Chapter 35: Systems of Differential Equations: A Starting Point

35.1 a. Plugging these formulas for x and y into the first equation we get

$$x' = 2y \quad \rightarrowtail \quad \frac{d}{dt}\left[\sin(2t) + \frac{1}{2}\right] = 2[\cos(2t)]$$

$$\hookrightarrow \qquad 2\cos(2t) + 0 = 2\cos(2t) \quad ,$$

which is true for all values of t. So the first equation in the system is satisfied.

Plugging these formulas for x and y into the second equation we get

$$y' = 1 - 2x \quad \rightarrowtail \quad \frac{d}{dt}[\cos(2t)] = 1 - 2\left[\sin(2t) + \frac{1}{2}\right]$$

$$\hookrightarrow \qquad -2\sin(2t) = -2\sin(2t) \quad ,$$

which is true for all values of t. So the second equation in the system is satisfied.

Since both equations in the system are satisfied by the given formulas of x and y, this function pair is a solution to the given system.

35.1 b. Plugging these formulas for x and y into the first equation we get

$$x' = 2y \quad \rightarrowtail \quad \frac{d}{dt}\left[e^{2t} - 1\right] = 2\left[e^{2t}\right]$$

$$\hookrightarrow \qquad 2e^{2t} + 0 = 2e^{2t} \quad ,$$

which is true for all values of t. So the first equation in the system is satisfied.

Plugging these formulas for x and y into the second equation we get

$$y' = 1 - 2x \quad \rightarrowtail \quad \frac{d}{dt}\left[e^{2t}\right] = 1 - 2\left[e^{2t} - 1\right]$$

$$\hookrightarrow \qquad 2e^{2t} = 3 - 2e^{2t} \quad \rightarrowtail \quad 4e^{2t} = 3 \quad ,$$

which is not true for all values of t. So the second equation in the system is not satisfied.

Since one equation in the system is not satisfied by the given formulas of x and y, this function pair is not a solution to the given system.

35.1 c. Plugging these formulas for x and y into the two equations, we get

$$x' = 2y \quad \rightarrowtail \quad \frac{d}{dt}\left[3\cos(2t) + \frac{1}{2}\right] = 2[-3\sin(2t)]$$

$$\hookrightarrow \qquad -6\sin(2t) + 0 = -6\sin(2t) \quad ,$$

and

$$y' = 1 - 2x$$

$$\hookrightarrow \qquad \frac{d}{dt}[-3\sin(2t)] = 1 - 2\left[3\cos(2t) + \frac{1}{2}\right]$$

$$\hookrightarrow \qquad -6\cos(2t) = -6\cos(2t) \quad .$$

Clearly, both equations in the system are satisfied by the given formulas of x and y (i.e., both equations remain true for all values of t). Thus, this function pair is a solution to the given system.

35.3 a. Plugging these formulas for x and y into the first equation we get

$$tx' + 2x = 15y$$

$$\hookrightarrow \quad t\frac{d}{dt}\left[3t^3\right] + 2 \cdot 3t^2 = 15\left[-t^3\right]$$

$$\hookrightarrow \quad 15t^3 = -15t^3 \quad ,$$

which is not true for any nonzero value of t . So this function pair is not a solution to the given system.

35.3 b. Plugging these formulas for x and y into the two equations, we get

$$tx' + 2x = 15y$$

$$\hookrightarrow \quad t\frac{d}{dt}\left[3t^3\right] + 2 \cdot 3t^2 = 15\left[t^3\right]$$

$$\hookrightarrow \quad 15t^3 = 15t^3 \quad ,$$

and

$$ty' = x \quad \rightarrowtail \quad t\frac{d}{dt}\left[t^3\right] = 3t^3$$

$$\hookrightarrow \quad 3t^3 = 3t^3 \quad .$$

So both equations in the system are satisfied by the given formulas of x and y , and, hence, this function pair is a solution to the given system.

35.3 c. Plugging these formulas for x and y into the two equations, we get

$$tx' + 2x = 15y$$

$$\hookrightarrow \quad t\frac{d}{dt}\left[-5t^{-5}\right] + 2\left[-5t^{-5}\right] = 15\left[t^{-5}\right]$$

$$\hookrightarrow \quad 15t^3 = 15t^3 \quad ,$$

and

$$ty' = x \quad \rightarrowtail \quad t\frac{d}{dt}\left[t^{-5}\right] = -5t^{-5}$$

$$\hookrightarrow \quad -5t^{-5} = -5t^{-5} \quad .$$

Since both equations in the system are satisfied by the given formulas of x and y , this function pair is a solution to the given system.

35.4 a. Plugging these formulas for x and y into the two equations, we get

$$
\begin{aligned}
x' &= x + 2y \\
\hookrightarrow \quad \frac{d}{dt}\left[c_1 e^{3t} + 2c_2 e^{-4t}\right] &= \left[c_1 e^{3t} + 2c_2 e^{-4t}\right] + 2\left[c_1 e^{3t} - 5c_2 e^{-4t}\right] \\
\hookrightarrow \quad 3c_1 e^{3t} + 2(-4)c_2 e^{-4t} &= (1+2)c_1 e^{3t} + (2-10)c_2 e^{-4t} \\
\hookrightarrow \quad 3c_1 e^{3t} - 8c_2 e^{-4t} &= 3c_1 e^{3t} - 8c_2 e^{-4t} \quad ,
\end{aligned}
$$

and

$$
\begin{aligned}
y' &= 5x - 2y \\
\hookrightarrow \quad \frac{d}{dt}\left[c_1 e^{3t} - 5c_2 e^{-4t}\right] &= 5\left[c_1 e^{3t} + 2c_2 e^{-4t}\right] - 2\left[c_1 e^{3t} - 5c_2 e^{-4t}\right] \\
\hookrightarrow \quad 3c_1 e^{3t} - 5(-4)c_2 e^{-4t} &= (5-2)c_1 e^{3t} + (10+10)c_2 e^{-4t} \\
\hookrightarrow \quad 3c_1 e^{3t} + 20c_2 e^{-4t} &= 3c_1 e^{3t} + 20c_2 e^{-4t} \quad .
\end{aligned}
$$

Clearly, both equations in the system are satisfied by the given formulas of x and y no matter what are the values of c_1 and c_2. So this function pair is a solution to the given system for any choice of constants c_1 and c_2.

35.4 b. From the initial conditions and given formulas for x and y, we have

$$7 = x(0) = c_1 e^{3\cdot 0} + 2c_2 e^{-4\cdot 0}$$

and

$$-7 = y(0) = c_1 e^{3\cdot 0} - 5c_2 e^{-4\cdot 0} \quad ,$$

which reduces to the simple algebraic system

$$
\begin{aligned}
c_1 + 2c_2 &= 7 \\
c_1 - 5c_2 &= -7
\end{aligned}
\quad . \tag{$\star$}
$$

Subtracting the second equation in system ($\star$) from the first:

$$(1-1)c_1 + (2-[-5])c_2 = 7-[-7] \quad \rightarrowtail \quad 7c_2 = 14 \quad \rightarrowtail \quad c_2 = 2 \quad .$$

Plugging this result back into the first equation in system ($\star$):

$$c_1 + 2\cdot 2 = 7 \quad \rightarrowtail \quad c_1 = 7 - 4 = 3 \quad .$$

So the solution to the initial-value problem is

$$x(t) = c_1 e^{3t} + 2c_2 e^{-4t} \qquad \text{and} \qquad y(t) = c_1 e^{3t} - 5c_2 e^{-4t}$$

with $c_1 = 3$ and $c_2 = 2$. That is,

$$x(t) = 3e^{3t} + 4e^{-4t} \qquad \text{and} \qquad y(t) = 3e^{3t} - 10e^{-4t} \quad .$$

35.5 a. Plugging these formulas for x and y into the two equations, we get

$$x' = 5x + 4y$$

$$\hookrightarrow \quad \frac{d}{dt}\left[c_1 e^{9t} - c_2 e^{-3t}\right] = 5\left[c_1 e^{9t} - c_2 e^{-3t}\right] + 4\left[c_1 e^{9t} + 2c_2 e^{-3t}\right]$$

$$\hookrightarrow \quad 9c_1 e^{9t} + 3c_2 e^{-3t} = (5+4)c_1 e^{9t} + (-5+8)c_2 e^{-3t}$$

$$\hookrightarrow \quad 9c_1 e^{9t} + 3c_2 e^{-3t} = 9c_1 e^{9t} + 3c_2 e^{-3t} \quad ,$$

and

$$y' = 8x + y$$

$$\hookrightarrow \quad \frac{d}{dt}\left[c_1 e^{9t} + 2c_2 e^{-3t}\right] = 8\left[c_1 e^{9t} - c_2 e^{-3t}\right] + \left[c_1 e^{9t} + 2c_2 e^{-3t}\right]$$

$$\hookrightarrow \quad 9c_1 e^{9t} + 2(-3)c_2 e^{-3t} = (8+1)c_1 e^{9t} + (-8+2)c_2 e^{-3t}$$

$$\hookrightarrow \quad 9c_1 e^{9t} - 6c_2 e^{-3t} = 9c_1 e^{9t} - 6c_2 e^{-3t} \quad .$$

Since both equations in the system are satisfied by the given formulas of x and y no matter what are the values of c_1 and c_2, this function pair is a solution to the given system for any choice of constants c_1 and c_2.

35.5 b. From the initial conditions and given formulas for x and y, we have the system

$$\begin{aligned} 0 &= x(0) = c_1 e^{9\cdot 0} - c_2 e^{-3\cdot 0} = c_1 - c_2 \\ 9 &= y(0) = c_1 e^{9\cdot 0} + 2c_2 e^{-3\cdot 0} = c_1 + 2c_2 \end{aligned} \quad . \qquad (\star)$$

From the first equation in system ($\star$), it is clear that $c_2 = c_1$. Using this in the second equation yields

$$9 = c_1 + 2c_1 = 3c_1 \quad \rightarrowtail \quad c_1 = 3 \quad .$$

Hence, $c_2 = c_1 = 3$, and the solution to the initial-value problem is

$$x(t) = 3e^{9t} - 3e^{-3t} \qquad \text{and} \qquad y(t) = 3e^{9t} + 2\cdot 3e^{-3t} = 3e^{9t} + 6e^{-3t} \quad .$$

35.7 a. Since the first equation, $x'' + x = 0$, has only one unknown function, we will solve it first. Fortunately, it is a simple second-order linear equation with constant coefficients. Writing down and solving the corresponding characteristic equation, we have

$$r^2 + 1 = 0 \quad \rightarrowtail \quad r = \pm i \quad ,$$

which, as we learned in chapter 15, means that

$$x(t) = c_1 \cos(x) + c_2 \sin(x) \quad .$$

With this, we can proceed to solve the second equation in the system:

$$y' \;=\; x \;=\; c_1\cos(x) \;+\; c_2\sin(x)$$

$$\hookrightarrow \quad y(t) \;=\; \int [c_1\cos(x) \;+\; c_2\sin(x)]\,dt \;=\; c_1\sin(x) \;-\; c_2\cos(x) \;+\; c_3 \quad .$$

So the solution to the system is the pair

$$x(t) = c_1\cos(t) + c_2\sin(t) \qquad \text{and} \qquad y(t) = c_1\sin(t) - c_2\cos(t) + c_3 \quad .$$

35.7 c. In this case, it is the last equation, $z' - 3z = 0$, that only involves one unknown function. It is a very simple first-order separable and linear equation whose solution is easily found to be

$$z(t) \;=\; Ae^{3t} \quad .$$

Since the first equation, $x' + 2x = 10z$ only involves the unknown function x and the now-known z, we solve that next. Using the formula just found for z, the equation becomes

$$x' \;+\; 2x \;=\; 10Ae^{3t} \quad .$$

This is a first-order linear equation with integrating factor

$$\mu(t) \;=\; e^{\int 2\,dt} \;=\; e^{2t} \quad .$$

Solving it:

$$e^{2t}\left[\frac{dx}{dt} + 2x\right] \;=\; e^{2t}\left[10Ae^{3t}\right]$$

$$\hookrightarrow \quad \underbrace{e^{2t}\frac{dx}{dt} + 2e^{2t}x}_{\frac{d}{dt}[e^{2t}x]} \;=\; 10Ae^{5t}$$

$$\hookrightarrow \quad e^{2t}x \;=\; \int 10Ae^{5t}\,dt \;=\; 2Ae^{5t} \;+\; B$$

$$\hookrightarrow \quad x(t) \;=\; 2Ae^{3t} \;+\; Be^{-2t} \quad .$$

Using the now-known formulas for x and z in the second equation of the system:

$$zy' \;+\; 5zy \;=\; 15x$$

$$\hookrightarrow \quad Ae^{3t}y' \;+\; 5Ae^{3t}y \;=\; 15\left[2Ae^{3t} + Be^{-2t}\right]$$

$$\hookrightarrow \quad y' \;+\; 5y \;=\; 15\left[2 + \frac{B}{A}e^{-5t}\right] \quad .$$

This is a first-order linear equation with integrating factor

$$\mu \;=\; e^{\int 5\,dt} \;=\; e^{5t} \quad .$$

Proceeding as usual with such equations:

$$e^{5t}\left[y' + 5y\right] = e^{5t}15\left[2 + \frac{B}{A}e^{-5t}\right]$$

$$\hookrightarrow \quad \underbrace{e^{5t}\frac{dy}{dt} + 5e^{5t}y}_{\frac{d}{dt}[e^{5t}y]} = 30e^{5t} + \frac{15B}{A}$$

$$\hookrightarrow \quad e^{5t}y = \int\left[30e^{5t} + \frac{15B}{A}\right]dt = 6e^{5t} + \frac{15B}{A}t + C$$

$$\hookrightarrow \quad y(t) = 6 + \left[\frac{15B}{A}t + C\right]e^{-5t} \quad .$$

Thus, the solution to this system is the triple

$$x(t) = 2Ae^{3t} + Be^{-2t} \quad , \quad y(t) = 6 + \left[\frac{15B}{A}t + C\right]e^{-5t}$$

and

$$z(t) = Ae^{3t} \quad .$$

35.8 a. For this tank system:

$$\begin{aligned}
x' = \frac{dx}{dt} &= \text{change in the amount of alcohol in tank } \mathcal{A} \text{ per minute} \\
&= \text{rate alcohol is pumped into tank } \mathcal{A} \text{ from the outside} \\
&\quad + \text{rate alcohol is pumped into tank } \mathcal{A} \text{ from tank } \mathcal{B} \\
&\quad - \text{rate alcohol is pumped from tank } \mathcal{A} \text{ into tank } \mathcal{B} \\
&\quad - \text{rate alcohol is drained from tank } \mathcal{A} \\
&= \left(I_{\mathcal{A}} \times \frac{C_{\mathcal{A}}}{100}\right) + \left(\gamma_1 \times \frac{y}{600}\right) - \left(\gamma_2 \times \frac{x}{1200}\right) - \left(O_{\mathcal{A}} \times \frac{x}{1200}\right) \\
&= \left(5 \times \frac{0}{100}\right) + \left(1 \times \frac{y}{600}\right) - \left(1 \times \frac{x}{1200}\right) - \left(5 \times \frac{x}{1200}\right) \\
&= -\frac{6}{1200}x + \frac{1}{600}y \quad ,
\end{aligned}$$

and

$$\begin{aligned}
y' = \frac{dy}{dt} &= \text{change in the amount of alcohol in tank } \mathcal{B} \text{ per minute} \\
&= \text{rate alcohol is pumped into tank } \mathcal{B} \text{ from the outside} \\
&\quad + \text{rate alcohol is pumped into tank } \mathcal{B} \text{ from tank } \mathcal{A} \\
&\quad - \text{rate alcohol is pumped from tank } \mathcal{B} \text{ into tank } \mathcal{A} \\
&\quad - \text{rate alcohol is drained from tank } \mathcal{B} \\
&= \left(I_{\mathcal{B}} \times \frac{C_{\mathcal{B}}}{100}\right) + \left(\gamma_2 \times \frac{x}{1200}\right) - \left(\gamma_1 \times \frac{y}{600}\right) - \left(O_{\mathcal{B}} \times \frac{y}{600}\right) \\
&= \left(3 \times \frac{100}{100}\right) + \left(1 \times \frac{x}{1200}\right) - \left(1 \times \frac{y}{600}\right) - \left(3 \times \frac{y}{600}\right) \\
&= 3 + \frac{1}{1200}x - \frac{4}{600}y \quad .
\end{aligned}$$

Together, these two equations form the system

$$\begin{aligned} x' &= -\frac{6}{1200}x + \frac{1}{600}y \\ y' &= 3 + \frac{1}{1200}x - \frac{4}{600}y \end{aligned} \quad .$$

35.9 a. Rewriting the differential equation as

$$y'' = -4y' - 2y \quad .$$

and letting $x = y'$ (so that $x' = y''$) leads to

$$x' = y'' = -4y' - 2y = -4x' - 2y \quad ,$$

giving us the standard first-order system

$$\begin{aligned} x' &= -4x' - 2y \\ y' &= x \end{aligned} \quad .$$

35.9 c. Letting $x = y'$ (so that $x' = y''$) leads to

$$x' = y'' = 4 - y^2 \quad ,$$

giving us the standard first-order system

$$\begin{aligned} x' &= 4 - y^2 \\ y' &= x \end{aligned} \quad .$$

35.9 e. Rewriting the differential equation as

$$y'' = t^{-2}\left[ty' - 10y\right] = t^{-1}y' - 10t^{-2}y \quad .$$

and letting $x = y'$ (so that $x' = y''$) leads to

$$x' = y'' = t^{-1}y' - 10t^{-2}y = t^{-1}x - 10t^{-2}y \quad ,$$

giving us the standard first-order system

$$\begin{aligned} x' &= t^{-1}x - 10t^{-2}y \\ y' &= x \end{aligned} \quad .$$

35.9 g. Rewriting the differential equation as

$$y''' = -2y'' + 3y' + 4y \quad .$$

and letting $x = y'$ (so that $x' = y''$) and $z = x' = y''$ (so that $z' = x'' = y'''$) leads to

$$z' = y''' = -2y'' + 3y' + 4y = -2z + 3x + 4y = 3x + 4y - 2z \quad ,$$

giving us the standard first-order system

$$\begin{aligned} x' &= z \\ y' &= x \\ z' &= 3x + 4y - 2z \end{aligned} \quad .$$

35.10 a. Taking the Laplace transform of the first equation:

$$\mathcal{L}\big[x'\big]\big|_s = \mathcal{L}[x]|_s + 2\mathcal{L}[y]|_s$$

$$\hookrightarrow \quad sX(s) - x(0) = X(s) + 2Y(s)$$

$$\hookrightarrow \quad sX(s) - 1 = X(s) + 2Y(s)$$

$$\hookrightarrow \quad [s-1]X(s) - 2Y(s) = 1 \quad .$$

Doing the same with the second equation:

$$\mathcal{L}\big[y'\big]\big|_s = 5\mathcal{L}[x]|_s - 2\mathcal{L}[y]|_s$$

$$\hookrightarrow \quad sY(s) - y(0) = 5X(s) - 2Y(s)$$

$$\hookrightarrow \quad sY(s) - 15 = 5X(s) - 2Y(s)$$

$$\hookrightarrow \quad -5X(s) + [s+2]Y(s) = 15 \quad .$$

So the transforms $X = \mathcal{L}[x]$ and $Y = \mathcal{L}[y]$ must satisfy the algebraic system

$$\begin{aligned} [s-1]X(s) - 2Y(s) &= 1 \\ -5X(s) + [s+2]Y(s) &= 15 \end{aligned} \quad . \qquad (\star)$$

To find $X(s)$, we add $s+2$ times the first equation to 2 times the second (to obtain an equation with no Y's), and then apply a bit more algebra (including the use of partial fractions, as described in section 25.2):

$$[(s-1)(s+2)]\,X(s) + 2[-5X(s)] = (s+2) + 2\cdot 15$$

$$\hookrightarrow \quad \underbrace{\left(s^2+s-12\right)}_{(s+4)(s-3)} X(s) = s + 32$$

$$\hookrightarrow \quad X(s) = \frac{s+32}{(s+4)(s-3)} = \cdots = \frac{-4}{s+4} + \frac{5}{s-3} \quad .$$

Taking the inverse transform of this:

$$x(t) = \mathcal{L}^{-1}[X]|_t = \mathcal{L}^{-1}\left[\frac{-4}{s+4}\right]\bigg|_t + \mathcal{L}^{-1}\left[\frac{5}{s-3}\right]\bigg|_t = -4e^{-4t} + 5e^{3t} \quad .$$

To find $Y(s)$, we add 5 times the first equation in system $(\star)$ to $s-1$ times the second (to obtain an equation with no X's), and then apply a bit more algebra (including the use of partial fractions):

$$-2\cdot 5Y(s) + (s-1)(s+2)Y(s) = 5 + 15(s-1)$$

$$\hookrightarrow \quad \underbrace{\left(s^2+s-12\right)}_{(s+4)(s-3)} Y(s) = 15s - 10$$

$$\hookrightarrow \quad Y(s) = \frac{15s-10}{(s+4)(s-3)} = \cdots = \frac{10}{s+4} + \frac{5}{s-3} \quad .$$

Taking the inverse transform of this:

$$y(t) = \mathcal{L}^{-1}[Y]|_t = \mathcal{L}^{-1}\left[\frac{10}{s+4}\right]\Big|_t + \mathcal{L}^{-1}\left[\frac{5}{s-3}\right]\Big|_t = 10e^{-4t} + 5e^{3t} .$$

Hence, the solution to the given system of differential equations with initial conditions is the pair

$$x(t) = -4e^{-4t} + 5e^{3t} \quad \text{and} \quad y(t) = 10e^{-4t} + 5e^{3t} .$$

35.10 c. Taking the Laplace transform of the first equation:

$$\mathcal{L}[x']\big|_s = 2\mathcal{L}[y]|_s \quad \rightarrowtail \quad sX(s) - x_0 = 2Y(s)$$

$$\hookrightarrow \qquad sX(s) - 2Y(s) = x_0 .$$

Doing the same with the second equation:

$$\mathcal{L}[y']\big|_s = -2\mathcal{L}[x]|_s \quad \rightarrowtail \quad sY(s) - y_0 = -2X(s)$$

$$\hookrightarrow \qquad 2X(s) + sY(s) = y_0 .$$

So the transforms $X = \mathcal{L}[x]$ and $Y = \mathcal{L}[y]$ must satisfy the algebraic system

$$\begin{aligned} sX(s) - 2Y(s) &= x_0 \\ 2X(s) + sY(s) &= y_0 \end{aligned} . \qquad (\star)$$

To find $X(s)$, we add s times the first equation to 2 times the second, and then apply a bit more algebra:

$$\left[s^2 + 2^2\right] X(s) + 0Y(s) = sx_0 + 2y_0$$

$$\hookrightarrow \qquad X(s) = \frac{x_0 s + 2y_0}{s^2 + 2^2} = x_0 \frac{s}{s^2 + 2^2} + y_0 \frac{2}{s^2 + 2^2} .$$

Taking the inverse transform of this:

$$x(t) = \mathcal{L}^{-1}\left[x_0 \frac{s}{s^2 + 2^2} + y_0 \frac{2}{s^2 + 2^2}\right]\Big|_t = x_0 \cos(2t) + y_0 \sin(2t) .$$

To find $Y(s)$, we add -2 times the first equation in system $(\star)$ to s times the second, and then apply a bit more algebra:

$$0X(s) + \left[2^2 + s^2\right] Y(s) = -2x_0 + sy_0$$

$$\hookrightarrow \qquad Y(s) = \frac{-2x_0 + y_0 s}{s^2 + 2^2} = y_0 \frac{s}{s^2 + 2^2} - x_0 \frac{2}{s^2 + 2^2} .$$

Taking the inverse transform of this:

$$y(t) = \mathcal{L}^{-1}\left[y_0 \frac{s}{s^2 + 2^2} - x_0 \frac{2}{s^2 + 2^2}\right]\Big|_t = y_0 \cos(2t) - x_0 \sin(2t) .$$

Hence, the solution to the given system of differential equations with initial conditions is the pair

$$x(t) = x_0 \cos(2t) + y_0 \sin(2t) \quad \text{and} \quad y(t) = y_0 \cos(2t) - x_0 \sin(2t) .$$

35.10 e. Taking the Laplace transforms of the two equations, we have

$$\mathcal{L}\big[x'\big]\big|_s = \mathcal{L}[4x - 13y]|_s \quad \rightarrowtail \quad sX(s) - 2 = 4X(s) - 13Y(x)$$

$$\hookrightarrow \qquad [s-4]X(s) + 13Y(s) = 2 \quad ,$$

and

$$\mathcal{L}\big[y'\big]\big|_s = \mathcal{L}[x]|_s \quad \rightarrowtail \quad sY(s) - 1 = X(s)$$

$$\hookrightarrow \qquad -X(s) + sY(s) = 1 \quad ,$$

giving us the algebraic system

$$\begin{aligned} [s-4]X(s) + 13Y(s) &= 2 \\ -X(s) + sY(s) &= 1 \end{aligned} \quad . \tag{$\star$}$$

For variety, we'll solve system ($\star$) by first solving the second equation for X,

$$X(s) = sY(s) - 1 \quad ,$$

and then plugging that result into the first equation, and solving for Y:

$$[s-4]\,[sY(s) - 1] + 13Y(s) = 2$$

$$\hookrightarrow \qquad \left[s^2 - 4s\right]Y(s) - s + 4 + 13Y(s) = 2$$

$$\hookrightarrow \qquad \underbrace{\left[s^2 - 4s + 13\right]}_{(s-2)^2+3^2} Y(s) = s - 2$$

$$\hookrightarrow \qquad Y(s) = \frac{s-2}{(s-2)^2 + 3^2} \quad .$$

To take the inverse transform of this, we will need the first shifting identity (see section 25.3). Here,

$$y(t) = \mathcal{L}^{-1}\left[\frac{s-2}{(s-2)^2 + 3^2}\right]\Bigg|_t = \mathcal{L}^{-1}[F(s-2)]|_t = e^{2t} f(t)$$

where

$$F(s-2) = \frac{s-2}{(s-2)^2 + 3^2} \quad .$$

Letting $X = s - 2$, this becomes

$$F(X) = \frac{X}{X^2 + 3^2} \quad .$$

So,

$$f(t) = \mathcal{L}^{-1}[F(s)]|_t = \mathcal{L}^{-1}\left[\frac{s}{s^2 + 3^2}\right]\Bigg|_t = \cos(3t) \quad ,$$

and

$$y(t) = e^{2t} f(t) = e^{2t}\cos(3t) \quad .$$

Similar computations could be done to find the formulas for $X(s)$ and $x(t)$. However, let us recall that the second equation in the original system is $y' = x$. Combining this with the above formula for y immediately yields

$$x(t) \;=\; y'(t) \;=\; \frac{d}{dt}\left[e^{2t}\cos(3t)\right] \;=\; 2e^{2t}\cos(3t) \;-\; 3e^{2t}\sin(3t) \quad .$$

So the solution to the original system with initial values is the pair

$$x(t) \;=\; e^{2t}[2\cos(3t) - 3\sin(3t)] \qquad \text{and} \qquad y(t) \;=\; e^{2t}\cos(3t) \quad .$$

35.10 g. Taking the Laplace transforms of the two differential equations:

$$\mathcal{L}\big[x'\big]\Big|_s \;=\; \mathcal{L}[8x + 2y - 17]|_s$$

$$\hookrightarrow \qquad sX(s) \;-\; 0 \;=\; 8X(s) \;+\; 2Y(s) \;-\; \frac{17}{s}$$

$$\hookrightarrow \qquad [s-8]X(s) \;-\; 2Y(s) \;=\; -\frac{17}{s} \quad .$$

and

$$\mathcal{L}\big[y'\big]\Big|_s \;=\; \mathcal{L}[4x + y - 13]|_s$$

$$\hookrightarrow \qquad sY(s) \;-\; 0 \;=\; 4X(s) \;+\; Y(s) \;-\; \frac{13}{s}$$

$$\hookrightarrow \qquad -4X(s) \;-\; (1-s)Y(s) \;=\; -\frac{13}{s} \quad .$$

So the algebraic system of transforms is

$$\begin{aligned} [s-8]X(s) \;-\; 2Y(s) \;&=\; -\frac{17}{s} \\ -4X(s) \;+\; (s-1)Y(s) \;&=\; -\frac{13}{s} \end{aligned} \quad . \qquad (\star)$$

Adding $s-1$ times the first equation above to 2 times the second, and then doing a little algebra:

$$[(s-1)(s-8)]\,X(s) \;-\; 2\cdot 4X(s) \;+\; 0Y(s) \;=\; -\frac{17(s-1)}{s} \;-\; \frac{2\cdot 13}{s}$$

$$\hookrightarrow \qquad \left[s^2 - 9s\right]X(s) \;=\; -\frac{17s+9}{s}$$

$$\hookrightarrow \qquad X(s) \;=\; -\frac{17s+9}{s^2(s-9)} \;=\; \cdots \;=\; \frac{-2}{s-9} \;+\; \frac{1}{s^2} \;+\; \frac{2}{s} \quad .$$

Thus,

$$x(t) \;=\; \mathcal{L}^{-1}\left[\frac{-2}{s-9} + \frac{1}{s^2} + \frac{2}{s}\right]\Big|_t \;=\; -2e^{9t} \;+\; t \;+\; 2 \quad .$$

Adding 4 times the first equation in system ($\star$) to $s-8$ times the second equation, and then doing a little algebra:

$$0X(s) \;-\; 2\cdot 4Y(s) \;+\; [(s-8)(s-1)]\,Y(s) \;=\; -\frac{4\cdot 17}{s} \;-\; \frac{13(s-8)}{s}$$

$$\hookrightarrow \qquad \left[s^2 - 9s\right]Y(s) \;=\; -\frac{13s-36}{s}$$

$$\hookrightarrow \qquad Y(s) \;=\; -\frac{13s-36}{s^2(s-9)} = \cdots \;=\; \frac{-1}{s-9} \;-\; \frac{4}{s^2} \;+\; \frac{1}{s} \quad .$$

Hence,

$$y(t) = \mathcal{L}^{-1}\left[\frac{-1}{s-9} - \frac{4}{s^2} + \frac{1}{s}\right]\Big|_t = -e^{9t} - 4t + 1 \quad ,$$

and the solution to the given initial-value problem is the pair

$$x(t) = -2e^{9t} + t + 2 \qquad \text{and} \qquad y(t) = -e^{9t} - 4t + 1 \quad .$$

35.10 i. Taking the Laplace transforms of the two differential equations:

$$\mathcal{L}\left[x'\right]\big|_s = \mathcal{L}\left[4x + 3y - 6e^{3t}\right]\Big|_s$$

$$\hookrightarrow \qquad sX(s) - 4 = 4X(s) + 3Y(s) - \frac{6}{s-3}$$

$$\hookrightarrow \qquad [s-4]X(s) - 3Y(s) = 4 - \frac{6}{s-3} \quad .$$

and

$$\mathcal{L}\left[y'\right]\big|_s = \mathcal{L}\left[x + 6y + 2e^{3t}\right]\Big|_s$$

$$\hookrightarrow \qquad sY(s) - 0 = X(s) + 6Y(s) + \frac{2}{s-3}$$

$$\hookrightarrow \qquad -X(s) + [s-6]Y(s) = \frac{2}{s-3} \quad .$$

So the algebraic system of transforms is

$$\begin{aligned} [s-4]X(s) - 3Y(s) &= 4 - \frac{6}{s-3} \\ -X(s) + [s-6]Y(s) &= \frac{2}{s-3} \end{aligned} \quad . \qquad (\star)$$

Solving system $(\star)$ for X:

$$[(s-6)(s-4)]\,X(s) - 3X(s) + 0Y(s) = 4(s-6) - \frac{6(s-6)}{s-3} + \frac{3\cdot 2}{s-3}$$

$$\hookrightarrow \qquad \underbrace{\left[s^2 - 10s + 21\right]}_{(s-7)(s-3)} X(s) = \frac{4s^2 - 42s + 114}{s-3}$$

$$\hookrightarrow \qquad X(s) = \frac{4s^2 - 42s + 114}{(s-7)(s-3)^2} = \cdots = \frac{1}{s-7} + \frac{3}{s-3} - \frac{6}{(s-3)^2} \quad .$$

So,

$$x(t) = \mathcal{L}^{-1}\left[\frac{1}{s-7} + \frac{3}{s-3} - \frac{6}{(s-3)^2}\right]\Big|_t = \cdots = e^{7t} + 3e^{3t} - 6te^{3t} \quad .$$

Solving system (⋆) for Y :

$$0X(s) - 3Y(s) + [(s-4)(s-6)]Y(s) = 4 - \frac{6}{s-3} + \frac{2(s-4)}{s-3}$$

$$\hookrightarrow \qquad \underbrace{\left[s^2 - 10s + 21\right]}_{(s-7)(s-3)} Y(s) = \frac{6s-26}{s-3}$$

$$\hookrightarrow \qquad Y(s) = \frac{6s-26}{(s-7)(s-3)^2} = \cdots = \frac{1}{s-7} - \frac{1}{s-3} + \frac{2}{(s-3)^2} \quad .$$

So,

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{s-7} - \frac{1}{s-3} + \frac{2}{(s-3)^2}\right]\Bigg|_t = \cdots = e^{7t} - e^{3t} + 2te^{3t} \quad ,$$

and the solution to the given initial-value problem is the pair

$$x(t) = e^{7t} + 3e^{3t} - 6te^{3t} \qquad \text{and} \qquad y(t) = e^{7t} - e^{3t} + 2te^{3t} \quad .$$

35.10 k. Taking the Laplace tranforms of both equations:

$$\mathcal{L}\left[x'\right]\big|_s = \mathcal{L}[4x - 13y]|_s \quad \rightarrowtail \quad sX(s) - 13 = 4X(s) - 13Y(s)$$

$$\hookrightarrow \qquad [s-4]X(s) + 13Y(s) = 13 \quad ,$$

and

$$\mathcal{L}\left[y'\right]\big|_s = \mathcal{L}[x + 19\cos(4t) - 13\sin(4t)]|_s$$

$$\hookrightarrow \qquad sY(s) - 0 = X(s) + \frac{19s}{s^2+4^2} - \frac{13\cdot 4}{s^2+4^2}$$

$$\hookrightarrow \qquad -X(s) + sY(s) = \frac{19s-52}{s^2+4^2} \quad .$$

So the algebraic system of transforms is

$$\begin{aligned} [s-4]X(s) + 13Y(s) &= 13 \\ -X(s) + sY(s) &= \frac{19s-52}{s^2+4^2} \end{aligned} \quad . \qquad (\star)$$

Solving system (⋆) for X :

$$[s(s-4)]X(s) + 13X(s) + 0Y(s) = 13s + \frac{13(19s-52)}{s^2+4^2}$$

$$\hookrightarrow \qquad \left[s^2 - 4s + 13\right]X(s) = 13\frac{s^3 - 3s + 52}{s^2+4^2}$$

$$\hookrightarrow \qquad X(s) = 13\frac{s^3-3s+52}{\left(s^2-4s+13\right)\left(s^2+4^2\right)} = \cdots = 13\left[\frac{s}{s^2+4^2} + \frac{4}{s^2+4^2}\right] \quad .$$

So,

$$x(t) = \mathcal{L}^{-1}\left[13\left[\frac{s}{s^2+4^2} + \frac{4}{s^2+4^2}\right]\right]\Bigg|_t = 13\left[\cos(4t) + \sin(4t)\right] \quad .$$

Solving system $(\star)$ for Y :

$$0X(s) \;+\; 13Y(s) \;+\; [s(s-4)]\,Y(s) \;=\; 13 \;+\; \frac{(s-4)(19s-52)}{s^2+4^2}$$

$$\hookrightarrow \qquad \left[s^2-4s+13\right]Y(s) \;=\; \frac{32s^2-128s+416}{s^2+4^2}$$

$$\hookrightarrow \qquad \left[s^2-4s+13\right]Y(s) \;=\; \frac{32\left[s^2-4s+13\right]}{s^2+4^2}$$

$$\hookrightarrow \qquad Y(s) \;=\; \frac{32\left(s^2-4s+13\right)}{\left(s^2-4s+13\right)\left(s^2+4^2\right)} \;=\; 8\frac{4}{s^2+4^2} \quad .$$

Thus,

$$y(t) \;=\; \mathcal{L}^{-1}\left[8\frac{4}{s^2+4^2}\right]\bigg|_t \;=\; 8\sin(4t) \quad ,$$

and the solution to the original initial-value problem is the pair

$$x(t) \;=\; 13\,[\cos(4t)+\sin(4t)] \qquad \text{and} \qquad y(t) \;=\; 8\sin(4t) \quad .$$

35.11. In exercise 35.5, it was verified that, for any choice of constants c_1 and c_2 ,

$$x(t) \;=\; c_1e^{9t} \;-\; c_2e^{-3t} \qquad \text{and} \qquad y(t) \;=\; c_1e^{9t} \;+\; 2c_2e^{-3t}$$

is a solution over $(-\infty,\infty)$ to the given system of differential equations. Moreover, the component functions in the system of differential equations, $5x+4y$ and $8x+y$, are clearly continuous and have continuous partial derivatives.

Now let A and B be any two real numbers, and consider the algebraic problem of finding C_1 and C_2 such that

$$\begin{aligned} A \;&=\; x(0) \;=\; C_1e^{9\cdot 0} \;-\; C_2e^{-3\cdot 0} \;=\; C_1 \;-\; C_2 \\ B \;&=\; y(0) \;=\; C_1e^{9\cdot 0} \;+\; 2C_2e^{-3\cdot 0} \;=\; C_1 \;+\; 2C_2 \end{aligned} \quad . \qquad (\star)$$

But this is an easily solved system. Subtracting the second equation from the first and solving for C_2 :

$$A \;-\; B \;=\; 0C_1 \;-\; (1+2)C_2 \quad\rightarrowtail\quad C_2 \;=\; \frac{B-A}{3} \quad .$$

Using this with the first equation in system $(\star)$ yields

$$A \;=\; C_1 \;-\; \frac{B-A}{3} \quad\rightarrowtail\quad C_1 \;=\; A \;+\; \frac{B-A}{3} \;=\; \frac{2A+B}{3} \quad .$$

Thus, for any pair A and B , we can clearly find C_1 and C_2 such that system $(\star)$ holds. Theorem 35.3 (with $t_0=0$) then assures us that the pair

$$x(t) \;=\; c_1e^{9t} \;-\; c_2e^{-3t} \qquad \text{and} \qquad y(t) \;=\; c_1e^{9t} \;+\; 2c_2e^{-3t}$$

is, in fact, a general solution.