33.2 a. Since \( x - x_0 = x - 0 = x \), the quasi-Euler form is simply
\[
x^2 y'' + x \frac{(-2)}{ \beta(x)} y' + \left( \frac{2 - x^2}{ \gamma(x)} \right) y = 0
\]

Here,
\[
\alpha(x) = \alpha(0) = 1 \quad , \quad \beta(0) = -2 \quad \text{and} \quad \gamma(0) = 2 - 0^2 = 2
\]

The indicial equation:
\[
\alpha(0) r(r - 1) + \beta(0) r + \gamma(0) = 0
\]
\[
\iff \quad 1 \cdot r(r - 1) + (-2) r + \left( 2 - 0^2 \right) = 0
\]
\[
\iff \quad r^2 - 3r + 2 = 0
\]

This factors to
\[
(r - 2)(r - 1) = 0
\]

So \( r_1 = 2 \) and \( r_2 = 1 \).

The corresponding (shifted) Euler equation:
\[
x^2 \alpha(0) y'' + x \beta(0) y' + \gamma(0) y = 0
\]
\[
\iff \quad x^2 y'' + x(-2) y' + 2 y = 0
\]

Its two main solutions:
\[
y_{\text{Euler, 1}}(x) = x^{r_1} = x^2 \quad \text{and} \quad y_{\text{Euler, 2}}(x) = x^{r_2} = x^1 = x
\]

The limits of these solutions as \( x \to x_0 \):
\[
\lim_{x \to x_0} \left| y_{\text{Euler, 1}}(x) \right| = \lim_{x \to x_0} \left| x^2 \right| = 0 \quad \text{and} \quad \lim_{x \to x_0} \left| y_{\text{Euler, 2}}(x) \right| = \lim_{x \to x_0} \left| x \right| = 0
\]

33.2 c. Multiplying through by \( x^2 \), we find that the quasi-Euler form is
\[
x^2 y'' + xy' + \frac{x^2}{ \gamma(x)} y = 0
\]

Here,
\[
\alpha(x) = \alpha(0) = 1 \quad , \quad \beta(x) = \beta(0) = 1 \quad \text{and} \quad \gamma(0) = 0^2 = 0
\]

The indicial equation:
\[
\alpha(0) r(r - 1) + \beta(0) r + \gamma(0) = 0
\]
\[
\iff \quad r(r - 1) + r + 0 = 0
\]
\[
\iff \quad r^2 = 0
\]
So \( r_1 = r_2 = 0 \).

The corresponding (shifted) Euler equation is
\[
x^2 \alpha(0) y'' + x \beta(0) y' + \gamma(0) y = 0 \quad \rightarrow \quad x^2 y'' + xy' = 0,
\]
and has solutions:
\[
y_{\text{Euler},1}(x) = x^{r_1} = x^0 = 1 \quad \text{and} \quad y_{\text{Euler},2}(x) = x^{r_1} \ln |x| = \ln |x|.
\]

The limits of these solutions as \( x \to x_0 \):
\[
\lim_{x \to 0} |y_{\text{Euler},1}(x)| = \lim_{x \to 0} 1 = 1 \quad \text{and} \quad \lim_{x \to 0} |y_{\text{Euler},2}(x)| = \lim_{x \to 0} |\ln |x|| = \infty.
\]

**33.2 e.** The quasi-Euler form is
\[
x^2 y'' + x \left(-5 - 2x\right) y' + \frac{9}{\beta(x)} y = 0
\]

Here,
\[
\alpha(x) = \alpha(0) = 1, \quad \beta(0) = -5 - 2 \cdot 0 = -5 \quad \text{and} \quad \gamma(0) = 9.
\]

The indicial equation:
\[
\alpha(0) r(r - 1) + \beta(0) r + \gamma(0) = 0
\]
\[
\quad \rightarrow \quad r(r - 1) - 5r + 9 = 0
\]
\[
\quad \rightarrow \quad r^2 - 6r + 9 = 0.
\]

So \( r_1 = r_2 = 3 \).

The corresponding (shifted) Euler equation is
\[
x^2 \alpha(0) y'' + x \beta(0) y' + \gamma(0) y = 0
\]
\[
\quad \rightarrow \quad x^2 y'' - 5xy' + 9y = 0,
\]
and has solutions:
\[
y_{\text{Euler},1}(x) = x^{r_1} = x^3 \quad \text{and} \quad y_{\text{Euler},2}(x) = x^{r_1} \ln |x|.
\]

The limits of these solutions as \( x \to x_0 \):
\[
\lim_{x \to 0} |y_{\text{Euler},1}(x)| = \lim_{x \to 0} |x^3| = 0
\]
and
\[
\lim_{x \to 0} |y_{\text{Euler},2}(x)| = \lim_{x \to 0} |x^3 \ln |x|| = 0.
\]
33.2 g. The quasi-Euler form is

\[ x^2 4y'' + x 8y' + \frac{(1 - 4x) y}{\gamma(x)} = 0 \]

Here,

\[ \alpha(x) = \alpha(0) = 4 , \quad \beta(x) = \beta(0) = 8 \quad \text{and} \quad \gamma(0) = 1 - 4 \cdot 0 = 1 . \]

The indicial equation:

\[ \alpha(0)r(r - 1) + \beta(0)r + \gamma(0) = 0 \]

\[ \iff 4r(r - 1) + 8r + 1 = 0 \]

\[ \iff \frac{4r^2 + 4r + 1}{(2r+1)^2} = 0 . \]

So \( r_1 = r_2 = -\frac{1}{2} \).

The corresponding (shifted) Euler equation is

\[ x^2 4y'' + 8y' + y = 0 \]

and has solutions:

\[ y_{\text{Euler}, 1}(x) = |x|^{-1/2} \quad \text{and} \quad y_{\text{Euler}, 2}(x) = |x|^{-1/2} \ln |x| . \]

The limits of these solutions as \( x \to x_0 \):

\[ \lim_{x \to 0} \left| y_{\text{Euler}, 1}(x) \right| = \lim_{x \to 0} \left| x |^{-1/2} \right| = \infty \]

and

\[ \lim_{x \to 0} \left| y_{\text{Euler}, 2}(x) \right| = \lim_{x \to 0} \left| x |^{-1/2} \ln |x| \right| = \infty . \]

33.2 i. Since \( x_0 = 0 \), we merely have to multiply through by \( x \) to obtain the quasi-Euler form

\[ x^2 y'' + x 4 \frac{y'}{\beta(x)} + \frac{12x}{(x+2)^2} y = 0 \]

(We could also multiply through by \((x+2)^2\). That would be recommended when finding the modified power series by the basic Frobenius method, but is not necessary for this problem since the above \( \gamma(x) \) is analytic at \( x = 0 \).) Using the above,

\[ \alpha(x) = \alpha(0) = 1 , \quad \beta(0) = 4 \quad \text{and} \quad \gamma(0) = \frac{12 \cdot 0}{(0+2)^2} = 0 . \]
The indicial equation:
\[ \alpha(0) r(r - 1) + \beta(0) r + \gamma(0) = 0 \]

\[ \iff \quad r(r - 1) + 4r = 0 \]

\[ \iff \quad \frac{r^2 + 3r}{r(r - 1)} = 0 . \]

So \( r_1 = 0 \) and \( r_2 = -3 \).

The corresponding (shifted) Euler equation is
\[ x^2 \alpha(0) y'' + x\beta(0) y' + \gamma(0) y = 0 \iff x^2 y'' + 4xy' = 0 , \]
and has solutions:
\( y_{Euler,1}(x) = x^{r_1} = x^0 = 1 \) and \( y_{Euler,2}(x) = x^{r_2} = x^{-3} \).

The limits of these solutions as \( x \to x_0 \):
\[ \lim_{x \to 0} |y_{Euler,1}(x)| = \lim_{x \to 0} |1| = 1 \quad \text{and} \quad \lim_{x \to 0} |y_{Euler,2}(x)| = \lim_{x \to 0} |x^{-3}| = \infty . \]

33.2 k. Multiplying by \( x - 3 \), we find the quasi-Euler form is
\[ (x - 3)^2 y'' + (x - 3)\underbrace{(x - 3)}_{\beta(x)} y' + (x - 3)^2 y = 0 \]

Here,
\( \alpha(x_0) = \alpha(3) = 1 \quad , \quad \beta(3) = 3 - 3 = 0 \quad \text{and} \quad \gamma(3) = 3 - 3 = 0 . \)

The indicial equation:
\[ \alpha(0) r(r - 1) + \beta(0) r + \gamma(0) = 0 \]

\[ \iff \quad r(r - 1) = 0 \]

\[ \iff \quad r^2 - r = 0 . \]

So \( r_1 = 1 \) and \( r_2 = 0 \).

The corresponding (shifted) Euler equation is
\[ x^2 \alpha(0) y'' + x\beta(0) y' + \gamma(0) y = 0 \]

\[ \iff \quad x^2 y'' = 0 , \]
and has solutions:
\( y_{Euler,1}(x) = x^{r_1} = (x - 3)^1 = x - 3 \)
and
\( y_{Euler,2}(x) = x^{r_2} = (x - 3)^0 = 1 \).

The limits of these solutions as \( x \to x_0 \):
\[ \lim_{x \to 0} |y_{Euler,1}(x)| = \lim_{x \to 0} |x - 3| = 0 \quad \text{and} \quad \lim_{x \to 0} |y_{Euler,2}(x)| = \lim_{x \to 0} |1| = 1 . \]
33.3. From section 32.6 on page 698, we know that \( y = Ay_+ + By_- \) where \( A \) and \( B \) are constants with at least one being nonzero, and with \( \{y_1, y_2\} \) being the fundamental set of solutions given by
\[
y_+(x) = (x - x_0)^{\lambda+i\omega} \sum_{k=0}^{\infty} a_k (x - x_0)^k \quad \text{with} \quad a_0 = 1
\]
and
\[
y_-(x) = (x - x_0)^{\lambda-i\omega} \sum_{k=0}^{\infty} a_k^* (x - x_0)^k \quad \text{with} \quad a_0^* = 1^* = 1.
\]
Moreover, as noted in section 33.3, as \( x \to x_0 \) the nonconstant terms in the power series factors approach 0 and we have
\[
\lim_{x \to x_0^+} y_{\pm}(x) = \lim_{x \to x_0^+} y_{\text{Euler,}\pm}(x)
\]
where
\[
y_{\text{Euler,}\pm}(x) = (x - x_0)^{\lambda+i\omega} = (x - x_0)^{\lambda} (x - x_0)^{i\omega} = (x - x_0)^{\lambda} \left[ \cos(\omega \ln |x - x_0|) \pm i \sin(\omega \ln |x - x_0|) \right] .
\]
Hence,
\[
\lim_{x \to x_0^+} y(x) = \lim_{x \to x_0^+} \left[ Ay_+(x) + By_-(x) \right] = \lim_{x \to x_0^+} \left[ A y_{\text{Euler,}+}(x) + B y_{\text{Euler,}-}(x) \right] = \lim_{x \to x_0^+} (x - x_0)^{\lambda} \left[ (A + B) \cos(\omega \ln |x - x_0|) + i (A - B) \sin(\omega \ln |x - x_0|) \right] .
\]
For the case where \( \lambda > 0 \), observe that
\[
| (A + B) \cos(\omega \ln |x - x_0|) + i (A - B) \sin(\omega \ln |x - x_0|) | \leq |A + B| | \cos(\omega \ln |x - x_0|) | + |A - B| | \sin(\omega \ln |x - x_0|) | \leq 1
\]
\[
\leq |A + B| + |A - B| .
\]
Thus, still assuming \( \lambda > 0 \)
\[
\lim_{x \to x_0^+} |y(x)| = \lim_{x \to x_0^+} |x - x_0|^\lambda |(A + B) \cos(\omega \ln |x - x_0|) + i (A - B) \sin(\omega \ln |x - x_0|) |
\]
\[
\leq \lim_{x \to x_0^+} |x - x_0|^\lambda (|A + B| + |A - B|)
\]
\[
= 0 (|A + B| + |A - B|) = 0 ,
\]
which immediately tells us that
\[
\lim_{x \to x_0^+} y(x) = 0 \quad \text{when} \quad \lambda > 0 .
\]
However, when \( \lambda = 0 \), \((x - x_0)^\lambda = 1\), and

\[
\lim_{x \to x_0^+} y(x) = \lim_{x \to x_0^+} [(A + B) \cos(\omega \ln |x - x_0|) + i(A - B) \sin(\omega \ln |x - x_0|)] .
\]

But because \( \ln |x - x_0| \to -\infty \) as \( x \to x_0 \), the last is equivalent to

\[
\lim_{x \to x_0^+} y(x) = \lim_{x \to -\infty} [(A + B) \cos(\omega X) + i(A - B) \sin(\omega X)] ,
\]

which does not converge since the sine and cosine terms repeatedly vary between 1 and \(-1\), as illustrated in figure 32.1a on page 669.

Finally, if \( \lambda < 0 \), then, as \( x \to x_0 \) from the right,

\[
(x - x_0)^\lambda \to +\infty
\]

while the sine and cosine terms in

\[
(A + B) \cos(\omega \ln |x - x_0|) + i(A - B) \sin(\omega \ln |x - x_0|)
\]

oscillates infinitely many times between 1 and \(-1\). Consequently, as \( x \to x_0 \) from the right,

\[
(x - x_0)^\lambda [(A + B) \cos(\omega \ln |x - x_0|) + i(A - B) \sin(\omega \ln |x - x_0|)]
\]

traces out an infinite number of ever increasing “wiggles”. Hence,

\[
\lim_{x \to x_0^+} y(x) = \lim_{x \to -\infty} (x - x_0)^\lambda [(A + B) \cos(\omega X) + i(A - B) \sin(\omega X)]
\]

does not exist if \( \lambda < 0 \).

Note: To make the cases where \( \lambda \leq 0 \) more explicit, compute the limits using the sequences

\[
x_n = x_0 + \exp\left(\frac{-n\pi i}{\alpha}\right) \quad \text{and} \quad x_n = x_0 + \exp\left(\frac{-(2n + 1)\pi i}{2\alpha}\right) .
\]

**33.5 a.** For \( x_0 = 1 \): We already know \( x = 1 \) is a singular point. Computing the limits from theorem 32.2 on page 674, we have

\[
\lim_{x \to x_0^+} \frac{b(x)}{a(x)} = \lim_{x \to 1^-} \frac{-x}{1 - x} = 1
\]

and

\[
\lim_{x \to x_0^+} \frac{c(x)}{a(x)} = \lim_{x \to 1^-} \frac{\lambda}{1 - x} = 0 .
\]

Since both limits are finite, theorem 32.2 assures us that \( x = 1 \) is a regular singular point.

For \( x_0 = -1 \): Repeat the above with \(-1\) replacing \( 1 \).

**33.5 b.** Exponents at \( x_0 = 1 \): First, we get the Chebyshev equation into quasi-Euler form:

\[
-(x - 1) \left[(1 - x^2)y'' - xy' + \lambda y\right] = -(x - 1) \cdot 0
\]

\[\leftrightarrow\]

\[-(x - 1)(x + 1)(1 - x)y'' + (x - 1)xy' - (x - 1)\lambda y = 0\]

\[\leftrightarrow\]

\[(x - 1)^2 \alpha(x)y'' + (x - 1)\beta(x)y' + \gamma(x)y = 0\]
where
\[ \alpha(x) = x + 1 \quad , \quad \beta(x) = x \quad \text{and} \quad \gamma(x) = \lambda(x - 1) . \]

Thus,
\[ \alpha(1) = 1 + 1 = 2 \quad , \quad \beta(1) = 1 \quad \text{and} \quad \gamma(1) = \lambda(1 - 1) = 0 , \]
and the indicial equation is
\[ r(r - 1)2 + r + 0 = 0 \quad \implies \quad \frac{2r^2 - r}{r(2r-1)} = 0 . \]

From this it follows that \( r_1 = \frac{1}{2} \) and \( r_2 = 0 \).

Similar computations yield the same exponents at \( x_0 = -1 \).

33.5 c. Since \( y \) is a nonpolynomial solution, we know it is given by a power series about 0 with radius of convergence \( R = 1 \). From theorem 33.5 on page 715 it follows that \( y \) must have a singular point at least at \( x = -1 \) or \( x = 1 \). That is, \( y \) is not analytic at one or both of these points.

In addition, because the equation’s exponents at both \( x_0 = 1 \) and \( x_0 = -1 \) are \( r_1 = \frac{1}{2} \) and \( r_2 = 0 \), we know that we can find modified power series solutions about both points via the basic Frobenius method,
\[ y_1^+(x) = (x - 1)^{1/2} \sum_{k=0}^{\infty} a_k^+(x - 1)^k \quad \text{with} \quad a_0^+ = 1 , \]
\[ y_2^+(x) = \sum_{k=0}^{\infty} b_k^+(x - 1)^k \quad \text{with} \quad b_0^+ = 1 , \]
\[ y_1^-(x) = (x + 1)^{1/2} \sum_{k=0}^{\infty} a_k^-(x + 1)^k \quad \text{with} \quad a_0^- = 1 \]

and\[ y_2^-(x) = \sum_{k=0}^{\infty} b_k^-(x + 1)^k \quad \text{with} \quad b_0^- = 1 . \]

Moreover, there must be constants \( A^\pm \) and \( B^\pm \) such that, for \(-1 < x < 1\),
\[ y(x) = A^+ y_1^+(x) + B^+ y_2^+(x) = A^- y_1^-(x) + B^- y_2^-(x) . \]

If \( A^+ = 0 = A^- \), then the above reduces to
\[ y(x) = B^+ \sum_{k=0}^{\infty} b_k^+(x - 1)^k = B^- \sum_{k=0}^{\infty} b_k^-(x + 1)^k . \]

But this would mean that \( y \) is analytic at both \( x_0 = 1 \) and \( x_0 = -1 \), which we already know is not the case. So we do not have \( A^+ = 0 = A^- \), at least one must be nonzero. If it is \( A^+ \) which is nonzero, then
\[ y'(x) = \frac{d}{dx} \left[ A^+ y_1^+(x) + B^+ y_2^+(x) \right] \]
\[ = \frac{d}{dx} \left[ A^+ \sum_{k=0}^{\infty} a_k^+(x - 1)^{k+1/2} + B^+ \sum_{k=0}^{\infty} b_k^+(x - 1)^k \right] \]
33.5 e. Let

\[ y = A^+(x - 1)^{-1/2} \sum_{k=0}^{\infty} a_k^+ \left( k + \frac{1}{2} \right) (x - 1)^k + B^+ \sum_{k=1}^{\infty} b_k^+ (x - 1)^k, \]

and

\[ \lim_{x \to 1^-} y'(x) = \lim_{x \to 1^-} A^+(x - 1)^{-1/2} \sum_{k=0}^{\infty} a_k^+ \left( k + \frac{1}{2} \right) (x - 1)^k + B^+ \sum_{k=1}^{\infty} b_k^+ (x - 1)^k. \]

Obviously, similar computations apply using \( y_1^- \) and \( y_2^- \) if, instead, it is \( A^- \) which is nonzero.

33.5 d. Because \( p_m \) is a solution to the differential equation, there are constants \( A^+ \) and \( B^+ \), not both 0, such that

\[ p_m(x) = A^+ y_1^+(x) + B^+ y_2^+(x) \quad \text{for} \quad -1 < x < 1 \]

where \( y_1^+ \) and \( y_2^+ \) are as above. But both \( p_m \) and \( y_2^+ \) are analytic at \( x = 1 \), while \( y_1^+ \) is not analytic at \( x = 1 \). From this it follows that \( A^+ = 0 \), \( B^+ \neq 0 \) and

\[ \lim_{x \to 1^-} p_m(x) = \lim_{x \to 1^-} B^+ y_2^+(x) = B^+ \lim_{x \to 1^-} \sum_{k=0}^{\infty} b_k^+ (x - 1)^k = B^+ b_0^+ = B^+ \neq 0. \]

33.5 e. We must have \( T_m(x) = c_m p_m(x) \) where \( 1 = T_m(1) = c_m p_m(1) \). Solving for \( c_m \) yields

\[ T_m(x) = \frac{p_m(x)}{p_m(1)}. \]

So,

\[ T_0(x) = \frac{p_0(x)}{p_0(1)} = \frac{1}{1} = 1, \]

\[ T_1(x) = \frac{p_1(x)}{p_1(1)} = \frac{x}{1} = x, \]

\[ T_2(x) = \frac{p_2(x)}{p_2(1)} = \frac{1 - 2x^2}{1 - 2 \cdot 1^2} = 2x^2 - 1, \]

\[ T_3(x) = \frac{p_3(x)}{p_3(1)} = \frac{x - \frac{4}{3} x^3}{1 - \frac{4}{3} \cdot 1^3} = 4x^3 - 3x, \]

\[ \vdots \]

33.5 f. Let \( y \) be a nontrivial solution with bounded first derivatives on \((-1, 1)\) to the Chebyshev equation with parameter \( \lambda \). Since \( |y'| \) is bounded by some finite value \( M \), we must have

\[ |y'(x)| \leq M \quad \text{for all} \quad -1 < x < 1. \]
which makes it impossible to have
\[
\lim_{x \to 1^-} |y'(x)| = \infty \quad \text{or} \quad \lim_{x \to 1^+} |y'(x)| = \infty.
\]
This, by our work in part (c), means that \( y \) cannot be nonpolynomial. So \( y \) must be a polynomial solution to the Chebyshev equation. And, as noted at the start of this exercise set, this means that \( \lambda = m^2 \) for some nonnegative integer \( m \), and that \( y \) is a constant multiple \( c \) of \( p_m \). But then, using the results from part (e),
\[
y(x) = cp_m(x) = CT_m(x) \quad \text{where} \quad C = cp_m(1).
\]

33.6a. We first note that the first coefficient in the given series for \( y_1 \) is \( b_y(0)^2 = 1 \), as desired. And since \( r_1 = r_2 \), the appropriate form for \( y_1(x) \) with \( x > 0 \) is
\[
y_2(x) = y_1(x) \ln|x| + x^{1+r_1} \sum_{k=0}^{\infty} b_k x^k
\]
\[
= y_1(x) \ln|x| + x^{3/2} \sum_{k=0}^{\infty} b_k x^k = Y_1(x) + Y_2(x)
\]
where, for convenience, we are letting
\[
Y_1(x) = y_1(x) \ln|x| \quad \text{and} \quad Y_2(x) = x^{3/2} \sum_{k=0}^{\infty} b_k x^k.
\]
Plugging the last formula for \( Y_2 \) into the differential equation we get
\[
0 = 4x^2 y_2'' + (1 - 4x) y_2
\]
\[
= 4x^2 [Y_1 + Y_2]'' + (1 - 4x) [Y_1 + Y_2],
\]
which can be rewritten as
\[
0 = \left[4x^2 Y_1'' + (1 - 4x) Y_1\right] + \left[4x^2 Y_2'' + (1 - 4x) Y_2\right]. \quad (\star)
\]
Differentiating \( Y_1 \) twice gives
\[
Y_1' = \frac{d}{dx} [y_1(x) \ln|x|] = y_1'(x) \ln|x| + y_1(x) x^{-1}
\]
and
\[
Y_1'' = \frac{d}{dx} \left[y_1'(x) \ln|x| + y_1(x) x^{-1}\right]
\]
\[
= y_1''(x) \ln|x| + 2y_1'(x) x^{-1} - y_1(x) x^{-2}.
\]
This and the fact that \( y_1 \) is the given series solution to the differential equation yields
\[
4x^2 Y_1'' + (1 - 4x) Y_1 = 4x^2 \left[y_1''(x) \ln|x| + 2y_1'(x) x^{-1} - y_1(x) x^{-2}\right]
\]
\[
+ (1 - 4x) y_1(x) \ln|x|
\]
\[
= \left[4x^2 y_1''(x) + (1 - 4x) y_1(x) \ln|x|\right] \ln|x| + 8x y_1'(x) - 4y_1(x)
\]
\[ \begin{align*}
&= 8x y_1'(x) - 4y_1(x) \\
&= 8x \frac{d}{dx} \left[ x^{1/2} \sum_{k=0}^{\infty} \frac{1}{(k)!^2} x^k \right] - 4x^{1/2} \sum_{k=0}^{\infty} \frac{1}{(k)!^2} x^k \\
&= \ldots \\
&= x^{1/2} \sum_{k=0}^{\infty} \frac{8k}{(k)!^2} x^k ,
\end{align*} \]

which, since the first term in the last series happens to be 0, reduces to

\[ 4x^2 Y_1'' + (1 - 4x) Y_1 = x^{1/2} \sum_{k=1}^{\infty} \frac{8k}{(k)!^2} x^k . \tag{**} \]

Next, using computations that should be second nature by now, we have

\[ \begin{align*}
&= 4x^2 \frac{d^2}{dx^2} \left[ x^{3/2} \sum_{k=0}^{\infty} b_k x^k \right] + (1 - 4x)x^{3/2} \sum_{k=0}^{\infty} b_k x^k \\
&= \ldots \\
&= x^{1/2} \sum_{k=0}^{\infty} b_k [(2k + 3)(2k + 1) + 1] x^{k+1} + x^{1/2} \sum_{k=0}^{\infty} (-4)b_k x^{k+2} .
\end{align*} \]

Combining the last results with equations (***) and (**), and dividing out the common \( x^{1/2} \) factor:

\[ \begin{align*}
0 &= x^{-1/2} \left( 4x^2 Y_1'' + (1 - 4x) Y_1 \right) + \left[ 4x^2 Y_2'' + (1 - 4x) Y_2 \right] \\
&= \sum_{k=1}^{\infty} \frac{8k}{(k)!^2} x^k + \sum_{n=0}^{\infty} b_k [(2k + 3)(2k + 1) + 1] x^{k+1} + \sum_{k=0}^{\infty} (-4)b_k x^{k+2} \\
&= \ldots \\
&= [8 + 4b_0] x^0 + \sum_{n=2}^{\infty} \left[ \frac{8n}{(n!)^2} + b_{n-1} 4n^2 - 4b_{n-2} \right] x^n
\end{align*} \]

Because each term in the last series must be zero, we have

\[ b_0 = -2 \]

and, for \( n \geq 2 \),

\[ \frac{8n}{(n!)^2} + b_{n-1} 4n^2 - 4b_{n-2} = 0 \quad \Rightarrow \quad b_{n-1} = \frac{b_{n-2}}{n^2} - \frac{2}{n(n!)^2} . \]

Letting \( k = n - 1 \) this becomes the recursion formula

\[ b_k = \frac{b_{k-1}}{(k+1)^2} - \frac{2}{(k+1)((k+1)!)^2} \quad \text{for} \quad n \geq 1 . \]
Applying the above:

\[ b_0 = -2 , \]
\[ b_1 = \frac{b_{1-1}}{(1+1)^2} - \frac{2}{(1+1)(1+1)!^2} = \frac{b_0}{4} - \frac{1}{4} = -\frac{2}{4} - \frac{1}{4} = -\frac{3}{4} , \]
\[ b_2 = \frac{b_{2-1}}{(2+1)^2} - \frac{2}{(2+1)(2+1)!^2} = \frac{b_1}{3^2} - \frac{2}{3(3)!^2} = -\frac{3}{4 \cdot 9} - \frac{2}{3(4 \cdot 9)}\]
\[ = \ldots = -\frac{11}{108} , \]
\[ b_3 = \frac{b_{3-1}}{(3+1)^2} - \frac{2}{(3+1)(3+1)!^2} = \frac{b_2}{4^2} - \frac{2}{4(4)!^2} = -\frac{11}{108 \cdot 4^2} - \frac{2}{4(4)!^2}\]
\[ = \ldots = -\frac{25}{3,456} , \]
\[ b_4 = \frac{b_{4-1}}{(4+1)^2} - \frac{2}{(4+1)(4+1)!^2} = \frac{b_3}{5^2} - \frac{2}{5(5)!^2} = -\frac{25}{3,456 \cdot 25} - \frac{2}{5(5)!^2}\]
\[ = \ldots = -\frac{137}{432,000} , \]
\[ \vdots \]

Hence,

\[ y_2(x) = y_1(x) \ln |x| + x^{3/2} \sum_{k=0}^{\infty} b_k x^k \]
\[ = y_1(x) \ln |x| + x^{3/2} \left[ -2 - \frac{3}{4} x - \frac{11}{108} x^2 - \frac{25}{3,456} x^3 - \frac{137}{432,000} x^4 + \ldots \right] . \]

**33.6 c.** Since the exponents differ by an integer and \( r_2 = 0 \), we set

\[ y_2(x) = \mu y_1(x) \ln |x| + |x|^{r_2} \sum_{k=0}^{\infty} b_k x^k \]
\[ = \mu y_1(x) \ln |x| + \sum_{k=0}^{\infty} b_k x^k = \mu Y_1(x) + Y_2(x) \]

where \( b_0 = 1 , b_{2-0} = b_2 \) is arbitrary,

\[ Y_1(x) = y_1(x) \ln |x| \quad \text{and} \quad Y_2(x) = \sum_{k=0}^{\infty} b_k x^k . \]

Plugging the last formula for \( y_2 \) into the differential equation we get

\[ 0 = x^2 y_2'' - \left( x + x^2 \right) y_2' + 4x y_2 \]
\[ = x^2 \left[ \mu Y_1 + Y_2 \right]'' - \left( x + x^2 \right) \left[ \mu Y_1 + Y_2 \right]' + 4x \left[ \mu Y_1 + Y_2 \right] . \]
which can be rewritten as

\[ 0 = \mu \left[ x^2 Y_1'' - \left( x + x^2 \right) Y_1' + 4x Y_1 \right] \]

\[ + \left[ x^2 Y_2'' - \left( x + x^2 \right) Y_2' + 4x Y_2 \right]. \]  

\[ \text{(*)} \]

Differentiating \( Y_1 \) twice gives

\[ Y_1' = \frac{d}{dx} \left[ y_1(x) \ln |x| \right] = y_1'(x) \ln |x| + y_1(x)x^{-1} \]

and

\[ Y_1'' = \frac{d}{dx} \left[ y_1'(x) \ln |x| + y_1(x)x^{-1} \right] \]

\[ = y_1''(x) \ln |x| + 2y_1'(x)x^{-1} - y_1(x)x^{-2} \].

This and the fact that \( y_1 \) is the given series solution to the differential equation yields

\[ x^2 Y_1'' - \left( x + x^2 \right) Y_1' + 4x Y_1 \]

\[ = x^2 \left[ y_1''(x) \ln |x| + 2y_1'(x)x^{-1} - y_1(x)x^{-2} \right] \]

\[ - \left( x + x^2 \right) \left[ y_1'(x) \ln |x| + y_1(x)x^{-1} \right] + 4x [y_1(x) \ln |x|] \]

\[ = \left[ x^2 y_1''(x) - \left( x + x^2 \right) y_1'(x) + 4y_1(x) \right] \ln |x| \]

\[ + 2xy_1'(x) - y_1(x) - (1 + x) y_1(x) \]

\[ = 2xy_1'(x) - (2 + x)y_1(x) \]

\[ = 2x \frac{d}{dx} \left[ x^2 - \frac{2}{3} x^3 + \frac{1}{12} x^4 \right] - (2 + x) \left[ x^2 - \frac{2}{3} x^3 + \frac{1}{12} x^4 \right], \]

which ultimately reduces to

\[ x^2 Y_1'' - \left( x + x^2 \right) Y_1' + 4x Y_1 = 2x^2 - \frac{11}{3} x^3 + \frac{7}{6} x^4 - \frac{1}{12} x^5. \]  

\[ \text{(**)} \]

Doing similar computations with \( Y_2 \):

\[ x^2 Y_2'' - \left( x + x^2 \right) Y_2' + 4x Y_2 \]

\[ = x^2 \frac{d^2}{dx^2} \left[ \sum_{k=0}^{\infty} b_k x^k \right] - \left( x + x^2 \right) \frac{d}{dx} \left[ \sum_{k=0}^{\infty} b_k x^k \right] + 4x \sum_{k=0}^{\infty} b_k x^k \]

\[ = \cdots \]

\[ = \sum_{k=2}^{\infty} b_k k(k-1) x^{k-1} + \sum_{k=1}^{\infty} -b_k k x^k + \sum_{k=1}^{\infty} -b_k k x^{k+1} + \sum_{k=0}^{\infty} 4b_k x^{k+1}. \]

Combining the last results with equations (**) and (**):

\[ 0 = \mu \left[ x^2 Y_1'' - \left( x + x^2 \right) Y_1' + 4x Y_1 \right] \]

\[ + \left[ x^2 Y_2'' - \left( x + x^2 \right) Y_2' + 4x Y_2 \right]. \]
\[
\begin{align*}
&= \mu \left[ 2x^2 - \frac{11}{3}x^3 + \frac{7}{6}x^4 - \frac{1}{12}x^5 \right] \\
&\quad + \sum_{k=2}^{n} b_k (k-1)x^k + \sum_{k=1}^{n+1} -b_k k x^k + \sum_{k=1}^{n+1} -b_k k x^{k+1} + \sum_{k=0}^{n+1} 4b_k x^{k+1} \\
&= \cdots \\
&= \mu \left[ 2x^2 - \frac{11}{3}x^3 + \frac{7}{6}x^4 - \frac{1}{12}x^5 \right] \\
&\quad + [4b_0 - b_1] x^1 + \sum_{n=2}^{\infty} [b_n n(n-2) + b_{n-1}(5-n)] x^n \\
&= \cdots \\
&= [4b_0 - b_1] x^1 + [2\mu + 3b_1] x^2 \\
&\quad + \left[ -\frac{11}{3} \mu + 3b_3 + 2b_2 \right] x^3 + \left[ \frac{7}{6} \mu + 8b_4 + b_3 \right] x^4 \\
&\quad + \left[ -\frac{1}{12} \mu + 15b_5 \right] x^5 + \sum_{n=6}^{\infty} [b_n n(n-2) + b_{n-1}(5-n)] x^n .
\end{align*}
\]

Remember that \(b_0 = 1\), and that \(b_2\) is arbitrary and can be set equal to 0. Remembering this and carrying out the arithmetic, we can reduce the above to

\[
0 = [4 - b_1] x^1 + [2\mu + 3b_1] x^2 + \left[ -\frac{11}{3} \mu + 3b_3 \right] x^3 \\
&\quad + \left[ \frac{7}{6} \mu + 8b_4 + b_3 \right] x^4 + \left[ -\frac{1}{12} \mu + 15b_5 \right] x^5 \\
&\quad + \sum_{n=6}^{\infty} [b_n n(n-2) + b_{n-1}(5-n)] x^n .
\]

Since each term on the right must be zero:

\[
\begin{align*}
4 - b_1 &= 0 \Rightarrow b_1 = 4 , \\
2\mu + 3b_1 &= 0 \Rightarrow \mu = -\frac{3b_1}{2} = -\frac{3 \cdot 4}{2} = -6 , \\
-\frac{11}{3} \mu + 3b_3 &= 0 \Rightarrow b_3 = \frac{11\mu}{3 \cdot 3} = \frac{11(-6)}{3 \cdot 3} = -\frac{22}{3} , \\
\frac{7}{6} \mu + 8b_4 + b_3 &= 0 \Rightarrow b_4 = -\frac{1}{8} \left[ \frac{7\mu}{6} + b_3 \right] = -\frac{1}{8} \left[ \frac{7(-6)}{6} - \frac{22}{3} \right] = -\frac{43}{24} , \\
-\frac{1}{12} \mu + 15b_5 &= 0 \Rightarrow b_5 = \frac{\mu}{12 \cdot 15} = -\frac{6}{12 \cdot 15} = -\frac{1}{30} ,
\end{align*}
\]

and for \(n \geq 6\)

\[
b_n n(n-2) + b_{n-1}(5-n) = 0 \Rightarrow b_n = \frac{n - 5}{n(n-2)} b_{n-1} .
\]
Thus,

\[ y_2(x) = \mu y_1(x) \ln |x| + \sum_{k=0}^{\infty} b_k x^k \]

\[ = \mu y_1(x) \ln |x| + \left[ b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + \cdots \right] \]

\[ = 6y_1(x) \ln |x| + \left[ 1 + 4x + 0x^2 - \frac{22}{3} x^3 + \frac{43}{24} x^4 + \cdots \right] . \]