Chapter 33: The Big Theorem on the Frobenius Method, with Applications

33.2 a. Since $x - x_0 = x - 0 = x$, the quasi-Euler form is simply

$$x^2 y'' + x \underbrace{(-2)}_{\beta(x)} y' + \underbrace{\left(2 - x^2\right)}_{\gamma(x)} y = 0$$

Here,

$$\alpha(x) = \alpha(0) = 1 \quad , \quad \beta(0) = -2 \quad \text{and} \quad \gamma(0) = 2 - 0^2 = 2 \quad .$$

The indicial equation:

$$\alpha(0)r(r-1) + \beta(0)r + \gamma(0) = 0$$

$$\hookrightarrow \qquad 1 \cdot r(r-1) + (-2)r + \left(2 - 0^2\right) = 0$$

$$\hookrightarrow \qquad r^2 - 3r + 2 = 0 \quad .$$

This factors to

$$(r-2)(r-1) = 0 \quad .$$

So $r_1 = 2$ and $r_2 = 1$.

The corresponding (shifted) Euler equation:

$$x^2 \alpha(0) y'' + x\beta(0) y' + \gamma(0) y = 0$$

$$\hookrightarrow \qquad x^2 y'' + x(-2) y' + 2y = 0 \quad .$$

Its two main solutions:

$$y_{\text{Euler},1}(x) = x^{r_1} = x^2 \quad \text{and} \quad y_{\text{Euler},2}(x) = x^{r_2} = x^1 = x \quad .$$

The limits of these solutions as $x \to x_0$:

$$\lim_{x\to 0} \left|y_{\text{Euler},1}(x)\right| = \lim_{x\to 0} \left|x^2\right| = 0 \quad \text{and} \quad \lim_{x\to 0} \left|y_{\text{Euler},2}(x)\right| = \lim_{x\to 0} |x| = 0 \quad .$$

33.2 c. Multiplying through by x^2 , we find that the quasi-Euler form is

$$x^2 y'' + xy' + \underbrace{x^2}_{\gamma(x)} y = 0$$

Here,

$$\alpha(x) = \alpha(0) = 1 \quad , \quad \beta(x) = \beta(0) = 1 \quad \text{and} \quad \gamma(0) = 0^2 = 0 \quad .$$

The indicial equation:

$$\alpha(0)r(r-1) + \beta(0)r + \gamma(0) = 0$$

$$\hookrightarrow \qquad r(r-1) + r + 0 = 0$$

$$\hookrightarrow \qquad r^2 = 0 \quad .$$

So $r_1 = r_2 = 0$.

The corresponding (shifted) Euler equation is

$$x^2\alpha(0)y'' + x\beta(0)y' + \gamma(0)y = 0 \quad \rightarrowtail \quad x^2y'' + xy' = 0 \quad ,$$

and has solutions:

$$y_{\text{Euler},1}(x) = x^{r_1} = x^0 = 1 \qquad \text{and} \qquad y_{\text{Euler},2}(x) = x^{r_1}\ln|x| = \ln|x| \quad .$$

The limits of these solutions as $x \to x_0$:

$$\lim_{x\to 0}\left|y_{\text{Euler},1}(x)\right| = \lim_{x\to 0} 1 = 1 \qquad \text{and} \qquad \lim_{x\to 0}\left|y_{\text{Euler},2}(x)\right| = \lim_{x\to 0}|\ln|x|| = \infty \quad .$$

33.2 e. The quasi-Euler form is

$$x^2y'' + x\underbrace{(-5-2x)}_{\beta(x)}y' + \underbrace{9}_{\gamma(x)}y = 0$$

Here,

$$\alpha(x) = \alpha(0) = 1 \quad , \quad \beta(0) = -5 - 2\cdot 0 = -5 \quad \text{and} \quad \gamma(0) = 9 \quad .$$

The indicial equation:

$$\alpha(0)r(r-1) + \beta(0)r + \gamma(0) = 0$$

$$\hookrightarrow \qquad r(r-1) - 5r + 9 = 0$$

$$\hookrightarrow \qquad \underbrace{r^2 - 6r + 9}_{(r-3)^2} = 0 \quad .$$

So $r_1 = r_2 = 3$.

The corresponding (shifted) Euler equation is

$$x^2\alpha(0)y'' + x\beta(0)y' + \gamma(0)y = 0$$

$$\hookrightarrow \qquad x^2y'' - 5xy' + 9y = 0 \quad ,$$

and has solutions:

$$y_{\text{Euler},1}(x) = x^{r_1} = x^3 \qquad \text{and} \qquad y_{\text{Euler},2}(x) = x^{r_1}\ln|x| \quad .$$

The limits of these solutions as $x \to x_0$:

$$\lim_{x\to 0}\left|y_{\text{Euler},1}(x)\right| = \lim_{x\to 0}\left|x^3\right| = 0$$

and

$$\lim_{x\to 0}\left|y_{\text{Euler},2}(x)\right| = \lim_{x\to 0}\left|x^3\ln|x|\right| = 0 \quad .$$

33.2 g. The quasi-Euler form is

$$x^2 4y'' + x8y' + \underbrace{(1-4x)}_{\gamma(x)}y = 0$$

Here,

$$\alpha(x) = \alpha(0) = 4 \quad , \quad \beta(x) = \beta(0) = 8 \quad \text{and} \quad \gamma(0) = 1 - 4\cdot 0 = 1 \quad .$$

The indicial equation:

$$\alpha(0)r(r-1) + \beta(0)r + \gamma(0) = 0$$

$$\hookrightarrow \qquad 4r(r-1) + 8r + 1 = 0$$

$$\hookrightarrow \qquad \underbrace{4r^2 + 4r + 1}_{(2r+1)^2} = 0 \quad .$$

So $r_1 = r_2 = -\frac{1}{2}$.

The corresponding (shifted) Euler equation is

$$x^2\alpha(0)y'' + x\beta(0)y' + \gamma(0)y = 0$$

$$\hookrightarrow \qquad x^2 4y'' + 8y' + y = 0 \quad ,$$

and has solutions:

$$y_{\text{Euler},1}(x) = |x|^{-1/2} \quad \text{and} \quad y_{\text{Euler},2}(x) = |x|^{-1/2}\ln|x| \quad .$$

The limits of these solutions as $x \to x_0$:

$$\lim_{x\to 0}\left|y_{\text{Euler},1}(x)\right| = \lim_{x\to 0}\left||x|^{-1/2}\right| = \infty$$

and

$$\lim_{x\to 0}\left|y_{\text{Euler},2}(x)\right| = \lim_{x\to 0}\left||x|^{-1/2}\ln|x|\right| = \infty \quad .$$

33.2 i. Since $x_0 = 0$, we merely have to multiply through by x to obtain the quasi-Euler form

$$x^2y'' + x\underbrace{4}_{\beta(x)}y' + \underbrace{\frac{12x}{(x+2)^2}}_{\gamma(x)}y = 0$$

(We could also multiply through by $(x+2)^2$. That would be recommended when finding the modified power series by the basic Frobenius method, but is not necessary for this problem since the above $\gamma(x)$ is analytic at $x = 0$.) Using the above,

$$\alpha(x) = \alpha(0) = 1 \quad , \quad \beta(0) = 4 \quad \text{and} \quad \gamma(0) = \frac{12\cdot 0}{(0+2)^2} = 0 \quad .$$

The indicial equation:

$$\alpha(0)r(r-1) + \beta(0)r + \gamma(0) = 0$$

$$\hookrightarrow \quad r(r-1) + 4r = 0$$

$$\hookrightarrow \quad \underbrace{r^2 + 3r}_{r(r-[-3])} = 0 \quad .$$

So $r_1 = 0$ and $r_2 = -3$.

The corresponding (shifted) Euler equation is

$$x^2\alpha(0)y'' + x\beta(0)y' + \gamma(0)y = 0 \quad \rightarrowtail \quad x^2y'' + 4xy' = 0 \quad ,$$

and has solutions:

$$y_{\text{Euler},1}(x) = x^{r_1} = x^0 = 1 \quad \text{and} \quad y_{\text{Euler},2}(x) = x^{r_2} = x^{-3} \quad .$$

The limits of these solutions as $x \to x_0$:

$$\lim_{x\to 0}\left|y_{\text{Euler},1}(x)\right| = \lim_{x\to 0}|1| = 1 \quad \text{and} \quad \lim_{x\to 0}\left|y_{\text{Euler},2}(x)\right| = \lim_{x\to 0}\left|x^{-3}\right| = \infty \quad .$$

33.2 k. Multiplying by $x-3$, we find the quasi-Euler form is

$$(x-3)^2y'' + (x-3)\underbrace{(x-3)}_{\beta(x)}y' + \underbrace{(x-3)}_{\gamma(x)}y = 0$$

Here,

$$\alpha(x_0) = \alpha(3) = 1 \quad , \quad \beta(3) = 3-3 = 0 \quad \text{and} \quad \gamma(3) = 3-3 = 0 \quad .$$

The indicial equation:

$$\alpha(0)r(r-1) + \beta(0)r + \gamma(0) = 0$$

$$\hookrightarrow \quad r(r-1) = 0$$

$$\hookrightarrow \quad r^2 - r = 0 \quad .$$

So $r_1 = 1$ and $r_2 = 0$.

The corresponding (shifted) Euler equation is

$$x^2\alpha(0)y'' + x\beta(0)y' + \gamma(0)y = 0$$

$$\hookrightarrow \quad x^2y'' = 0 \quad ,$$

and has solutions:

$$y_{\text{Euler},1}(x) = x^{r_1} = (x-3)^1 = x - 3$$

and

$$y_{\text{Euler},2}(x) = x^{r_2} = (x-3)^0 = 1 \quad .$$

The limits of these solutions as $x \to x_0$:

$$\lim_{x\to 0}\left|y_{\text{Euler},1}(x)\right| = \lim_{x\to 0}|x-3| = 0 \quad \text{and} \quad \lim_{x\to 0}\left|y_{\text{Euler},2}(x)\right| = \lim_{x\to 0}|1| = 1 \quad .$$

33.3. From section 32.6 on page 698, we know that $y = Ay_+ + By_-$ where A and B are constants with at least one being nonzero, and with $\{y_1, y_2\}$ being the fundamental set of solutions given by

$$y_+(x) = (x - x_0)^{\lambda + i\omega} \sum_{k=0}^{\infty} \alpha_k (x - x_0)^k \qquad \text{with} \quad \alpha_0 = 1$$

and

$$y_-(x) = (x - x_0)^{\lambda - i\omega} \sum_{k=0}^{\infty} {\alpha_k}^* (x - x_0)^k \qquad \text{with} \quad {\alpha_0}^* = 1^* = 1 \quad .$$

Moreover, as noted in section 33.3, as $x \to x_0$ the nonconstant terms in the power series factors approach 0 and we have

$$\lim_{x \to x_0^+} y_\pm(x) = \lim_{x \to x_0^+} y_{\text{Euler},\pm}(x)$$

where

$$\begin{aligned} y_{\text{Euler},\pm}(x) &= (x - x_0)^{\lambda + i\omega} \\ &= (x - x_0)^{\lambda} (x - x_0)^{i\omega} \\ &= (x - x_0)^{\lambda} \left[\cos(\omega \ln |x - x_0|) \pm i \sin(\omega \ln |x - x_0|) \right] \quad . \end{aligned}$$

Hence,

$$\begin{aligned} \lim_{x \to x_0^+} y(x) &= \lim_{x \to x_0^+} \left[Ay_+(x) + By_-(x) \right] \\ &= \lim_{x \to x_0^+} \left[Ay_{\text{Euler},+}(x) + By_{\text{Euler},-}(x) \right] \\ &= \lim_{x \to x_0^+} (x - x_0)^{\lambda} \Big[(A + B) \cos(\omega \ln |x - x_0|) \\ &\qquad\qquad + i(A - B) \sin(\omega \ln |x - x_0|) \Big] \quad . \end{aligned}$$

For the case where $\lambda > 0$, observe that

$$\begin{aligned} &|(A + B) \cos(\omega \ln |x - x_0|) + i(A - B) \sin(\omega \ln |x - x_0|)| \\ &\qquad \leq |A + B| \underbrace{|\cos(\omega \ln |x - x_0|)|}_{\leq 1} + |A - B| \underbrace{|\sin(\omega \ln |x - x_0|)|}_{\leq 1} \\ &\qquad \leq |A + B| + |A - B| \quad . \end{aligned}$$

Thus, still assuming $\lambda > 0$

$$\begin{aligned} \lim_{x \to x_0^+} |y(x)| &= \lim_{x \to x_0^+} |x - x_0|^{\lambda} \Big| (A + B) \cos(\omega \ln |x - x_0|) \\ &\qquad\qquad + i(A - B) \sin(\omega \ln |x - x_0|) \Big| \\ &\leq \lim_{x \to x_0^+} |x - x_0|^{\lambda} \left(|A + B| + |A - B| \right) \\ &= 0 \left(|A + B| + |A - B| \right) = 0 \quad , \end{aligned}$$

which immediately tells us that

$$\lim_{x \to x_0^+} y(x) = 0 \qquad \text{when} \quad \lambda > 0 \quad .$$

However, when $\lambda = 0$, $(x - x_0)^\lambda = 1$, and

$$\lim_{x \to x_0^+} y(x) = \lim_{x \to x_0^+} [(A + B)\cos(\omega \ln |x - x_0|) + i(A - B)\sin(\omega \ln |x - x_0|)] \quad .$$

But because $\ln |x - x_0| \to -\infty$ as $x \to x_0$, the last is equivalent to

$$\lim_{x \to x_0^+} y(x) = \lim_{X \to -\infty} [(A + B)\cos(\omega X) + i(A - B)\sin(\omega X)] \quad ,$$

which does not converge since the sine and cosine terms repeatedly vary between 1 and -1, as illustrated in figure 32.1a on page 669.

Finally, if $\lambda < 0$, then, as $x \to x_0$ from the right,

$$(x - x_0)^\lambda \to +\infty$$

while the sine and cosine terms in

$$(A + B)\cos(\omega \ln |x - x_0|) + i(A - B)\sin(\omega \ln |x - x_0|)$$

oscillates infinitely many times between 1 and -1. Consequently, as $x \to x_0$ from the right,

$$(x - x_0)^\lambda [(A + B)\cos(\omega \ln |x - x_0|) + i(A - B)\sin(\omega \ln |x - x_0|)]$$

traces out an infinite number of ever increasing "wiggles". Hence,

$$\lim_{x \to x_0^+} y(x) = \lim_{X \to -\infty} (x - x_0)^\lambda [(A + B)\cos(\omega X) + i(A - B)\sin(\omega X)]$$

does not exist if $\lambda < 0$.

Note: To make the cases where $\lambda \leq 0$ more explicit, compute the limits using the sequences

$$x_n = x_0 + \exp\left(-\frac{n\pi}{\omega}\right) \qquad \text{and} \qquad x_n = x_0 + \exp\left(-\frac{(2n+1)\pi}{2\omega}\right) \quad .$$

33.5 a. For $x_0 = 1$: We already know $x = 1$ is a singular point. Computing the limits from theorem 32.2 on page 674, we have

$$\lim_{x \to x_0} (x - x_0)\frac{b(x)}{a(x)} = \lim_{x \to 1} (x - 1)\frac{-x}{1 - x} = 1$$

and

$$\lim_{x \to x_0} (x - x_0)^2 \frac{c(x)}{a(x)} = \lim_{x \to 1} (x - 1)^2 \frac{\lambda}{1 - x} = 0 \quad .$$

Since both limits are finite, theorem 32.2 assures us that $x = 1$ is a regular singular point.

For $x_0 = -1$: Repeat the above with -1 replacing 1.

33.5 b. Exponents at $x_0 = 1$: First, we get the Chebyshev equation into quasi-Euler form:

$$-(x - 1)\left[(1 - x^2)y'' - xy' + \lambda y\right] = -(x - 1) \cdot 0$$

$$\hookrightarrow \quad -(x - 1)(x + 1)(1 - x)y'' + (x - 1)xy' - (x - 1)\lambda y = 0$$

$$\hookrightarrow \quad (x - 1)^2\alpha(x)y'' + (x - 1)\beta(x)y' + \gamma(x)y = 0$$

where

$$\alpha(x) = x + 1 \quad , \quad \beta(x) = x \quad \text{and} \quad \gamma(x) = \lambda(x-1) \quad .$$

Thus,

$$\alpha(1) = 1 + 1 = 2 \quad , \quad \beta(1) = 1 \quad \text{and} \quad \gamma(1) = \lambda(1-1) = 0 \quad ,$$

and the indicial equation is

$$r(r-1)2 + r + 0 = 0 \quad \rightarrowtail \quad \underbrace{2r^2 - r}_{r(2r-1)} = 0 \quad .$$

From this it follows that $r_1 = \frac{1}{2}$ and $r_2 = 0$.

Similar computations yield the same exponents at $x_0 = -1$.

33.5 c. Since y is a nonpolynomial solution, we know it is given by a power series about 0 with radius of convergence $R = 1$. From theorem 33.5 on page 715 it follows that y must have a singular point at least at $x = -1$ or $x = 1$. That is, y is not analytic at one or both of these points.

In addition, because the equation's exponents at both $x_0 = 1$ and $x_0 = -1$ are $r_1 = {}^1\!/_2$ and $r_2 = 0$, we know that we can find modified power series solutions about both points via the basic Frobenius method,

$$y_1^+(x) = (x-1)^{1/2} \sum_{k=0}^{\infty} a_k^+ (x-1)^k \quad \text{with} \quad a_0^+ = 1 \quad ,$$

$$y_2^+(x) = \sum_{k=0}^{\infty} b_k^+ (x-1)^k \quad \text{with} \quad b_0^+ = 1 \quad ,$$

$$y_1^-(x) = (x+1)^{1/2} \sum_{k=0}^{\infty} a_k^- (x+1)^k \quad \text{with} \quad a_0^- = 1$$

and

$$y_2^-(x) = \sum_{k=0}^{\infty} b_k^- (x+1)^k \quad \text{with} \quad b_0^- = 1 \quad .$$

Moreover, there must be constants $A^{\pm}$ and $B^{\pm}$ such that, for $-1 < x < 1$,

$$y(x) = A^+ y_1^+(x) + B^+ y_2^+(x) = A^- y_1^-(x) + B^- y_2^-(x) \quad .$$

If $A^+ = 0 = A^-$, then the above reduces to

$$y(x) = B^+ \sum_{k=0}^{\infty} b_k^+ (x-1)^k = B^- \sum_{k=0}^{\infty} b_k^- (x+1)^k \quad .$$

But this would mean that y is analytic at both $x_0 = 1$ and $x_0 = -1$, which we already know is not the case. So we do not have $A^+ = 0 = A^-$, at least one must be nonzero. If it is A^+ which is nonzero, then

$$y'(x) = \frac{d}{dx}\left[A^+ y_1^+(x) + B^+ y_2^+(x)\right]$$

$$= \frac{d}{dx}\left[A^+ \sum_{k=0}^{\infty} a_k^+ (x-1)^{k+1/2} + B^+ \sum_{k=0}^{\infty} b_k^+ (x-1)^k\right]$$

$$= \cdots$$

$$= A^+(x-1)^{-1/2}\sum_{k=0}^{\infty} a_k^+\left(k+\tfrac{1}{2}\right)(x-1)^k \ + \ B^+\sum_{k=1}^{\infty} b_k^+ k(x-1)^k \quad ,$$

and

$$\lim_{x\to 1^-}\left|y'(x)\right| \ = \ \lim_{x\to 1^-}\left|A^+(x-1)^{-1/2}\sum_{k=0}^{\infty} a_k^+\left(k+\tfrac{1}{2}\right)(x-1)^k \ + \ B^+\sum_{k=1}^{\infty} b_k^+ k(x-1)^k\right|$$

$$= \cdots$$

$$= \lim_{x\to 1^-}\left|A^+(x-1)^{-1/2}\right| \ = \ +\infty \quad .$$

Obviously, similar computations apply using y_1^- and y_2^- if, instead, it is A^- which is nonzero.

33.5 d. Because p_m is a solution to the differential equation, there are constants A^+ and B^+, not both 0, such that

$$p_m(x) \ = \ A^+y_1^+(x) \ + \ B^+y_2^+(x) \qquad \text{for} \quad -1 < x < 1$$

where y_1^+ and y_2^+ are as above. But both p_m and y_2^+ are analytic at $x = 1$, while y_1^+ is not analytic at $x = 1$. From this it follows that $A^+ = 0$, $B^+ \neq 0$ and

$$\lim_{x\to 1^-} p_m(x) \ = \ \lim_{x\to 1^-} B^+y_2^+(x) \ = \ B^+\lim_{x\to 1^-}\sum_{k=0}^{\infty} b_k^+(x-1)^k \ = \ B^+b_0^+ \ = \ B^+ \ \neq \ 0 \quad .$$

33.5 e. We must have $T_m(x) \ = \ c_m p_m(x)$ where $1 = T_m(1) = c_m p_m(1)$. Solving for c_m yields

$$T_m(x) \ = \ \frac{p_m(x)}{p_m(1)} \quad .$$

So,

$$T_0(x) \ = \ \frac{p_0(x)}{p_0(1)} \ = \ \frac{1}{1} \ = \ 1 \quad ,$$

$$T_1(x) \ = \ \frac{p_1(x)}{p_1(1)} \ = \ \frac{x}{1} \ = \ x \quad ,$$

$$T_2(x) \ = \ \frac{p_2(x)}{p_2(1)} \ = \ \frac{1-2x^2}{1-2\cdot 1^2} \ = \ 2x^2 \ - \ 1 \quad ,$$

$$T_3(x) \ = \ \frac{p_3(x)}{p_3(1)} \ = \ \frac{x-\frac{4}{3}x^3}{1-\frac{4}{3}1^3} \ = \ 4x^3 \ - \ 3x \quad ,$$

$$\vdots$$

33.5 f. Let y be a nontrivial solution with bounded first derivatives on $(-1, 1)$ to the Chebyshev equation with parameter λ. Since $\left|y'\right|$ is bounded by some finite value M, we must have

$$\left|y'(x)\right| \ \leq \ M \qquad \text{for all} \quad -1 < x < 1 \quad ,$$

which makes it impossible to have

$$\lim_{x\to 1^-} \left|y'(x)\right| = \infty \qquad \text{or} \qquad \lim_{x\to -1^+} \left|y'(x)\right| = \infty \quad .$$

This, by our work in part **c**, means that y can not be nonpolynomial. So y must be a polynomial solution to the Chebyshev equation. And, as noted at the start of this exercise set, this means that $\lambda = m^2$ for some nonnegative integer m, and that y is a constant multiple c of p_m. But then, using the results from part **e**,

$$y(x) = cp_m(x) = CT_m(x) \qquad \text{where} \quad C = cp_m(1) \quad .$$

33.6 a. We first note that the first coefficient in the given series for y_1 is $1/(0!)^2 = 1$, as desired. And since $r_1 = r_2$, the appropriate form for $y_1(x)$ with $x > 0$ is

$$\begin{aligned} y_2(x) &= y_1(x)\ln|x| + x^{1+r_1}\sum_{k=0}^{\infty} b_k x^k \\ &= y_1(x)\ln|x| + x^{3/2}\sum_{k=0}^{\infty} b_k x^k = Y_1(x) + Y_2(x) \end{aligned}$$

where, for convenience, we are letting

$$Y_1(x) = y_1(x)\ln|x| \qquad \text{and} \qquad Y_2(x) = x^{3/2}\sum_{k=0}^{\infty} b_k x^k \quad .$$

Plugging the last formula for y_2 into the differential equation we get

$$\begin{aligned} 0 &= 4x^2 y_2'' + (1-4x)y_2 \\ &= 4x^2[Y_1+Y_2]'' + (1-4x)[Y_1+Y_2] \quad , \end{aligned}$$

which can be rewritten as

$$0 = \left[4x^2 Y_1'' + (1-4x)Y_1\right] + \left[4x^2 Y_2'' + (1-4x)Y_2\right] \quad . \qquad (\star)$$

Differentiating Y_1 twice gives

$$Y_1' = \frac{d}{dx}[y_1(x)\ln|x|] = y_1'(x)\ln|x| + y_1(x)x^{-1}$$

and

$$\begin{aligned} Y_1'' &= \frac{d}{dx}\left[y_1'(x)\ln|x| + y_1(x)x^{-1}\right] \\ &= y_1''(x)\ln|x| + 2y_1'(x)x^{-1} - y_1(x)x^{-2} \quad . \end{aligned}$$

This and the fact that y_1 is the given series solution to the differential equation yields

$$\begin{aligned} 4x^2 Y_1'' + (1-4x)Y_1 &= 4x^2\left[y_1''(x)\ln|x| + 2y_1'(x)x^{-1} - y_1(x)x^{-2}\right] \\ &\qquad + (1-4x)y_1(x)\ln|x| \\ &= \underbrace{\left[4x^2 y_1''(x) + (1-4x)y_1(x)\right]}_{0}\ln|x| + 8xy_1'(x) - 4y_1(x) \end{aligned}$$

$$
\begin{aligned}
&= 8x{y_1}'(x) - 4y_1(x) \\
&= 8x\frac{d}{dx}\left[x^{1/2}\sum_{k=0}^{\infty}\frac{1}{(k!)^2}x^k\right] - 4x^{1/2}\sum_{k=0}^{\infty}\frac{1}{(k!)^2}x^k \\
&= \cdots \\
&= x^{1/2}\sum_{k=0}^{\infty}\frac{8k}{(k!)^2}x^k \quad ,
\end{aligned}
$$

which, since the first term in the last series happens to be 0, reduces to

$$
4x^2{Y_1}'' + (1-4x)Y_1 = x^{1/2}\sum_{k=1}^{\infty}\frac{8k}{(k!)^2}x^k \quad . \tag{$\star\star$}
$$

Next, using computations that should be second nature by now, we have

$$
\begin{aligned}
&4x^2{Y_2}'' + (1-4x)Y_2 \\
&\quad = 4x^2\frac{d^2}{dx^2}\left[x^{3/2}\sum_{k=0}^{\infty}b_k x^k\right] + (1-4x)x^{3/2}\sum_{k=0}^{\infty}b_k x^k \\
&\quad = \cdots \\
&\quad = x^{1/2}\sum_{k=0}^{\infty}b_k\left[(2k+3)(2k+1)+1\right]x^{k+1} + x^{1/2}\sum_{k=0}^{\infty}(-4)b_k x^{k+2} \quad .
\end{aligned}
$$

Combining the last results with equations ($\star$) and ($\star\star$), and dividing out the common $x^{1/2}$ factor:

$$
\begin{aligned}
0 &= x^{-1/2}\left(\left[4x^2{Y_1}'' + (1-4x)Y_1\right] + \left[4x^2{Y_2}'' + (1-4x)Y_2\right]\right) \\
&= \underbrace{\sum_{k=1}^{\infty}\frac{8k}{(k!)^2}x^k}_{n=k} + \underbrace{\sum_{k=0}^{\infty}b_k\left[(2k+3)(2k+1)+1\right]x^{k+1}}_{n=k+1} + \underbrace{\sum_{k=0}^{\infty}(-4)b_k x^{k+2}}_{n=k+2} \\
&= \cdots \\
&= [8+4b_0]x^0 + \sum_{n=2}^{\infty}\left[\frac{8n}{(n!)^2} + b_{n-1}4n^2 - 4b_{n-2}\right]x^n
\end{aligned}
$$

Because each term in the last series must be zero, we have

$$
b_0 = -2
$$

and, for $n \geq 2$,

$$
\frac{8n}{(n!)^2} + b_{n-1}4n^2 - 4b_{n-2} = 0 \quad \rightarrowtail \quad b_{n-1} = \frac{b_{n-2}}{n^2} - \frac{2}{n(n!)^2} \quad .
$$

Letting $k = n-1$ this becomes the recursion formula

$$
b_k = \frac{b_{k-1}}{(k+1)^2} - \frac{2}{(k+1)([k+1]!)^2} \qquad \text{for} \quad n \geq 1 \quad .
$$

Applying the above:

$$
\begin{aligned}
b_0 &= -2 \quad , \\
b_1 &= \frac{b_{1-1}}{(1+1)^2} - \frac{2}{(1+1)([1+1]!)^2} = \frac{b_0}{4} - \frac{1}{4} = -\frac{2}{4} - \frac{1}{4} = -\frac{3}{4} \quad , \\
b_2 &= \frac{b_{2-1}}{(2+1)^2} - \frac{2}{(2+1)([2+1]!)^2} = \frac{b_1}{3^2} - \frac{2}{3(3!)^2} = -\frac{3}{4 \cdot 9} - \frac{2}{3(4 \cdot 9)} \\
&= \cdots = -\frac{11}{108} \quad , \\
b_3 &= \frac{b_{3-1}}{(3+1)^2} - \frac{2}{(3+1)([3+1]!)^2} = \frac{b_2}{4^2} - \frac{2}{4(4!)^2} = -\frac{11}{108 \cdot 4^2} - \frac{2}{4(4!)^2} \\
&= \cdots = -\frac{25}{3{,}456} \quad , \\
b_4 &= \frac{b_{4-1}}{(4+1)^2} - \frac{2}{(4+1)([4+1]!)^2} = \frac{b_4}{5^2} - \frac{2}{5(5!)^2} = -\frac{25}{3{,}456 \cdot 25} - \frac{2}{5(5!)^2} \\
&= \cdots = -\frac{137}{432{,}000} \quad , \\
&\ \vdots
\end{aligned}
$$

Hence,

$$
\begin{aligned}
y_2(x) &= y_1(x) \ln|x| + x^{3/2} \sum_{k=0}^{\infty} b_k x^k \\
&= y_1(x) \ln|x| + x^{3/2} \left[-2 - \frac{3}{4}x - \frac{11}{108}x^2 - \frac{25}{3{,}456}x^3 - \frac{137}{432{,}000}x^4 + \cdots \right] \quad .
\end{aligned}
$$

33.6 c. Since the exponents differ by an integer and $r_2 = 0$, we set

$$
\begin{aligned}
y_2(x) &= \mu y_1(x) \ln|x| + |x|^{r_2} \sum_{k=0}^{\infty} b_k x^k \\
&= \mu y_1(x) \ln|x| + \sum_{k=0}^{\infty} b_k x^k = \mu Y_1(x) + Y_2(x)
\end{aligned}
$$

where $b_0 = 1$, $b_{2-0} = b_2$ is arbitrary,

$$
Y_1(x) = y_1(x) \ln|x| \qquad \text{and} \qquad Y_2(x) = \sum_{k=0}^{\infty} b_k x^k \quad .
$$

Plugging the last formula for y_2 into the differential equation we get

$$
\begin{aligned}
0 &= x^2 y_2'' - \left(x + x^2\right) y_2' + 4x y_2 \\
&= x^2 [\mu Y_1 + Y_2]'' - \left(x + x^2\right) [\mu Y_1 + Y_2]' + 4x [\mu Y_1 + Y_2] \quad ,
\end{aligned}
$$

which can be rewritten as

$$0 = \mu\left[x^2 Y_1'' - \left(x + x^2\right) Y_1' + 4x Y_1\right] + \left[x^2 Y_2'' - \left(x + x^2\right) Y_2' + 4x Y_2\right] \quad . \qquad (\star)$$

Differentiating Y_1 twice gives

$$Y_1' = \frac{d}{dx}\left[y_1(x)\ln|x|\right] = y_1'(x)\ln|x| + y_1(x)x^{-1}$$

and

$$Y_1'' = \frac{d}{dx}\left[y_1'(x)\ln|x| + y_1(x)x^{-1}\right] = y_1''(x)\ln|x| + 2y_1'(x)x^{-1} - y_1(x)x^{-2} \quad .$$

This and the fact that y_1 is the given series solution to the differential equation yields

$$\begin{aligned}
x^2 Y_1'' &- \left(x + x^2\right) Y_1' + 4x Y_1 \\
&= x^2\left[y_1''(x)\ln|x| + 2y_1'(x)x^{-1} - y_1(x)x^{-2}\right] \\
&\quad - \left(x + x^2\right)\left[y_1'(x)\ln|x| + y_1(x)x^{-1}\right] + 4x\left[y_1(x)\ln|x|\right] \\
&= \underbrace{\left[x^2 y_1''(x) - \left(x + x^2\right) y_1'(x) + 4y_1(x)\right]}_{0}\ln|x| \\
&\quad + 2x y_1'(x) - y_1(x) - (1 + x)\, y_1(x) \\
&= 2x y_1'(x) - (2 + x) y_1(x) \\
&= 2x\frac{d}{dx}\left[x^2 - \tfrac{2}{3}x^3 + \tfrac{1}{12}x^4\right] - (2 + x)\left[x^2 - \tfrac{2}{3}x^3 + \tfrac{1}{12}x^4\right] \quad ,
\end{aligned}$$

which ultimately reduces to

$$x^2 Y_1'' - \left(x + x^2\right) Y_1' + 4x Y_1 = 2x^2 - \frac{11}{3}x^3 + \frac{7}{6}x^4 - \frac{1}{12}x^5 \quad . \qquad (\star\star)$$

Doing similar computations with Y_2:

$$\begin{aligned}
x^2 Y_2'' &- \left(x + x^2\right) Y_2' + 4x Y_2 \\
&= x^2\frac{d^2}{dx^2}\left[\sum_{k=0}^{\infty} b_k x^k\right] - \left(x + x^2\right)\frac{d}{dx}\left[\sum_{k=0}^{\infty} b_k x^k\right] + 4x\sum_{k=0}^{\infty} b_k x^k \\
&= \cdots \\
&= \sum_{k=2}^{\infty} b_k k(k-1)x^k + \sum_{k=1}^{\infty} -b_k k x^k + \sum_{k=1}^{\infty} -b_k k x^{k+1} + \sum_{k=0}^{\infty} 4b_k x^{k+1} \quad .
\end{aligned}$$

Combining the last results with equations ($\star$) and ($\star\star$):

$$\begin{aligned}
0 = \mu&\left[x^2 Y_1'' - \left(x + x^2\right) Y_1' + 4x Y_1\right] \\
&+ \left[x^2 Y_2'' - \left(x + x^2\right) Y_2' + 4x Y_2\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mu\left[2x^2 - \frac{11}{3}x^3 + \frac{7}{6}x^4 - \frac{1}{12}x^5\right] \\
&\qquad + \underbrace{\sum_{k=2}^{\infty} b_k k(k-1)x^k}_{n=k} + \underbrace{\sum_{k=1}^{\infty} -b_k k x^k}_{n=k} + \underbrace{\sum_{k=1}^{\infty} -b_k k x^{k+1}}_{n=k+1} + \underbrace{\sum_{k=0}^{\infty} 4b_k x^{k+1}}_{n=k+1} \\
&= \cdots \\
&= \mu\left[2x^2 - \frac{11}{3}x^3 + \frac{7}{6}x^4 - \frac{1}{12}x^5\right] \\
&\qquad + [4b_0 - b_1]\,x^1 + \sum_{n=2}^{\infty}\left[b_n n(n-2) + b_{n-1}(5-n)\right]x^n \\
&= \cdots \\
&= [4b_0 - b_1]\,x^1 + [2\mu + 3b_1]\,x^2 \\
&\qquad + \left[-\frac{11}{3}\mu + 3b_3 + 2b_2\right]x^3 + \left[\frac{7}{6}\mu + 8b_4 + b_3\right]x^4 \\
&\qquad + \left[-\frac{1}{12}\mu + 15b_5\right]x^5 + \sum_{n=6}^{\infty}\left[b_n n(n-2) + b_{n-1}(5-n)\right]x^n \quad .
\end{aligned}
$$

Remember that $b_0 = 1$, and that b_2 is arbitrary and can be set equal to 0. Remembering this and carrying out the arithmetic, we can reduce the above to

$$
\begin{aligned}
0 &= [4 - b_1]\,x^1 + [2\mu + 3b_1]\,x^2 + \left[-\frac{11}{3}\mu + 3b_3\right]x^3 \\
&\qquad + \left[\frac{7}{6}\mu + 8b_4 + b_3\right]x^4 + \left[-\frac{1}{12}\mu + 15b_5\right]x^5 \\
&\qquad + \sum_{n=6}^{\infty}\left[b_n n(n-2) + b_{n-1}(5-n)\right]x^n
\end{aligned}
$$

Since each term on the right must be zero:

$$
\begin{aligned}
4 - b_1 = 0 \quad &\rightarrowtail \quad b_1 = 4 \quad , \\
2\mu + 3b_1 = 0 \quad &\rightarrowtail \quad \mu = -\frac{3b_1}{2} = -\frac{3\cdot 4}{2} = -6 \quad , \\
-\frac{11}{3}\mu + 3b_3 = 0 \quad &\rightarrowtail \quad b_3 = \frac{11\mu}{3\cdot 3} = \frac{11(-6)}{3\cdot 3} = -\frac{22}{3} \quad , \\
\frac{7}{6}\mu + 8b_4 + b_3 = 0 \quad &\rightarrowtail \quad b_4 = -\frac{1}{8}\left[\frac{7\mu}{6} + b_3\right] = -\frac{1}{8}\left[\frac{7(-6)}{6} - \frac{22}{3}\right] = \frac{43}{24} \quad , \\
-\frac{1}{12}\mu + 15b_5 = 0 \quad &\rightarrowtail \quad b_5 = \frac{\mu}{12\cdot 15} = \frac{-6}{12\cdot 15} = -\frac{1}{30} \quad ,
\end{aligned}
$$

and for $n \geq 6$

$$
b_n n(n-2) + b_{n-1}(5-n) = 0 \quad \rightarrowtail \quad b_n = \frac{n-5}{n(n-2)}b_{n-1} \quad .
$$

Thus,

$$\begin{aligned} y_2(x) &= \mu y_1(x) \ln|x| + \sum_{k=0}^{\infty} b_k x^k \\ &= \mu y_1(x) \ln|x| + \left[b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + \cdots \right] \\ &= 6 y_1(x) \ln|x| + \left[1 + 4x + 0x^2 - \frac{22}{3} x^3 + \frac{43}{24} x^4 + \cdots \right] \quad . \end{aligned}$$