keep in mind that, because the critical point is stable but not asymptotically stable, the trajectories near the critical point must stay near that point, but cannot spiral into the critical point. This suggests that the trajectories form closed loops. Then do the same thing starting at point $(0, 1)$. The result should be similar to the sketch at the right.



36.7 c. Repeat what was done in the previous part using a number of different points. Try to get curves sketched throughout the region. The result should be similar to the sketch at the right.



36.7 d. Following the trajectory through point $(0, 0)$, sketched above, it appears that, as $t \to \infty$, $(x(t), y(t))$ "orbits" critical point $(1, 0)$ counterclockwise in a somewhat elliptic path.

36.9 a. For the critical points, we solve the algebraic system

$$\begin{aligned} 0 &= x + 4y^2 - 1 \\ 0 &= 2x - y - 2 \end{aligned} \quad .$$

From the first equation, we get $x = 1 - 4y^2$. Combined with the second equation, we have

$$0 = 2\left[1 - 4y^2\right] - y - 2 = -8y^2 - y = -y(8 + y) \quad ,$$

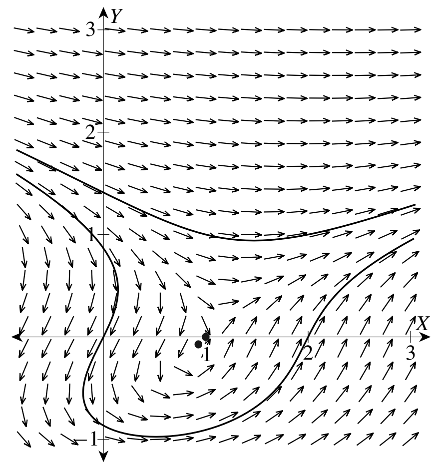
which tells us that $y = 0$ or $y = -\frac{1}{8}$.

If $y = 0$, the first equation becomes $0 = x - 1$. So $x = 1$, and $(1, 0)$ is a critical point.

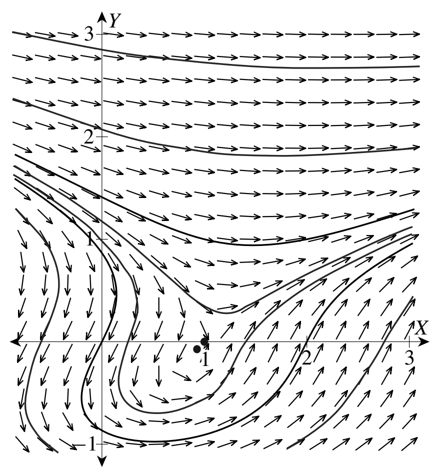
If $y = -\frac{1}{8}$ the first equation becomes $0 = x + 4 \cdot \frac{1}{8^2} - 1 = x = \frac{15}{16}$. So $x = \frac{15}{16}$, and $\left(\frac{15}{16}, -\frac{1}{8}\right)$ is a critical point.

There are no other solutions to the above algebraic system. So, there are two critical points, $(1, 0)$ and $\left(\frac{15}{16}, -\frac{1}{8}\right)$.

36.9 b. Starting at point $(0, 0)$ sketch the curve that best "follows" the arrows. Then do the same thing starting at point $(1, 1)$. The result should be similar to the sketch at the right.



36.9 c. Repeat what was done in the previous part using a number of different points. Try to get curves sketched throughout the region. The result should be similar to the sketch at the right.



36.11. Using the chain rule and fact that $\tilde{x}'(\tau) = f(\tilde{x}(\tau), \tilde{y}(\tau))$ for $\alpha < \tau < \beta$, we have

$$\begin{aligned} x'(t) &= \frac{d}{dt}[\tilde{x}(t - t_0)] \\ &= \tilde{x}'(t - t_0) \cdot \frac{d}{dt}[t - t_0] \\ &= \tilde{x}'(t - t_0) \\ &= f(\tilde{x}(t - t_0), \tilde{y}(t - t_0)) \\ &= f(x(t), y(t)) \qquad \text{whenever} \quad \alpha < t - t_0 < \beta \quad , \end{aligned}$$

verifying that $(x(t), y(t))$ satisfies the first equation in our system of differential equations when $\alpha + t_0 < t < \beta + t_0$.

Almost identical computations verify that the other differential equation in the system is also satisfied by $(x(t), y(t))$ when $\alpha + t_0 < t < \beta + t_0$. Thus, we have that the given (x, y) is a solution to the system of differential equations on the interval $(\alpha + t_0, \beta + t_0)$.

Confirming that the given (x, y) satisfies the desired initial condition is even easier:

$$\begin{bmatrix} x(t_0) \\ y(t_0) \end{bmatrix} = \begin{bmatrix} \tilde{x}(t_0 - t_0) \\ \tilde{y}(t_0 - t_0) \end{bmatrix} = \begin{bmatrix} \tilde{x}(0) \\ \tilde{y}(0) \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \quad .$$