Chapter 32: Modified Power Series Solutions and the Basic Method of Frobenius

32.2 a. Let $y = y(x) = (x-3)^r$. Then

$$
\begin{aligned}
& (x-3)^2 y'' \; - \; 2(x-3)y' \; + \; 2y \; = \; 0 \\
\hookrightarrow \quad & (x-3)^2 \left[(x-3)^r\right]'' \; - \; 2(x-3)\left[(x-3)^r\right]' \; + \; 2\left[(x-3)^r\right] \; = \; 0 \\
\hookrightarrow \quad & (x-3)^2 \left[r(r-1)(x-3)^{r-2}\right]'' \; - \; 2(x-3)\left[r(x-3)^{r-1}\right]' \\
& \qquad + 2\left[(x-3)^r\right] \; = \; 0 \\
\hookrightarrow \quad & (x-3)^r \left[r(r-1) - 2r + 2\right] \; = \; 0 \quad .
\end{aligned}
$$

Dividing out the $(x-3)^r$ and simplifying the rest yields

$$r^2 \; - \; 3r \; + \; 2 \; = \; 0 \quad ,$$

which factors to

$$(r-2)(r-1) \; = \; 0 \quad ,$$

Hence, $r = 2$ or $r = 1$, and a fundamental set of solutions is $\{y_1, y_2\}$ with

$$y_1(x) \; = \; (x-3)^2 \qquad \text{and} \qquad y_1(x) \; = \; (x-3)^1 \; = \; x \; - \; 3 \quad .$$

32.2 c. Letting $y = y(x) = (x-1)^r$,

$$
\begin{aligned}
0 \; &= \; (x-1)^2 y'' \; - \; 5(x-1)y' \; + \; 9y \\
&= \; (x-1)^2 \left[(x-1)^r\right]'' \; - \; 5(x-1)\left[(x-1)^r\right]' \; + \; 9\left[(x-1)^r\right] y \\
&= \; (x-1)^2 \left[r(r-1)(x-1)^{r-2}\right]'' \; - \; 5(x-1)\left[r(x-1)^{r-1}\right]' \; + \; 9\left[(x-3)^r\right] \\
&= \; (x-1)^r \left[r(r-1) - 5r + 9\right] \quad .
\end{aligned}
$$

Dividing out the $(x-1)^r$ leaves us with

$$0 \; = \; r(r-1) \; - \; 5r \; + \; 9 \; = \; r^2 \; - \; 6r \; + \; 9 \; = \; (r-3)^3 \quad .$$

Thus, $r = 3$ is the only solution to the indicial equation, and a fundamental pair of solutions is given by

$$y_1(x) \; = \; (x-1)^3 \qquad \text{and} \qquad y_2(x) \; = \; (x-1)^3 \ln|x-1| \quad .$$

32.2 e. Letting $y = y(x) = (x-5)^r$,

$$
\begin{aligned}
0 \; &= \; 3(x-5)^2 y'' \; - \; 4(x-5)y' \; + \; 2y \\
&= \; 3(x-5)^2 \left[(x-5)^r\right]'' \; - \; 4(x-5)\left[(x-5)^r\right]' \; + \; 2\left[(x-5)^r\right] \\
&= \; 3(x-5)^2 \left[r(r-1)(x-5)^{r-2}\right]'' \; - \; 4(x-5)\left[r(x-5)^{r-1}\right]' \; + \; 2\left[(x-5)^r\right] \\
&= \; (x-5)^r \underbrace{\left[3r(r-1) - r4 + 2\right]}_{3r^2 - 7r + 2} \quad .
\end{aligned}
$$

Dividing out the $(x-5)^r$ leaves us with

$$0 = 3r^2 - 7r + 2 \quad .$$

Hence,

$$r = \frac{-[-7] \pm \sqrt{(-7)^2 - 4 \cdot 3 \cdot 2}}{2 \cdot 3} = \frac{7 \pm 5}{6} \quad \rightarrowtail \quad r = 2 \quad \text{or} \quad r = \frac{1}{3} \quad .$$

So a fundamental pair of solutions is given by

$$y_1(x) = (x-5)^2 \qquad \text{and} \qquad y_2(x) = (x-5)^{1/3} = \sqrt[3]{x-5} \quad .$$

32.3 a. After multiplying through by $x-2$ and by $x+2$, the differential equation becomes

$$\underbrace{x^2(x-2)(x+2)}_{a(x)} y'' + \underbrace{x(x+2)}_{b(x)} y' + \underbrace{2(x-2)}_{c(x)} y = 0 \quad . \qquad (\star)$$

Each coefficient is a polynomial; there are no factors common to all three, and the first is 0 if and only if $x=0$, $x=2$ or $x=-2$. So these values of x are the singular points for the differential equation.

To determine whether each singular point is regular or not, we can either see if we can rewrite the equation in quasi-Euler form about that point or use the test from theorem 32.2 on page 674. We'll illustrate the use of both approaches.

For the singular point $x_0 = 0$, let us simply note that equation $(\star)$ is

$$(x-0)^2 \underbrace{(x-2)(x+2)}_{\alpha(x)} y'' + (x-0) \underbrace{(x+2)}_{\beta(x)} y' + \underbrace{2(x-2)}_{\gamma(x)} y = 0 \quad .$$

with

$$\alpha(0) = (0-2)(0+2) \neq 0 \quad .$$

This is a quasi-Euler form about $x_0 = 0$, and, so, x_0 is a regular singular point.

For the other two singular points, we will apply theorem 32.2, which involves the limits

$$\lim_{x \to x_0} (x-x_0) \frac{b(x)}{a(x)} = \lim_{x \to x_0} (x-x_0) \frac{x(x+2)}{x^2(x-2)(x+2)} = \lim_{x \to x_0} \frac{x-x_0}{x(x-2)}$$

and

$$\lim_{x \to x_0} (x-x_0)^2 \frac{c(x)}{a(x)} = \lim_{x \to x_0} (x-x_0)^2 \frac{2(x-2)}{x^2(x-2)(x+2)} = \lim_{x \to x_0} \frac{2(x-x_0)^2}{x^2(x+2)}$$

with x_0 being the singular point in question.

For $x_0 = 2$, we have

$$\lim_{x \to 2} (x-2) \frac{b(x)}{a(x)} = \lim_{x \to 2} \frac{x-2}{x(x-2)} = \lim_{x \to 2} \frac{1}{x} = \frac{1}{2}$$

and

$$\lim_{x \to 2} (x-2)^2 \frac{c(x)}{a(x)} = \lim_{x \to 2} \frac{2(x-2)^2}{x^2(x+2)} = 0 \quad .$$

Since both limits are finite, theorem 32.2 assures us that singular point $x_0 = 2$ is regular.

For $x_0 = -2$, we have

$$\lim_{x \to -2} (x-[-2]) \frac{b(x)}{a(x)} = \lim_{x \to -2} \frac{x+2}{x(x-2)} = 0$$

and

$$\lim_{x\to -2}(x-[-2])^2\frac{c(x)}{a(x)} \;=\; \lim_{x\to -2}\frac{2(x+2)^2}{x^2(x+2)} \;=\; \lim_{x\to -2}\frac{2(x+2)}{x^2} \;=\; 0 \quad .$$

Again, since both limits are finite, theorem 32.2 assures us that singular point $x_0 = -2$ is regular.

So all three singular points 0, 2 and -2 are regular singular points. The closest to 0 other than 0, itself, is either ± 2. Hence, the Frobenius radius about 0 is

$$R \;=\; |0-[\pm 2]| \;=\; 2 \quad .$$

32.3 c. Since all three coefficients are polynomials and clearly without common factors, each singular point x_0 where the first coefficient is 0:

$$0 \;=\; x^3 \;-\; x^4 \;=\; x^3(1-x) \quad \rightarrowtail \quad x_0 \;=\; 0 \quad \text{or} \quad x_0 \;=\; 1 \quad .$$

To apply theorem 32.2, we need

$$\lim_{x\to x_0}(x-x_0)\frac{b(x)}{a(x)} \;=\; \lim_{x\to x_0}(x-x_0)\frac{3x-1}{x^3-x^4} \;=\; \lim_{x\to x_0}\frac{(x-x_0)(3x-1)}{x^3(1-x)}$$

and

$$\lim_{x\to x_0}(x-x_0)^2\frac{c(x)}{a(x)} \;=\; \lim_{x\to x_0}(x-x_0)^2\frac{827}{x^3-x^4} \;=\; \lim_{x\to x_0}\frac{827(x-x_0)^2}{x^3(1-x)} \quad .$$

For $x_0 = 0$,

$$\lim_{x\to 0}(x-0)\frac{b(x)}{a(x)} \;=\; \lim_{x\to 0}\frac{(x-0)(3x-1)}{x^3(1-x)} \;=\; \lim_{x\to 0}\frac{(3x-1)}{x^2(1-x)} \quad ,$$

which is not finite. So, x_0 is an irregular singular point.

For $x_0 = 1$,

$$\lim_{x\to 1}(x-1)\frac{b(x)}{a(x)} \;=\; \lim_{x\to 1}\frac{(x-1)(3x-1)}{x^3(1-x)} \;=\; \lim_{x\to 1}\frac{-(3x-1)}{x^3} \;=\; -2$$

and

$$\lim_{x\to 1}(x-1)^2\frac{c(x)}{a(x)} \;=\; \lim_{x\to 1}\frac{827(x-1)^2}{x^3(1-x)} \;=\; \lim_{x\to 1}\frac{-827(x-1)}{x^3} \;=\; 0 \quad .$$

Since both limits are finite, we know $x_0 = 1$ is a regular singular point.

The closest singular point to 1 other than 1, itself, is 0. So the Frobenius radius about $x_0 = 1$ is

$$R \;=\; |1-0| \;=\; 1 \quad .$$

32.3 e. By a number of arguments, it should be clear that $x_0 = 3$ and $x_0 = 4$ are the singular points. For applying theorem 32.2, we have

$$\lim_{x\to x_0}(x-x_0)\frac{b(x)}{a(x)} \;=\; \lim_{x\to x_0}(x-x_0)\frac{1}{(x-3)^2} \;=\; \lim_{x\to x_0}\frac{x-x_0}{(x-3)^2}$$

and

$$\lim_{x\to x_0}(x-x_0)^2\frac{c(x)}{a(x)} \;=\; \lim_{x\to x_0}(x-x_0)^2\frac{1}{(x-4)^2} \;=\; \lim_{x\to x_0}\frac{(x-x_0)^2}{(x-4)^2} \quad .$$

For $x_0 = 3$,

$$\lim_{x\to 3}(x-3)\frac{b(x)}{a(x)} \;=\; \lim_{x\to 3}\frac{x-3}{(x-3)^2} \;=\; \lim_{x\to 3}\frac{1}{(x-3)} \quad ,$$

which is not finite, telling us that $x_0 = 3$ is an irregular singular point.

For $x_0 = 4$,

$$\lim_{x\to 4}(x-4)\frac{b(x)}{a(x)} = \lim_{x\to 4}\frac{x-4}{(x-3)^2} = 0$$

and

$$\lim_{x\to 4}(x-4)^2\frac{c(x)}{a(x)} = \lim_{x\to 4}\frac{(x-4)^2}{(x-4)^2} = \lim_{x\to 4} 1 = 1 \quad .$$

Both limits are finite, telling us that $x_0 = 4$ is a regular singular point.

Since the closest singular point to 4 other than 4, itself, is 3, the Frobenius radius about $x_0 = 4$ is

$$R = |4-3| = 1 \quad .$$

32.3 g. After multiplying through by x and by $1 + x^2$, the differential equation becomes

$$A(x)y'' + B(x)y' + C(x)y = 0 \qquad (\star)$$

where A, B and C are the polynomials

$$A(x) = x\left(1+x^2\right)\left(4x^2-1\right) \quad ,$$

$$B(x) = \left(1+x^2\right)(4x-2)$$

and

$$C(x) = x\left(1-x^2\right) \quad ,$$

which, after using a little algebra, can be written in fully-factored form

$$A(x) = 4x(x-i)(x+i)\left(x-\tfrac{1}{2}\right)\left(x+\tfrac{1}{2}\right) \quad ,$$

$$B(x) = 4(x-i)(x+i)\left(x-\tfrac{1}{2}\right)$$

and

$$C(x) = x(x-1)(x+1) \quad .$$

Since no factor is shared by all these polynomials, the singular points are the values of x for which $A(x) = 0$; namely,

$$0 \quad , \quad i \quad , \quad -i \quad , \quad \tfrac{1}{2} \quad \text{and} \quad -\tfrac{1}{2} \quad .$$

Rather than apply theorem 32.2, let us multiply equation $(\star)$ by x again, obtaining,

$$xA(x)y'' + xB(x)y' + xC(x)y = 0 \quad ,$$

which is

$$x^2\alpha(x)y'' + x\beta(x)y' + \gamma(x)y = 0 \qquad (\star\star)$$

with

$$xA(x) = x^2\underbrace{\left(1+x^2\right)\left(4x^2-1\right)}_{\alpha(x)} \quad ,$$

$$xB(x) = x\underbrace{\left(1+x^2\right)(4x-2)}_{\beta(x)}$$

and

$$x C(x) \;=\; \underbrace{x^2\left(1 - x^2\right)}_{\gamma(x)} \quad .$$

Noting that

$$\alpha(0) \;=\; \cdots \;=\; -1 \;\neq\; 0 \quad ,$$

we see that equation (⋆⋆) is our differential equation in quasi-Euler form about $x_0 = 0$. So, $x_0 = 0$ is a regular singular point.

Similar computations/arguments, will show that the other singular points are also regular. Alternatively, you can apply theorem 32.2 to verify that each singular point is regular.

Since the closest singular points to 0 other than 0, itself, are $\pm\frac{1}{2}$, the Frobenius radius of convergence about 0 is

$$F \;=\; \left|0 - \left[\pm\frac{1}{2}\right]\right| \;=\; \frac{1}{2} \quad .$$

32.4 a. The equation is already in desired form.

Since $x_0 = 0$, we set

$$y(x) \;=\; (x - 0)^r \sum_{k=0}^{\infty} a_k (x - 0)^k \;=\; \sum_{k=0}^{\infty} a_k x^{k+r} \quad .$$

Differentiating this and plugging the results into the differential equation, we get

$$y' \;=\; \frac{d}{dx}\sum_{k=0}^{\infty} a_k x^{k+r} \;=\; \sum_{k=0}^{\infty} \frac{d}{dx}\left[a_k x^{k+r}\right] \;=\; \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1} \quad ,$$

$$\begin{aligned}
y'' \;&=\; \frac{d}{dx}\sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1} \\
&=\; \sum_{k=0}^{\infty} \frac{d}{dx}\left[a_k (k+r) x^{k+r-1}\right] \;=\; \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-2}
\end{aligned}$$

and

$$\begin{aligned}
0 \;&=\; x^2 y'' \;-\; 2xy' \;+\; \left(x^2 + 2\right) y \\
&=\; x^2 \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-2} \;-\; 2x \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1} \\
&\qquad +\; \left(x^2 + 2\right) \sum_{k=0}^{\infty} a_k x^{k+r} \\
&=\; x^2 \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-2} \;-\; 2x \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1} \\
&\qquad +\; x^2 \sum_{k=0}^{\infty} a_k x^{k+r} \;+\; 2 \sum_{k=0}^{\infty} a_k x^{k+r}
\end{aligned}$$

$$= \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)x^{k+r} + \sum_{k=0}^{\infty}(-2)a_k(k+r)x^{k+r}$$
$$+ \sum_{k=0}^{\infty} a_k x^{k+r+2} + \sum_{k=0}^{\infty} 2a_k x^{k+r} \quad .$$

Dividing out the x^r and continuing:

$$0 = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)x^k + \sum_{k=0}^{\infty}(-2)a_k(k+r)x^k + \sum_{k=0}^{\infty} a_k x^{k+2} + \sum_{k=0}^{\infty} 2a_k x^k$$
$$= \underbrace{\sum_{k=0}^{\infty} a_k(k+r)(k+r-1)x^k}_{n=k} + \underbrace{\sum_{k=0}^{\infty}(-2)a_k(k+r)x^k}_{n=k} + \underbrace{\sum_{k=0}^{\infty} a_k x^{k+2}}_{n=k+2} + \underbrace{\sum_{k=0}^{\infty} 2a_k x^k}_{n=k}$$
$$= \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^n + \sum_{n=0}^{\infty}(-2)a_n(n+r)x^n + \sum_{n=2}^{\infty} a_{n-2}x^n + \sum_{n=0}^{\infty} 2a_n x^n$$
$$= \left[a_0(0+r)(0+r-1)x^0 + a_1(1+r)(1+r-1)x^1 + \sum_{n=2}^{\infty} a_n(n+r)(n+r-1)x^n\right]$$
$$+ \left[-2a_0(0+r)x^0 - 2a_1(1+r)x^1 + \sum_{n=2}^{\infty}(-2)a_n(n+r)x^n\right] + \sum_{n=2}^{\infty} a_{n-2}x^n$$
$$+ \left[2a_0x^0 + 2a_1x^1 + \sum_{n=2}^{\infty} 2a_n x^n\right]$$
$$= a_0\left[r(r-1) - 2r + 2\right]x^0 + a_1\left[(1+r)r - 2(1+r) + 2\right]x^1$$
$$+ \sum_{n=2}^{\infty}\left[a_n\left[(n+r)(n+r-1) - 2(n+r) + 2\right] + a_{n-2}\right]x^n \quad .$$

Simplifying slightly, we now have

$$0 = a_0\left[r^2 - 3r + 2\right]x^0 + a_1\left[r^2 - r\right]x^1$$
$$+ \sum_{n=2}^{\infty}\left[a_n\left[(n+r)^2 - 3(n+r) + 2\right] + a_{n-2}\right]x^n \quad . \tag{$\star$}$$

Each term on the right side of equation ($\star$) must be zero. In particular, the first term must be zero no matter what the arbitrary constant a_0 is. This yields the indicial equation

$$r^2 - 3r + 2 = 0 \quad ,$$

which factors to

$$(r-2)(r-1) = 0 \quad .$$

Hence, the equation's exponents are

$$r_1 = 2 \qquad \text{and} \qquad r_2 = 1 \quad .$$

Plugging the larger exponent, $r = r_1 = 2$, into equation ($\star$) (and simplifying the terms in the series a little more):

$$\begin{aligned}
0 \;&=\; a_0\Big[2^2 - 3\cdot 2 + 2\Big]x^0 \;+\; a_1\Big[2^2 - 2\Big]x^1 \\
&\qquad + \sum_{n=2}^{\infty}\Big[a_n\Big[\underbrace{(n+2)^2 - 3(n+2) + 2}_{\left((n+2)-2\right)\left((n+2)-1\right)}\Big] + a_{n-2}\Big]x^n \\
&=\; a_0[0]x^0 \;+\; 2a_1x^1 \;+\; \sum_{n=2}^{\infty}\big[a_n n(n+1) + a_{n-2}\big]x^n \quad .
\end{aligned}$$

Since each term must be zero, we have

$$2a_1 \;=\; 0 \quad\rightarrowtail\quad a_1 \;=\; 0$$

and, for $n \geq 2$,

$$a_n n(n+1) \;+\; a_{n-2} \;=\; 0 \quad\rightarrowtail\quad a_n \;=\; \frac{-1}{n(n+1)}a_{n-2} \quad .$$

So the recursion formula when $r = 2$ is

$$a_k \;=\; \frac{-1}{k(k+1)}a_{k-2} \qquad \text{for} \quad k \geq 2 \quad .$$

Applying the above:

$$\begin{aligned}
a_2 \;&=\; \frac{-1}{2(2+1)}a_{2-2} \;=\; \frac{-1}{3\cdot 2}a_0 \quad , \\
a_3 \;&=\; \frac{-1}{3(3+1)}a_{3-2} \;=\; \frac{-1}{4\cdot 3}a_1 \;=\; \frac{-1}{4\cdot 3}\cdot 0 \;=\; 0 \quad , \\
a_4 \;&=\; \frac{-1}{4(4+1)}a_{4-2} \;=\; \frac{-1}{5\cdot 4}a_2 \;=\; \frac{-1}{5\cdot 4}\cdot\frac{-1}{3\cdot 2}a_0 \;=\; \frac{(-1)^2}{5!}a_0 \quad , \\
a_5 \;&=\; \frac{-1}{5(5+1)}a_{5-2} \;=\; \frac{-1}{6\cdot 5}a_3 \;=\; \frac{-1}{6\cdot 5}\cdot 0 \;=\; 0 \quad , \\
a_6 \;&=\; \frac{-1}{6(6+1)}a_{6-2} \;=\; \frac{-1}{7\cdot 6}a_4 \;=\; \frac{-1}{7\cdot 6}\cdot\frac{(-1)^2}{5!}a_0 \;=\; \frac{(-1)^3}{7!}a_0 \quad , \\
&\;\;\vdots
\end{aligned}$$

In general, $a_k = 0$ if k is odd, and

$$a_{2m} \;=\; \frac{(-1)^m}{(2m+1)!}a_0 \qquad \text{for} \quad m \;=\; 1,\, 2,\, 3,\, \ldots \quad .$$

And since

$$a_{2\cdot 0} \;=\; \frac{(-1)^0}{(2\cdot 0+1)!}a_0 \;=\; \frac{1}{1!}a_0 \;=\; a_0 \quad ,$$

the above formula for a_{2m} also holds for $m = 0$. So, going back to the original series formula for y, we have

$$\begin{aligned}
y(x) \;&=\; x^{r_1}\sum_{k=0}^{\infty}a_kx^k \\
&=\; x^2\Big[a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6\cdots\Big]
\end{aligned}$$

$$= x^2 \left[a_0 + 0x - \frac{1}{3!}a_0 x^2 + 0x^3 + \frac{1}{5!}a_0 x^4 + 0x^5 - \frac{1}{7!}a_0 x^6 + \cdots \right]$$

$$= a_0 x^2 \left[1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \frac{1}{7!}x^6 + \cdots \right]$$

That is, $y(x) = ay_1(x)$ where a is an arbitrary constant, and

$$y_1(x) = x^2 \left[1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \frac{1}{7!}x^6 + \cdots \right] = x^2 \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m} \quad .$$

Note that, in fact,

$$y_1(x) = x^2 \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m} = x \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1} = x \sin(x) \quad .$$

Next, we plug the smaller exponent, $r = r_2 = 1$ into equation ($\star$) (and further simplify the terms):

$$0 = a_0 \left[r^2 - 3r + 2 \right] x^0 + a_1 \left[r^2 - r \right] x^1$$

$$+ \sum_{n=2}^{\infty} \left[a_n \left[(n+r)^2 - 3(n+r) + 2 \right] + a_{n-2} \right] x^n$$

$$= a_0 \left[1^2 - 3 \cdot 1 + 2 \right] x^0 + a_1 \left[1^2 - 1 \right] x^1$$

$$+ \sum_{n=2}^{\infty} \left[a_n \Big[\underbrace{(n+1)^2 - 3(n+1) + 2}_{\big((n+1)-2\big)\big((n+1)-1\big)} \Big] + a_{n-2} \right] x^n$$

$$= a_0[0]x^0 + a_1[0]x^1 + \sum_{n=2}^{\infty} \left[a_n(n-1)n + a_{n-2} \right] x^n \quad .$$

Thus,

$$0a_0 = 0 \quad , \quad 0a_1 = 0$$

and, for $n \geq 2$,

$$a_n(n-1)n + a_{n-2} = 0 \quad \rightarrowtail \quad a_n = \frac{-1}{n(n-1)} a_{n-2} \quad .$$

The first two equations tells us that both a_0 and a_1 can be arbitrary, but, following the advice given in step 8 on page 685 we will keep a_0 as arbitrary and set

$$a_1 = 0 \quad .$$

From above, we also have the recursion formula

$$a_k = \frac{-1}{k(k-1)} a_{k-2} \quad \text{for} \quad k = 2, 3, 4, 5, \ldots \quad .$$

Using the above,

$$a_2 = \frac{-1}{2(2-1)} a_{2-2} = \frac{-1}{2 \cdot 1} a_0 \quad ,$$

$$a_3 = \frac{-1}{3(3-1)} a_{3-2} = \frac{-1}{3 \cdot 2} a_1 = \frac{-1}{3 \cdot 2} \cdot 0 = 0 \quad ,$$

$$\begin{aligned}
a_4 &= \frac{-1}{4(4-1)}a_{4-2} = \frac{-1}{4\cdot 3}a_2 = \frac{-1}{4\cdot 3}\cdot\frac{-1}{2\cdot 1}a_0 = \frac{(-1)^2}{4!}a_0 \quad ,\\
a_5 &= \frac{-1}{5(4-1)}a_{5-2} = \frac{-1}{5\cdot 4}a_3 = \frac{-1}{5\cdot 4}\cdot 0 = 0 \quad ,\\
a_6 &= \frac{-1}{6(6-1)}a_{6-2} = \frac{-1}{6\cdot 5}a_4 = \frac{-1}{6\cdot 5}\cdot\frac{(-1)^2}{4!}a_0 = \frac{(-1)^3}{6!}a_0 \quad ,\\
a_7 &= \frac{-1}{7(6-1)}a_{7-2} = \frac{-1}{7\cdot 6}a_5 = \frac{-1}{7\cdot 6}\cdot 0 = 0 \quad ,\\
a_8 &= \frac{-1}{8(8-1)}a_{8-2} = \frac{-1}{8\cdot 7}a_6 = \frac{-1}{8\cdot 7}\cdot\frac{(-1)^4}{6!}a_0 = \frac{(-1)^4}{8!}a_0 \quad ,\\
&\vdots
\end{aligned}$$

In general, $a_k = 0$ if k is odd, and

$$a_{2m} = \frac{(-1)^m}{(2m)!}a_0 \qquad \text{for} \quad m = 0,\ 1,\ 2,\ 3,\ \ldots \quad .$$

Thus,

$$\begin{aligned}
y(x) &= x^{r_1}\sum_{k=0}^{\infty} a_k x^k\\
&= x^1\left[a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6\cdots\right]\\
&= x\left[a_0 + 0x - \frac{1}{2!}a_0x^2 + 0x^3 + \frac{1}{4!}a_0x^4 + 0x^5 - \frac{1}{6!}a_0x^6 + \cdots\right]\\
&= a_0x\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right]
\end{aligned}$$

That is, $y(x) = ay_2(x)$ where a is an arbitrary constant, and

$$y_1(x) = x\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right] = x\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}x^{2m} \quad .$$

Note that, in fact,

$$y_2(x) = x\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}x^{2m} = x\cos(x) \quad .$$

Finally, the general solution to this differential equation is given by

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

where y_1 and y_2 are as described above.

32.4 c. The equation is already in desired form.

Since $x_0 = 0$, we set

$$y(x) = (x-0)^r\sum_{k=0}^{\infty}a_k(x-0)^k = \sum_{k=0}^{\infty}a_kx^{k+r} \quad .$$

Differentiating this and plugging the results into the differential equation, we get

$$y' = \sum_{k=0}^{\infty}a_k(k+r)x^{k+r-1} \qquad , \qquad y'' = \sum_{k=0}^{\infty}a_k(k+r)(k+r-1)x^{k+r-2}$$

and

$$
\begin{aligned}
0 &= x^2y'' + xy' + (4x-4)y \\
&= x^2y'' + xy' + 4xy - 4y \\
&= x^2\sum_{k=0}^{\infty} a_k(k+r)(k+r-1)x^{k+r-2} + x\sum_{k=0}^{\infty} a_k(k+r)x^{k+r-1} \\
&\quad + 4x\sum_{k=0}^{\infty} a_k x^{k+r} - 4\sum_{k=0}^{\infty} a_k x^{k+r} \\
&= \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)x^{k+r} + \sum_{k=0}^{\infty} a_k(k+r)x^{k+r} \\
&\quad + \sum_{k=0}^{\infty} 4a_k x^{k+r+1} + \sum_{k=0}^{\infty}(-4)a_k x^{k+r} \quad .
\end{aligned}
$$

Dividing out the x^r and continuing:

$$
\begin{aligned}
0 &= \underbrace{\sum_{k=0}^{\infty} a_k(k+r)(k+r-1)x^k}_{n=k} + \underbrace{\sum_{k=0}^{\infty} a_k(k+r)x^k}_{n=k} \\
&\quad + \underbrace{\sum_{k=0}^{\infty} 4a_k x^{k+1}}_{n=k+1} + \underbrace{\sum_{k=0}^{\infty}(-4)a_k x^k}_{n=k} \\
&= \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^n + \sum_{n=0}^{\infty} a_n(n+r)x^n \\
&\quad + \sum_{n=1}^{\infty} 4a_{n-1}x^n + \sum_{n=0}^{\infty}(-4)a_n x^n \\
&= \left[a_0(0+r)(0+r-1)x^0 + \sum_{n=1}^{\infty} a_n(n+r)(n+r-1)x^n\right] \\
&\quad + \left[a_0(0+r)x^0 + \sum_{n=2}^{\infty} a_n(n+r)x^n\right] + \sum_{n=1}^{\infty} 4a_{n-1}x^n \\
&\quad + \left[-4a_0x^0 + \sum_{n=1}^{\infty}(-4)a_n x^n\right] \\
&= a_0\,[r(r-1)+r-4]\,x^0 \\
&\quad + \sum_{n=1}^{\infty}\Big[a_n\,[(n+r)(n+r-1)+(n+r)-4] + 4a_{n-1}\Big]x^n \quad .
\end{aligned}
$$

Simplifying slightly, we now have

$$0 \;=\; a_0\left[r^2-4\right]x^0 \;+\; \sum_{n=1}^{\infty}\left[a_n\left[(n+r)^2-4\right]+4a_{n-1}\right]x^n \quad . \qquad (\star)$$

Each term on the right side of equation ($\star$) must be zero. In particular, the first term must be zero no matter what the arbitrary constant a_0 is. This yields the indicial equation

$$r^2 \;-\; 4 \;=\; 0 \quad ,$$

which factors to

$$(r-2)(r+2) \;=\; 0 \quad .$$

Hence, the equation's exponents are

$$r_1 \;=\; 2 \qquad \text{and} \qquad r_2 \;=\; -2 \quad .$$

Plugging the larger exponent, $r = r_1 = 2$, into equation ($\star$) (and simplifying the terms in the series a little more):

$$0 \;=\; a_0\left[2^2-4\right]x^0 \;+\; \sum_{n=1}^{\infty}\left[a_n\left[(n+2)^2-4\right]+4a_{n-1}\right]x^n$$

$$=\; a_0[0]x^0 \;+\; \sum_{n=1}^{\infty}\left[a_n(n+4)n+4a_{n-1}\right]x^n \quad .$$

Since each term must be zero, we have, for $n \geq 1$,

$$a_n(n+4)n \;+\; 4a_{n-1} \;=\; 0 \quad \rightarrowtail \quad a_n \;=\; \frac{4}{(n+4)n}a_{n-1} \quad .$$

So the recursion formula when $r = 2$ is

$$a_k \;=\; \frac{-4}{(k+4)k}a_{k-1} \qquad \text{for} \quad k \geq 1 \quad .$$

Applying the above:

$$a_1 \;=\; \frac{-4}{(1+4)1}a_{1-1} \;=\; \frac{-4}{5\cdot 1}a_0 \quad ,$$

$$a_2 \;=\; \frac{-4}{(2+4)2}a_{2-1} \;=\; \frac{-4}{6\cdot 2}a_1 \;=\; \frac{-4}{6\cdot 2}\cdot\frac{-4}{5\cdot 1}a_0 \;=\; \frac{(-4)^2}{(6\cdot 5)(2\cdot 1)}a_0 \quad ,$$

$$a_3 \;=\; \frac{-4}{(3+4)3}a_{3-1} \;=\; \frac{-4}{7\cdot 3}a_2 \;=\; \frac{-4}{7\cdot 3}\cdot\frac{(-4)^2}{(6\cdot 5)(2\cdot 1)}a_0 \;=\; \frac{(-4)^3}{(7\cdot 6\cdot 5)(3\cdot 2\cdot 1)}a_0$$

$$=\; \frac{(-4)^3 4\cdot 3\cdot 2\cdot 1}{(7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1)(3\cdot 2\cdot 1)}a_0 \;=\; \frac{(-4)^3 4!}{7!\,3!}a_0 \quad ,$$

$$a_4 \;=\; \frac{-4}{(4+4)4}a_{4-1} \;=\; \frac{-4}{(4+4)4}a_3 \;=\; \frac{-4}{8\cdot 4}\cdot\frac{(-4)^3 4!}{7!\,3!)}a_0 \;=\; \frac{(-4)^4 4!}{8!\,4!}a_0 \quad ,$$

$$\vdots$$

In general, you can verify that

$$a_k \;=\; \frac{(-4)^k 4!}{(k+4)!\,k!}a_0 \qquad \text{for} \quad k \;=\; 0,\ 1,\ 2,\ 3,\ \ldots \quad .$$

So,

$$\begin{aligned}
y(x) &= x^{r_1}\sum_{k=0}^{\infty} a_k x^k \\
&= x^2\left[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \cdots\right] \\
&= x^2\left[a_0 + \frac{-4}{5\cdot 1}a_0 x + \frac{(-4)^2}{(6\cdot 5)(2\cdot 1)}a_0 x^2 + \frac{(-4)^3}{(7\cdot 6\cdot 5)(3\cdot 2\cdot 1)}a_0 x^3 + \cdots\right] \\
&= a_0 x^2\left[1 - \frac{4}{5}x + \frac{4^2}{(6\cdot 5)(2\cdot 1)}x^2 - \frac{(-4)^3}{(7\cdot 6\cdot 5)(3\cdot 2\cdot 1)}x^3 + \cdots\right] \quad .
\end{aligned}$$

Equivalently, $y(x) = ay_1(x)$ where a is an arbitrary constant, and

$$\begin{aligned}
y_1(x) &= x^2\left[1 - \frac{4}{5}x + \frac{4^2}{(6\cdot 5)(2\cdot 1)}x^2 - \frac{(-4)^3}{(7\cdot 6\cdot 5)(3\cdot 2\cdot 1)}x^3 + \cdots\right] \\
&= \cdots = x^2\sum_{k=0}^{\infty}\frac{(-4)^k 4!}{(k+4)!(k!)}x^k \quad .
\end{aligned}$$

Next, we plug the smaller exponent, $r = r_2 = -2$ into equation ($\star$):

$$\begin{aligned}
0 &= a_0\left[r^2 - 4\right]x^0 + \sum_{n=1}^{\infty}\left[a_n\left[(n+r)^2 - 4\right] + 4a_{n-1}\right]x^n \\
&= a_0\left[(-2)^2 - 4\right]x^0 + \sum_{n=1}^{\infty}\left[a_n\left[(n-2)^2 - 4\right] + 4a_{n-1}\right]x^n \\
&= a_0 0 x^0 + \sum_{n=1}^{\infty}\left[a_n n(n-4) + 4a_{n-2}\right]x^n \quad .
\end{aligned}$$

Thus, for $n \geq 1$,

$$a_n n(n-4) + 4a_{n-2} = 0 \quad \rightarrowtail \quad a_n = \frac{-4}{n(n-4)}a_{n-1} \quad .$$

This gives our recursion formula

$$a_k = \frac{-4}{k(k-4)}a_{k-1} \qquad \text{for} \quad k = 1, 2, 3, 4, 5, \ldots \quad .$$

(The fact that we get zero in our denominator when $k = 4$ should be a warning that we might not get another series solution.)

Using the above,

$$\begin{aligned}
a_1 &= \frac{-4}{1(1-4)}a_{k-1} = \frac{4}{3}a_0 \quad , \\
a_2 &= \frac{-4}{2(2-4)}a_{k-1} = 1\cdot a_1 = \frac{4}{3}a_0 \quad , \\
a_3 &= \frac{-4}{3(3-4)}a_{k-1} = \frac{4}{3}a_2 = \frac{4}{3}\cdot\frac{4}{3}a_0 = \frac{4^2}{3^2}a_0 \quad , \\
a_4 &= \frac{-4}{4(4-4)}a_{4-1} = \frac{-4}{0}a_3 = \frac{-4}{0}\cdot\frac{4^2}{3^2}a_0 = \text{“}\infty\text{”} \quad .
\end{aligned}$$

Since a_4 "blows up" when $a_0 \neq 0$, there cannot be a series solution of the form

$$y(x) = x^{r_2} \sum_{k=0}^{\infty} a_k x^k \qquad \text{with} \quad a_0 \neq 0 \quad .$$

32.4 e. First, multiply the differential equation by $1 - x$ to get it into the desired form,

$$\underbrace{\left(x^2 - x^3\right)}_{x^2(1-x)} y'' + \underbrace{\left(x^2 - x\right)}_{x(1-x)} y' + y = 0 \quad .$$

Then, set

$$y(x) = x^r \sum_{k=0}^{\infty} a_k (x-0)^k = \sum_{k=0}^{\infty} a_k x^{k+r} \quad .$$

Differentiating this and plugging the results into the differential equation, we get

$$y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1} \qquad , \qquad y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-2}$$

and

$$\begin{aligned}
0 &= \left(x^2 - x^3\right) y'' + \left(x^2 - x\right) y' + y \\
&= x^2 y'' - x^3 y'' + x^2 y' - x y' + y \\
&= x^2 \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-2} - x^3 \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-2} \\
&\quad + x^2 \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1} - x \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1} + \sum_{k=0}^{\infty} a_k x^{k+r} \\
&= \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r} + \sum_{k=0}^{\infty} (-1) a_k (k+r)(k+r-1) x^{k+r+1} \\
&\quad + \sum_{k=0}^{\infty} a_k (k+r) x^{k+r+1} + \sum_{k=0}^{\infty} (-1) a_k (k+r) x^{k+r} + \sum_{k=0}^{\infty} a_k x^{k+r} \quad .
\end{aligned}$$

Dividing out the x^r and continuing:

$$\begin{aligned}
0 &= \underbrace{\sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^k}_{n=k} + \underbrace{\sum_{k=0}^{\infty} (-1) a_k (k+r)(k+r-1) x^{k+1}}_{n=k+1} \\
&\quad + \underbrace{\sum_{k=0}^{\infty} a_k (k+r) x^{k+1}}_{n=k+1} + \underbrace{\sum_{k=0}^{\infty} (-1) a_k (k+r) x^k}_{n=k} + \underbrace{\sum_{k=0}^{\infty} a_k x^k}_{n=k}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^n \;+\; \sum_{n=1}^{\infty}(-1)a_{n-1}(n-1+r)(n-2+r)x^n \\
&\qquad + \sum_{n=1}^{\infty} a_{n-1}(n-1+r)x^n \;+\; \sum_{n=0}^{\infty}(-1)a_n(n+r)x^n \;+\; \sum_{n=0}^{\infty} a_n x^{n+r} \\
&= \left[a_0(0+r)(0+r-1)x^0 + \sum_{n=1}^{\infty} a_n(n+r)(n+r-1)x^n\right] \\
&\qquad + \sum_{n=1}^{\infty}(-1)a_{n-1}(n-1+r)(n-2+r)x^n \;+\; \sum_{n=1}^{\infty} a_{n-1}(n-1+r)x^n \\
&\qquad + \left[-a_0(0+r)x^0 + \sum_{n=1}^{\infty}(-1)a_n(n+r)x^n\right] \;+\; \left[a_0x^0 + \sum_{n=1}^{\infty} a_n x^{n+r}\right] \\
&= a_0\,[r(r-1) - r + 1]\,x^0 \\
&\qquad + \sum_{n=1}^{\infty}\big[a_n\,[(n+r)(n+r-1) - (n+r) + 1] \\
&\qquad\qquad + a_{n-1}\,[-(n-1+r)(n-2+r) + (n-1+r)]\big]x^n \quad ,
\end{aligned}
$$

which, after a bit of algebra, simplifies to

$$
\begin{aligned}
0 \;&=\; a_0\left[r^2 - 2r + 1\right]x^0 \\
&\qquad + \sum_{n=1}^{\infty}\big[a_n(n+r-1)^2 - a_{n-1}(n+r-1)(n+r-3)\big]x^n \quad .
\end{aligned}
\tag{$\star$}
$$

From the first term, we get the indicial equation

$$
0 \;=\; \underbrace{r^2 - 2r + 1}_{(r-1)^2} \quad .
$$

So $r_1 = r_2 = r = 1$.

Letting $r = 1$, equation ($\star$) becomes

$$
\begin{aligned}
0 \;&=\; a_0\left[1^2 - 2\cdot 1 + 1\right]x^0 \;+\; \sum_{n=1}^{\infty}\big[a_n(n+1-1)^2 - a_{n-1}(n+1-1)(n+1-3)\big]x^n \\
&=\; a_0[0]x^0 \;+\; \sum_{n=1}^{\infty}\big[a_n n^2 - a_{n-1}n(n-2)\big]x^n \quad .
\end{aligned}
$$

So, for $n \geq 1$,

$$
a_n n^2 \;-\; a_{n-1}n(n-2) \;=\; 0 \quad\rightarrowtail\quad a_n \;=\; \frac{n-2}{n}a_{n-1} \quad .
$$

This is our recursion formula. Using it:

$$
\begin{aligned}
a_1 \;&=\; \frac{1-2}{1}a_{1-1} \;=\; -a_0 \quad , \\
a_2 \;&=\; \frac{2-2}{2}a_{2-1} \;=\; 0a_1 \;=\; 0 \quad ,
\end{aligned}
$$

$$a_3 = \frac{3-2}{3}a_{3-1} = \frac{1}{3}a_2 = \frac{1}{3}0 = 0 \quad ,$$
$$a_4 = \frac{4-2}{4}a_{4-1} = \frac{2}{4}a_3 = \frac{2}{4}0 = 0 \quad ,$$
$$\vdots$$

Clearly, $a_k = 0$ for $k \geq 2$. So

$$\begin{aligned} y(x) &= x^{r_1}\sum_{k=0}^{\infty} a_k x^k \\ &= x\left[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \cdots\right] \\ &= x\left[a_0 - a_0 x + 0x^2 + \cdots\right] = a_0 y_1(x) \end{aligned}$$

where

$$y_1(x) = x[1-x] = x - x^2 \quad .$$

Since there is only one value of r, no other series solution can be found by the basic Frobenius method.

32.4 g. First, multiply the differential equation by x^2 to get it into the form

$$x^2y'' + xy' + \left[x^2 - 1\right]y = 0 \quad .$$

Then, set

$$y(x) = x^r\sum_{k=0}^{\infty} a_k(x-0)^k = \sum_{k=0}^{\infty} a_k x^{k+r} \quad .$$

Differentiating this and plugging the results into the differential equation, we get

$$y' = \sum_{k=0}^{\infty} a_k(k+r)x^{k+r-1} \quad , \quad y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)x^{k+r-2}$$

and

$$\begin{aligned} 0 &= x^2y'' + xy' + \left[x^2-1\right]y \\ &= x^2\sum_{k=0}^{\infty} a_k(k+r)(k+r-1)x^{k+r-2} + x\sum_{k=0}^{\infty} a_k(k+r)x^{k+r-1} \\ &\quad + \left[x^2-1\right]\sum_{k=0}^{\infty} a_k x^{k+r} \\ &= \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)x^{k+r} + \sum_{k=0}^{\infty} a_k(k+r)x^{k+r} \\ &\quad + \sum_{k=0}^{\infty} a_k x^{k+r+2} + \sum_{k=0}^{\infty}(-1)a_k x^{k+r} \quad . \end{aligned}$$

Dividing out the x^r and continuing:

$$\begin{aligned}
0 \;&=\; \underbrace{\sum_{k=0}^{\infty} a_k(k+r)(k+r-1)x^k}_{n=k} \;+\; \underbrace{\sum_{k=0}^{\infty} a_k(k+r)x^k}_{n=k} \;+\; \underbrace{\sum_{k=0}^{\infty} a_k x^{k+2}}_{n=k+2} \;+\; \underbrace{\sum_{k=0}^{\infty}(-1)a_k x^k}_{n=k} \\
&=\; \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^n \;+\; \sum_{n=0}^{\infty} a_n(n+r)x^n \;+\; \sum_{n=0}^{\infty}(-1)a_n x^n \\
&=\; \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^n \;+\; \sum_{n=0}^{\infty} a_n(n+r)x^n \;+\; \sum_{n=2}^{\infty} a_{n-2}x^n \;+\; \sum_{n=0}^{\infty}(-1)a_n x^n \\
&=\; \left[a_0(0+r)(0+r-1)x^0 + a_1(1+r)(1+r-1)x^1 + \sum_{n=2}^{\infty} a_n(n+r)(n+r-1)x^n\right] \\
&\qquad +\; \left[a_0(0+r)x^0 + a_1(1+r)x^1 + \sum_{n=2}^{\infty} a_n(n+r)x^n\right] \;+\; \sum_{n=2}^{\infty} a_{n-2}x^n \\
&\qquad +\; \left[-\,a_0x^0 - a_1x^1 + \sum_{n=2}^{\infty}(-1)a_n x^n\right] \\
&=\; a_0\,[r(r-1)+r-1]\,x^0 \;+\; a_1\,[(1+r)r+(1+r)-1]\,x^1 \\
&\qquad +\; \sum_{n=2}^{\infty}\big[a_n\,[(n+r)(n+r-1)+(n+r)-1] \;+\; a_{n-2}\big]\,x^n \quad ,
\end{aligned}$$

which simplifies to

$$0 \;=\; a_0\left[r^2-1\right]x^0 \;+\; a_1\left[r^2+2r\right]x^1 \;+\; \sum_{n=2}^{\infty}\left[a_n\left[(n+r)^2-1\right] \;+\; a_{n-2}\right]x^n \quad . \quad (\star)$$

From the first term, we get the indicial equation

$$0 \;=\; r^2 - 1 \quad ,$$

which has solutions $r = 1$ and $r = -1$. So the equation's exponents are

$$r_1 \;=\; 1 \qquad \text{and} \qquad r_2 \;=\; -1 \quad .$$

Plugging the larger exponent, $r = r_1 = 1$, into equation $(\star)$:

$$\begin{aligned}
0 \;&=\; a_0\,[0]\,x^0 \;+\; a_1\,[1+2]\,x^1 \;+\; \sum_{n=2}^{\infty}\left[a_n\left[(n+1)^2-1\right] \;+\; a_{n-2}\right]x^n \\
&=\; 3a_1x^1 \;+\; \sum_{n=2}^{\infty}\big[n(n+2)a_n + a_{n-2}\big]\,x^n \quad .
\end{aligned}$$

Since each term must be zero, we have $a_1 = 0$ and, for $n \geq 2$,

$$n(n+2)a_n \;+\; a_{n-2} \;=\; 0 \quad \rightarrowtail \quad a_n \;=\; \frac{-1}{n(n+2)}a_{n-2} \quad .$$

So the recursion formula when $r = 1$ is

$$a_k = \frac{-1}{k(k+2)} a_{k-2} \qquad \text{for} \quad k \geq 2 \quad .$$

Applying the above:

$$\begin{aligned}
a_1 &= 0 \quad , \\
a_2 &= \frac{-1}{2(2+2)} a_{2-2} = \frac{-1}{2 \cdot 4} a_0 \quad , \\
a_3 &= \frac{-1}{3(3+2)} a_{3-2} = \frac{-1}{3 \cdot 5} a_1 = \frac{-1}{3 \cdot 5} 0 = 0 \quad , \\
a_4 &= \frac{-1}{4(4+2)} a_{4-2} = \frac{-1}{4 \cdot 6} a_2 = \frac{-1}{4 \cdot 6} \cdot \frac{-1}{2 \cdot 4} a_0 = \frac{(-1)^2}{(4 \cdot 2)(6 \cdot 4)} a_0 \\
&= \frac{(-1)^2}{([2 \cdot 2] \cdot [2 \cdot 1])([2 \cdot 3] \cdot [2 \cdot 2])} a_0 = \frac{(-1)^2}{2^4 (2 \cdot 1)(3 \cdot 2)} a_0 = \frac{(-1)^2}{2^4 2!\, 3!} a_0 \quad , \\
a_5 &= \frac{-1}{5(5+2)} a_{5-2} = \frac{-1}{5 \cdot 7} a_3 = \frac{-1}{5 \cdot 7} 0 = 0 \quad , \\
a_6 &= \frac{-1}{6(6+2)} a_{6-2} = \frac{-1}{6 \cdot 8} a_4 = \frac{-1}{6 \cdot 8} \cdot \frac{(-1)^2}{(4 \cdot 2)(6 \cdot 4)} a_0 = \frac{(-1)^3}{(6 \cdot 4 \cdot 2)(8 \cdot 6 \cdot 4)} a_0 \\
&= \frac{(-1)^3}{\big([2 \cdot 3] \cdot [2 \cdot 2] \cdot [2 \cdot 1]\big)\big([2 \cdot 4] \cdot [2 \cdot 3] \cdot [2 \cdot 2]\big)} a_0 = \frac{(-1)^3}{2^6 3!\, 4!} a_0 \quad , \\
&\vdots
\end{aligned}$$

For odd values of k, we clearly have $a_k = 0$. For even values of k, you can verify that

$$a_k = a_{2m} = \frac{(-1)^m}{2^{2m} m!\, (m+1)!} a_0 \quad .$$

Thus,

$$\begin{aligned}
y(x) &= x^{r_1} \sum_{k=0}^{\infty} a_k x^k \\
&= x^1 \left[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \cdots \right] \\
&= x \Bigg[a_0 + 0x + \frac{-1}{2 \cdot 4} a_0 x^2 + 0x^3 + \frac{(-1)^2}{(4 \cdot 2)(6 \cdot 4)} a_0 x^4 + 0x^3 \\
&\qquad + \frac{(-1)^3}{(6 \cdot 4 \cdot 2)(8 \cdot 6 \cdot 4)} a_0 x^6 + \cdots \Bigg] \\
&= a_0 y_1(x)
\end{aligned}$$

where

$$\begin{aligned}
y_1(x) &= x \left[1 + \frac{-1}{2 \cdot 4} x^2 + \frac{(-1)^2}{(4 \cdot 2)(6 \cdot 4)} x^4 + \frac{(-1)^3}{(6 \cdot 4 \cdot 2)(8 \cdot 6 \cdot 4)} x^6 + \cdots \right] \\
&= \cdots = x \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m} m!(m+1)!} x^{2m} \quad .
\end{aligned}$$

Next, we plug the smaller exponent, $r = r_2 = -1$ into equation ($\star$):

$$0 = a_0\left[(-1)^2 - 1\right]x^0 + a_1\left[(-1)^2 + 2(-1)\right]x^1$$

$$+ \sum_{n=2}^{\infty}\left[a_n\left[(n-1)^2 - 1\right] + a_{n-2}\right]x^n$$

$$= a_0\,[0]\,x^0 + a_1\,[-1]\,x^1 + \sum_{n=2}^{\infty}\left[a_n\,[n(n-2)] + a_{n-2}\right]x^n \quad .$$

Thus, $a_1 = 0$ and, for $n \geq 2$,

$$a_n\,[n(n-2)] + a_{n-2} = 0 \quad \rightarrowtail \quad a_n = \frac{-1}{n(n-2)}a_{n-2} \quad .$$

This give our recursion formula

$$a_k = \frac{-1}{k(k-2)}a_{k-2} \qquad \text{for} \quad k = 2, 3, 4, 5, \ldots \quad .$$

(The fact that we get zero in our denominator when $k = 4$ should be a warning that we might not get another series solution.)

Applying the above:

$$a_1 = 0 \quad ,$$

but

$$a_2 = \frac{-1}{2(2-2)}a_{2-2} = \frac{-1}{0}a_0 = \text{“}\infty\text{”} \quad .$$

Since a_2 “blows up” when $a_0 \neq 0$, there cannot be a series solution of the form

$$y(x) = x^{r_2}\sum_{k=0}^{\infty}a_k x^k \qquad \text{with} \quad a_0 \neq 0 \quad .$$

32.4 i. The equation is already in desired form.

Since $x_0 = 0$, we set

$$y(x) = (x-0)^r\sum_{k=0}^{\infty}a_k(x-0)^k = \sum_{k=0}^{\infty}a_k x^{k+r} \quad .$$

Differentiating this and plugging the results into the differential equation, we get

$$y' = \sum_{k=0}^{\infty}a_k(k+r)x^{k+r-1} \qquad , \qquad y'' = \sum_{k=0}^{\infty}a_k(k+r)(k+r-1)x^{k+r-2}$$

and

$$0 = x^2y'' - \left(5x + 2x^2\right)y' + (9+4x)\,y$$

$$= x^2\sum_{k=0}^{\infty}a_k(k+r)(k+r-1)x^{k+r-2} - \left(5x+2x^2\right)\sum_{k=0}^{\infty}a_k(k+r)x^{k+r-1}$$

$$+ (9+4x)\sum_{k=0}^{\infty}a_k x^{k+r}$$

$$= \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)x^{k+r} + \sum_{k=0}^{\infty}(-5)a_k(k+r)x^{k+r}$$
$$+ \sum_{k=0}^{\infty}(-2)a_k(k+r)x^{k+r+1} + \sum_{k=0}^{\infty} 9a_k x^{k+r} + \sum_{k=0}^{\infty} 4a_k x^{k+r+1} \quad .$$

Dividing out x^r and continuing:

$$0 = \underbrace{\sum_{k=0}^{\infty} a_k(k+r)(k+r-1)x^k}_{n=k} + \underbrace{\sum_{k=0}^{\infty}(-5)a_k(k+r)x^k}_{n=k}$$
$$+ \underbrace{\sum_{k=0}^{\infty}(-2)a_k(k+r)x^{k+1}}_{n=k+1} + \underbrace{\sum_{k=0}^{\infty} 9a_k x^k}_{n=k} + \underbrace{\sum_{k=0}^{\infty} 4a_k x^{k+1}}_{n=k+1}$$

$$= \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^n + \sum_{n=0}^{\infty}(-5)a_n(n+r)x^n$$
$$+ \sum_{n=1}^{\infty}(-2)a_{n-1}(n-1+r)x^n + \sum_{n=0}^{\infty} 9a_n x^n + \sum_{n=1}^{\infty} 4a_{n-1}x^n$$

$$= \left[a_0(0+r)(0+r-1)x^0 + \sum_{n=1}^{\infty} a_n(n+r)(n+r-1)x^n\right]$$
$$+ \left[-5a_0(0+r)x^0 + \sum_{n=1}^{\infty}(-5)a_n(n+r)x^n\right]$$
$$+ \sum_{n=1}^{\infty}(-2)a_{n-1}(n-1+r)x^n + \left[9a_0x^0 + \sum_{n=1}^{\infty} 9a_n x^n\right] + \sum_{n=1}^{\infty} 4a_{n-1}x^n$$

$$= a_0\,[r(r-1) - 5r + 9]\,x^0$$
$$+ \sum_{n=1}^{\infty}\big[a_n\,[(n+r)(n+r-1) - 5(n+r) + 9] + a_{n-1}\,[-2(n-1+r) + 4]\big]x^n \quad ,$$

which, after a little algebra, simplifies to

$$0 = a_0\left[r^2 - 6r + 9\right]x^0 + \sum_{n=1}^{\infty}\left[a_n(n+r-3)^2 - a_{n-1}2(n+r-3)\right]x^n \quad . \qquad (\star)$$

From the first term, we get the indicial equation

$$0 = \underbrace{r^2 - 6r + 9}_{(r-3)^2} \quad .$$

So the equation's exponents are given by $r_1 = r_2 = r = 3$.

Plugging $r = r_1 = 3$ into equation ($\star$), we get

$$0 = a_0\left[3^2 - 6\cdot 3 + 9\right]x^0 + \sum_{n=1}^{\infty}\left[a_n(n+3-3)^2 - a_{n-1}2(n+3-3)\right]x^n$$

$$= a_0\,[0]\,x^0 + \sum_{n=1}^{\infty}\left[a_n n^2 - a_{n-1}2n\right]x^n$$

Since each term must be zero, we must have, for $n \geq 1$,

$$a_n n^2 - a_{n-1}2n = 0 \quad \rightarrowtail \quad a_n = \frac{2}{n)}a_{n-1} \quad .$$

So the recursion formula is

$$a_k = \frac{2}{k}a_{k-1} \qquad \text{for} \quad k \geq 1 \quad .$$

Applying the above:

$$a_1 = \frac{2}{1}a_{1-1} = \frac{2}{1}a_0 \quad ,$$

$$a_2 = \frac{2}{2}a_{2-1} = \frac{2}{2}a_1 = \frac{2}{2}\cdot\frac{2}{1}a_0 = \frac{2^2}{2\cdot 1}a_0 \quad ,$$

$$a_3 = \frac{2}{3}a_{3-1} = \frac{2}{3}a_2 = \frac{2}{3}\cdot\frac{2^2}{2\cdot 1}a_0 = \frac{2^3}{3!}a_0 \quad ,$$

$$a_5 = \frac{2}{5}a_{5-1} = \frac{2}{5}a_4 = \frac{2}{5}\cdot\frac{2^4}{4!}a_0 = \frac{2^5}{5!}a_0 \quad ,$$

$$\vdots$$

Clearly,

$$a_k = \frac{2^k}{k!}a_0 \qquad \text{for} \quad k = 0,\ 1,\ 2,\ 3\ldots \quad .$$

Thus,

$$y(x) = x^{r_1}\sum_{k=0}^{\infty}a_k x^k$$

$$= x^3\left[a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots\right]$$

$$= x^3\left[a_0 + \frac{2}{1}a_0x + \frac{2^2}{2!}a_0 + \frac{2^3}{3!}a_0x^3 + \frac{2^4}{4!}a_0x^4 + \frac{2^5}{5!}a_0x^5 + \cdots\right]$$

$$= a_0y_1(x)$$

where

$$y_1(x) = x^3\left[1 + \frac{2}{1}x + \frac{2^2}{2!} + \frac{2^3}{3!}x^3 + \frac{2^4}{4!}x^4 + \frac{2^5}{5!}x^5 + \cdots\right]$$

$$= x^3\sum_{k=0}^{\infty}\frac{2^k}{k!}x^k = x^3\sum_{k=0}^{\infty}\frac{1}{k!}(2x)^k = x^3e^{2x} \quad .$$

Since there is only one value of r, no other series solution can be found by the basic Frobenius method.

32.4 k. The equation is already in desired form.

Since $x_0 = 0$, we set

$$y(x) = (x-0)^r \sum_{k=0}^{\infty} a_k (x-0)^k = \sum_{k=0}^{\infty} a_k x^{k+r} \quad .$$

Differentiating this and plugging the results into the differential equation, we get

$$y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1} \quad , \quad y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-2}$$

and

$$\begin{aligned} 0 &= x^2 y'' - \left(x + x^2\right) y' + 4xy \\ &= x^2 y'' - xy - x^2 y' + 4xy \\ &= \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r} + \sum_{k=0}^{\infty} (-1) a_k (k+r) x^{k+r} \\ &\quad + \sum_{k=0}^{\infty} (-1) a_k (k+r) x^{k+r+1} + \sum_{k=0}^{\infty} 4 a_k x^{k+r+1} \quad . \end{aligned}$$

Dividing out x^r and continuing:

$$\begin{aligned} 0 &= \underbrace{\sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^k}_{n=k} + \underbrace{\sum_{k=0}^{\infty} (-1) a_k (k+r) x^k}_{n=k} \\ &\quad + \underbrace{\sum_{k=0}^{\infty} (-1) a_k (k+r) x^{k+1}}_{n=k+1} + \underbrace{\sum_{k=0}^{\infty} 4 a_k x^{k+1}}_{n=k+1} \\ &= \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^n + \sum_{n=0}^{\infty} (-1) a_n (n+r) x^n \\ &\quad + \sum_{n=1}^{\infty} (-1) a_{n-1} (n-1+r) x^n + \sum_{n=1}^{\infty} 4 a_{n-1} x^n \\ &= \left[a_0 (0+r)(0+r-1) x^0 + \sum_{n=1}^{\infty} a_n (n+r)(n+r-1) x^n \right] \\ &\quad + \left[-a_0 (0+r) x^0 + \sum_{n=1}^{\infty} (-1) a_n (n+r) x^n \right] \\ &\quad + \sum_{n=1}^{\infty} (-1) a_{n-1} (n-1+r) x^n + \sum_{n=1}^{\infty} 4 a_{n-1} x^n \end{aligned}$$

$$= a_0\,[r(r-1) - r]\,x^0$$
$$+ \sum_{n=0}^{\infty}\Big[a_n\,[(n+r)(n+r-1) - (n+r)] \;+\; a_{n-1}\,[-(n-1+r)+4]\Big]\,x^n \quad ,$$

which simplifies to

$$0 \;=\; a_0\Big[r^2 - 2r\Big]x^0 \;+\; \sum_{n=0}^{\infty}\Big[a_n(n+r)(n+r-2) - a_{n-1}(n+r-5)\Big]x^n \quad . \qquad (\star)$$

From the first term, we get the indicial equation

$$0 \;=\; \underbrace{r^2 - 2r}_{r(r-2)} \quad .$$

So the equation's exponents are $r_1 = 2$ and $r_2 = 0$.

Plugging the larger exponent, $r = r_1 = 2$, into equation ($\star$), we get

$$0 \;=\; a_0\Big[2^2 - 2\cdot 2\Big]x^0 \;+\; \sum_{n=0}^{\infty}\Big[a_n(n+2)(n+2-2) - a_{n-1}(n+2-5)\Big]x^n$$
$$= a_0\,[0]\,x^0 \;+\; \sum_{n=0}^{\infty}\Big[a_n(n+2)n - a_{n-1}(n-3)\Big]x^n$$

So, for $n \geq 1$,

$$a_n(n+2)n - a_{n-1}(n-3) \quad \rightarrowtail \quad a_n \;=\; \frac{n-3}{(n+2)n}a_{n-1} \quad .$$

That is, the recursion formula is

$$a_k \;=\; \frac{k-3}{(k+2)k}a_{k-1} \qquad \text{for} \quad k \geq 1 \quad .$$

Applying this:

$$a_1 \;=\; \frac{1-3}{(1+2)1}a_{1-1} \;=\; \frac{-2}{(3)1}a_0 \quad ,$$
$$a_2 \;=\; \frac{2-3}{(2+2)2}a_{2-1} \;=\; \frac{-1}{(4)2}a_1 \;=\; \frac{-1}{(4)(2)}\cdot\frac{-2}{(3)1}a_0 \;=\; \frac{(-1)(-2)}{(4\cdot 3)(2\cdot 1)}a_0 \quad ,$$
$$a_3 \;=\; \frac{3-3}{(3+2)3}a_{3-1} \;=\; \frac{0}{(5)3}a_2 \;=\; 0 \quad ,$$
$$a_4 \;=\; \frac{4-3}{(4+2)4}a_{4-1} \;=\; \frac{1}{(6)4}a_3 \;=\; \frac{1}{(6)4}\cdot 0 \;=\; 0 \quad ,$$
$$\vdots$$

Clearly, $a_k = 0$ for $k \geq 3$. Thus,

$$y(x) \;=\; x^{r_1}\sum_{k=0}^{\infty}a_k x^k$$
$$= x^2\Big[a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots\Big]$$

$$= x^2 \left[a_0 + \frac{-2}{(3)1} a_0 x + \frac{(-1)(-2)}{(4 \cdot 3)(2 \cdot 1)} a_0 x^2 + 0x^3 + 0x^4 + \cdots \right]$$
$$= a_0 y_1(x)$$

where

$$y_1(x) = x^2 - \frac{2}{3}x^3 + \frac{1}{12}x^4 \quad .$$

Next, we use $r = r_2 = 0$ in equation ($\star$), obtaining

$$0 = a_0 \left[0^2 - 2 \cdot 0 \right] x^0 + \sum_{n=0}^{\infty} \left[a_n(n+0)(n+0-2) - a_{n-1}(n+0-5) \right] x^n$$

$$= a_0[0]x^0 + \sum_{n=0}^{\infty} \left[a_n n(n-2) - a_{n-1}(n-5) \right] x^n \quad .$$

So,

$$a_n n(n-2) - a_{n-1}(n-5) \quad \rightarrowtail \quad a_n = \frac{n-5}{n(n-2)} a_{n-1} \quad .$$

That is, the recursion formula is

$$a_k = \frac{k-5}{k(k-2)} a_{k-1} \qquad \text{for} \quad k \geq 1 \quad .$$

Applying this:

$$a_1 = \frac{1-5}{1(1-2)} a_{1-1} = \frac{-4}{1(-1)} a_0$$

and

$$a_2 = \frac{2-5}{2(2-2)} a_{2-1} = \frac{-3}{2 \cdot 0} a_1 = \frac{-3}{2 \cdot 0} \cdot \frac{-4}{1(-1)} a_0 = \text{“}\infty\text{”} \quad .$$

Since the recursion formula corresponding to r_2 "blows up" at $k = 2$, there can be no solution of the form

$$y(x) = x^{r_2} \sum_{k=0}^{\infty} a_k x^k \quad .$$

32.4 m. The equation is already in desired form.

Since $x_0 = 0$, we set

$$y(x) = (x-0)^r \sum_{k=0}^{\infty} a_k (x-0)^k = \sum_{k=0}^{\infty} a_k x^{k+r} \quad .$$

Differentiating this and plugging the results into the differential equation, we get

$$y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1} \qquad , \qquad y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-2}$$

and

$$0 = x^2 y'' + \left(x - x^4 \right) y' + 3x^3 y$$

$$= x^2 y'' + xy - x^4 y' + 3x^3 y$$

$$= x^2 \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-2} \ + \ x \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1}$$

$$- \ x^4 \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1} \ + \ 3x^3 \sum_{k=0}^{\infty} a_k x^{k+r}$$

$$= \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r} \ + \ \sum_{k=0}^{\infty} a_k (k+r) x^{k+r}$$

$$+ \ \sum_{k=0}^{\infty} (-1) a_k (k+r) x^{k+r+3} \ + \ \sum_{k=0}^{\infty} 3 a_k x^{k+r+3} \quad .$$

Dividing out the x^r and continuing:

$$0 \ = \ \underbrace{\sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^k}_{n=k} \ + \ \underbrace{\sum_{k=0}^{\infty} a_k (k+r) x^k}_{n=k}$$

$$+ \ \underbrace{\sum_{k=0}^{\infty} (-1) a_k (k+r) x^{k+3}}_{n=k+3} \ + \ \underbrace{\sum_{k=0}^{\infty} 3 a_k x^{k+3}}_{n=k+3}$$

$$= \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^n \ + \ \sum_{n=0}^{\infty} a_n (n+r) x^n$$

$$+ \ \sum_{n=3}^{\infty} (-1) a_{n-3} (n-3+r) x^n \ + \ \sum_{n=3}^{\infty} 3 a_{n-3} x^n$$

$$= \Bigg[a_0 (0+r)(0+r-1) x^0 \ + \ a_1 (1+r)(1+r-1) x^1 \ + \ a_2 (2+r)(2+r-1) x^2$$

$$+ \ \sum_{n=3}^{\infty} a_n (n+r)(n+r-1) x^n \Bigg]$$

$$+ \ \Bigg[a_0 (0+r) x^0 \ + \ a_1 (1+r) x^1 \ + \ a_2 (2+r) x^2 \ + \ \sum_{n=3}^{\infty} a_n (n+r) x^n \Bigg]$$

$$+ \ \sum_{n=3}^{\infty} (-1) a_{n-3} (n-3+r) x^n \ + \ \sum_{n=3}^{\infty} 3 a_{n-3} x^n$$

$$= \ a_0 \left[r(r-1) + r \right] x^0 \ + \ a_1 \left[(1+r)r + (1+r) \right] x^1$$

$$+ \ a_2 \left[(2+r)(1+r) + (2+r) \right] x^2$$

$$+ \ \sum_{n=3}^{\infty} \Big[a_n \left[(n+r)(n+r-1) + (n+r) \right] \ + \ a_{n-3} \left[(-1)(n-3+r) + 3 \right] \Big] x^n \quad ,$$

which simplifies to

$$\begin{aligned} 0 \;=\; & a_0 r^2 x^0 \;+\; a_1(1+r)^2 x^1 \;+\; a_2(2+r)^2 x^2 \\ & + \sum_{n=3}^{\infty} \Big[a_n(n+r)^2 \;-\; a_{n-3}(n+r-6) \Big] x^n \end{aligned} \qquad (\star)$$

From the first term, we get the indicial equation

$$r^2 \;=\; 0 \quad .$$

which immediately tells us that the exponents of the equation are given by $r_1 = r_2 = r = 0$. With this value of r, equation $(\star)$ reduces to

$$0 \;=\; a_1 x^1 \;+\; a_2 4x^2 \;+\; \sum_{n=3}^{\infty} \Big[a_n n^2 \;-\; a_{n-3}(n-6) \Big] x^n$$

Since each term on the right must be zero, we have $a_1 = 0$, $a_2 = 0$ and, for $n \geq 3$,

$$a_n n^2 \;-\; a_{n-3}(n-6) \;=\; 0 \quad \rightarrowtail \quad a_n \;=\; \frac{n-6}{n^2} a_{n-3} \quad .$$

So the recursion formula is

$$a_k \;=\; \frac{k-6}{k^2} a_{k-3} \qquad \text{for} \quad k \geq 3 \quad .$$

Applying the above:

$$\begin{aligned} a_1 \;&=\; 0 \quad , \\ a_2 \;&=\; 0 \quad , \\ a_3 \;&=\; \frac{3-6}{3^2} a_{3-3} \;=\; \frac{-3}{3^2} a_0 \;=\; \frac{-1}{3} a_0 \quad , \\ a_4 \;&=\; \frac{4-6}{4^2} a_{4-3} \;=\; \frac{-2}{4^2} a_1 \;=\; \frac{-2}{4^2} \cdot 0 \;=\; 0 \quad , \\ a_5 \;&=\; \frac{5-6}{5^2} a_{5-3} \;=\; \frac{-1}{5^2} a_2 \;=\; \frac{-1}{5^2} \cdot 0 \;=\; 0 \quad , \\ a_6 \;&=\; \frac{6-6}{6^2} a_{6-3} \;=\; \frac{0}{6^2} a_3 \;=\; \frac{0}{6^2} \cdot \frac{-1}{3} a_0 \;=\; 0 \quad , \\ &\vdots \end{aligned}$$

Since $a_4 = a_5 = a_6 = 0$, it should be clear that the recursion formula will yield $a_k = 0$ for $k > 6$. Hence, we have

$$\begin{aligned} y(x) \;&=\; x^{r_1} \sum_{k=0}^{\infty} a_k x^k \\ &=\; x^0 \Big[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots \Big] \\ &=\; a_0 \;+\; 0x \;+\; 0x^2 \;-\; \frac{1}{3} a_0 x^3 \;+\; 0 \\ &=\; a_0 y_1(x) \end{aligned}$$

with $y_1(x) \;=\; 1 - \frac{1}{3} x^3$.

And since $r_2 = r_1$, the basic Frobenius method does not yield any othe solutions.

32.5 a. When $x = 3$ the first coefficient is zero and the the last coefficient is nonzero. So $x_0 = 3$ is a singular point. Moreover, we can easily get the equation into quasi-Euler form by multiplying through by $x - 3$:

$$(x-3)^2 y'' + (x-3)^2 y' + (x-3)y = 0$$

So $x_0 = 3$ is a regular singular point.

To simplify further computations, we let

$$Y(X) = y(x) \qquad \text{with} \quad X = x - 3 \quad .$$

The above differential equation then becomes

$$X^2 Y'' + X^2 Y' + XY = 0 \quad .$$

Since $X_0 = x_0 - 3 = 3 - 3 = 0$, we set

$$Y(X) = (X-0)^r \sum_{k=0}^{\infty} a_k (X-0)^k = \sum_{k=0}^{\infty} a_k X^{k+r} \quad .$$

Differentiating this and plugging the results into the last differential equation, we get

$$Y' = \sum_{k=0}^{\infty} a_k (k+r) X^{k+r-1} \quad , \quad Y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) X^{k+r-2}$$

and

$$\begin{aligned}
0 &= X^2 Y'' + X^2 Y' + XY \\
&= X^2 \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) X^{k+r-2} + X^2 \sum_{k=0}^{\infty} a_k (k+r) X^{k+r-1} + X \sum_{k=0}^{\infty} a_k X^{k+r} \\
&= \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) X^{k+r} + \sum_{k=0}^{\infty} a_k (k+r) X^{k+r+1} + \sum_{k=0}^{\infty} a_k X^{k+r+1}
\end{aligned}$$

Dividing out the X^r and continuing:

$$\begin{aligned}
0 &= \underbrace{\sum_{k=0}^{\infty} a_k (k+r)(k+r-1) X^k}_{n=k} + \underbrace{\sum_{k=0}^{\infty} a_k (k+r) X^{k+1}}_{n=k+1} + \underbrace{\sum_{k=0}^{\infty} a_k X^{k+1}}_{n=k+1} \\
&= \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) X^n + \sum_{n=1}^{\infty} a_{n-1} (n-1+r) X^n + \sum_{n=1}^{\infty} a_{n-1} X^n \\
&= \left[a_0 r(r-1) x^0 + \sum_{n=1}^{\infty} a_n (n+r)(n+r-1) X^n \right] \\
&\qquad + \sum_{n=1}^{\infty} a_{n-1} (n-1+r) X^n + \sum_{n=1}^{\infty} a_{n-1} X^n \quad ,
\end{aligned}$$

which simplifies to

$$0 = a_0\left[r^2 - r\right]x^0 \;+\; \sum_{n=1}^{\infty}\left[a_n(n+r-1) \;+\; a_{n-1}\right](n+r)X^n \qquad (\star)$$

From the first term, we get the indicial equation

$$0 \;=\; \underbrace{r^2 \;-\; r}_{r(r-1)} \quad ,$$

which tells us that the equation's exponents are $r_1 = 1$ and $r_2 = 0$.

Letting $r = r_1 = 1$ in equation ($\star$) yields:

$$0 \;=\; a_0\,[0]\,x^0 \;+\; \sum_{n=0}^{\infty}\left[a_n n \;+\; a_{n-1}\right]nX^n \quad .$$

Since each term must be zero, we have, for $n \geq 1$

$$\left[a_n n \;+\; a_{n-1}\right]n \;=\; 0 \quad \rightarrowtail \quad a_n \;=\; \frac{-1}{n}a_{n-1} \quad .$$

So our recursion formula is

$$a_k \;=\; \frac{-1}{k}a_{k-1} \qquad \text{for} \quad k \geq 1 \quad .$$

Applying the above:

$$\begin{aligned}
a_1 &= \frac{-1}{1}a_{1-1} = \frac{-1}{1}a_0 \quad , \\
a_2 &= \frac{-1}{2}a_{2-1} = \frac{-1}{2}a_1 = \frac{-1}{2}\cdot\frac{-1}{1}a_0 = \frac{(-1)^2}{2\cdot 1}a_0 \quad , \\
a_3 &= \frac{-1}{3}a_{3-1} = \frac{-1}{3}a_2 = \frac{-1}{3}\cdot\frac{(-1)^2}{2\cdot 1}a_0 = \frac{(-1)^3}{3!}a_0 \quad , \\
a_4 &= \frac{-1}{4}a_{4-1} = \frac{-1}{4}a_3 = \frac{-1}{4}\cdot\frac{(-1)^3}{3!}a_0 = \frac{(-1)^4}{4!}a_0 \quad , \\
&\;\;\vdots
\end{aligned}$$

Clearly,

$$a_k \;=\; \frac{(-1)^k}{k!}a_0 \qquad \text{for} \quad k \;=\; 0,\ 1,\ 2,\ \ldots \quad ,$$

Hence, corresponding to r_1, we have

$$Y(X) \;=\; X^{r_1}\sum_{k=0}^{\infty}a_kX^k \;=\; X^1\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}a_0X^k \;=\; a_0Y_1(X)$$

where

$$\begin{aligned}
Y_1(X) &= X\left[1 - \frac{1}{2}X + \frac{1}{2\cdot 1}X^2 - \frac{1}{3\cdot 2\cdot 1}X^3 + \cdots\right] \\
&= X\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}X^k \;=\; X\sum_{k=0}^{\infty}\frac{1}{k!}(-X)^k \;=\; Xe^{-X} \quad .
\end{aligned}$$

Recalling that $y(x) = Y(X)$ with $X = x - 3$, we see that the corresponding solution to our original differential equation is $y(x) = a_0 y_1(x)$ where

$$\begin{aligned} y_1(x) &= Y_1(x-3) \\ &= (x-3)\left[1 - \frac{1}{2}(x-3) + \frac{1}{2\cdot 1}(x-3)^2 - \frac{1}{3\cdot 2\cdot 1}(x-3)^3 + \cdots\right] \\ &= (x-3)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(x-3)^k \\ &= (x-3)e^{-(x-3)} \quad . \end{aligned}$$

Next we seek the solutions corresponding to $r = r_2 = 0$. Using this value for r in equation ($\star$) yields

$$\begin{aligned} 0 &= a_0\left[0^2 - 0\right]x^{-1} + \sum_{n=1}^{\infty}\left[a_n(n+0-1) + a_{n-1}\right](n+0)X^n \\ &= \sum_{n=1}^{\infty}\left[a_n(n-1) + a_{n-1}\right]nX^n \end{aligned}$$

So, for $n \geq 1$, we must have

$$\left[a_n(n-1) + a_{n-1}\right]n = 0 \quad \rightarrowtail \quad a_n = \frac{-1}{n-1}a_{n-1} \quad ,$$

and the recursion formula is

$$a_k = \frac{-1}{k-1}a_{k-1} \qquad \text{for} \quad k = 1, 2, 3, \ldots \quad .$$

Unfortunately, this recursion formula blows up with its first usage,

$$a_1 = \frac{-1}{1-1}a_{1-1} = \frac{-1}{0}a_0 \quad .$$

Hence, the basic method of Frobenius does not yield the second set of solutions.

32.5 c. It should be clear that $x_0 = 1$ is a singular point. To apply theorem 32.2, we compute

$$\lim_{x\to x_0}(x-x_0)\frac{b(x)}{a(x)} = \lim_{x\to 1}(x-1)\frac{0}{4} = 0$$

and

$$\lim_{x\to x_0}(x-x_0)^2\frac{c(x)}{a(x)} = \lim_{x\to 1}(x-1)^2\frac{(4x-3)/(x-1)^2}{4} = \lim_{x\to 1}\frac{(4x-3)}{4} = \frac{1}{4} \quad .$$

Since both limits are finite, theorem 32.2 assures us that $x_0 = 1$ is a regular singular point.

To simplify further computations, we let

$$Y(X) = y(x) \qquad \text{with} \quad X = x - 1 \quad .$$

The differential equation in question then becomes

$$4Y'' + \frac{4(X+1)-3}{X^2}Y = 0 \quad .$$

Doing a little algebra, we get this to desired form:

$$4X^2Y'' + (4X+1)Y = 0 \quad .$$

Since $X_0 = x_0 - 1 = 1 - 1 = 0$, we set

$$Y(X) = (X-0)^r \sum_{k=0}^{\infty} a_k (X-0)^k = \sum_{k=0}^{\infty} a_k X^{k+r} \quad .$$

Differentiating this and plugging the results into the differential equation, we get

$$Y' = \sum_{k=0}^{\infty} a_k (k+r) X^{k+r-1} \quad , \quad Y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) X^{k+r-2}$$

and

$$\begin{aligned} 0 &= 4X^2Y'' + (4X+1)Y \\ &= 4X^2 \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) X^{k+r-2} + (4X+1) \sum_{k=0}^{\infty} a_k X^{k+r} \\ &= \sum_{k=0}^{\infty} 4a_k (k+r)(k+r-1) X^{k+r} + \sum_{k=0}^{\infty} 4a_k X^{k+r+1} + \sum_{k=0}^{\infty} a_k X^{k+r} \quad . \end{aligned}$$

Dividing out X^r and continuing:

$$\begin{aligned} 0 &= \underbrace{\sum_{k=0}^{\infty} 4a_k (k+r)(k+r-1) X^k}_{n=k} + \underbrace{\sum_{k=0}^{\infty} 4a_k X^{k+1}}_{n=k+1} + \underbrace{\sum_{k=0}^{\infty} a_k X^k}_{n=k} \\ &= \sum_{n=0}^{\infty} 4a_n (n+r)(n+r-1) X^n + \sum_{k=1}^{\infty} 4a_{n-1} X^n + \sum_{n=0}^{\infty} a_n X^n \\ &= \left[a_0 4(0+r)(0+r-1) X^0 + \sum_{n=1}^{\infty} 4a_n (n+r)(n+r-1) X^n \right] + \sum_{k=1}^{\infty} 4a_{n-1} X^n \\ &\quad + \left[a_0 X^0 + \sum_{n=1}^{\infty} a_n X^n \right] \\ &= a_0 \left[4r(r-1) + 1 \right] X^0 + \sum_{n=1}^{\infty} \left[a_n \left[4(n+r)(n+r-1) + 1 \right] + 4a_{n-1} \right] X^n \quad , \end{aligned}$$

which simplifies to

$$0 = a_0 \left[4r^2 - 4r + 1 \right] X^0 + \sum_{n=1}^{\infty} \left[a_n \left[4(n+r)(n+r-1) + 1 \right] + 4a_{n-1} \right] X^n \quad . \quad (\star)$$

From the first term, we get the indicial equation

$$0 = 4r^2 - 4r + 1 \quad ,$$

whose solution is

$$r = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 4 \cdot 2}}{2 \cdot 4} = \frac{1}{2} \quad .$$

So the exponents of the equation are given by

$$r_1 = r_2 = \frac{1}{2} \quad .$$

Plugging $r = r_1 = \frac{1}{2}$ into equation (⋆), we get

$$\begin{aligned} 0 &= a_0 \, [0] \, X^0 + \sum_{n=1}^{\infty} \left[a_n \left[4 \left(n + \tfrac{1}{2} \right) \left(n + \tfrac{1}{2} - 1 \right) + 1 \right] + 4a_{n-1} \right] X^n \\ &= \sum_{n=1}^{\infty} \left[4a_n n^2 + 4a_{n-1} \right] X^n \quad , \end{aligned}$$

telling us that, for $n \geq 1$,

$$4a_n n^2 + 4a_{n-1} = 0 \quad \rightarrowtail \quad a_n = \frac{-1}{n^2} a_{n-1} \quad .$$

So,

$$a_k = \frac{-1}{k^2} a_{k-1} \qquad \text{for} \quad k = 1, 2, 3, \ldots$$

is our recursion formula, and

$$\begin{aligned} a_1 &= \frac{-1}{1^2} a_{1-1} = \frac{-1}{1^2} a_0 \quad , \\ a_2 &= \frac{-1}{2^2} a_{2-1} = \frac{-1}{2^2} a_1 = \frac{-1}{2^2} \cdot \frac{-1}{1^2} a_0 = \frac{(-1)^2}{(2 \cdot 1)^2} a_0 \quad , \\ a_3 &= \frac{-1}{3^2} a_{3-1} = \frac{-1}{3^2} a_2 = \frac{-1}{3^2} \cdot \frac{-1}{(2 \cdot 1)^2} a_0 = \frac{(-1)^3}{(3 \cdot 2 \cdot 1)^2} a_0 = \frac{(-1)^3}{(3!)^2} a_0 \quad , \\ a_4 &= \frac{-1}{4^2} a_{4-1} = \frac{-1}{4^2} a_3 = \frac{-1}{4^2} \cdot \frac{-1}{(3!)^2} a_0 = \frac{(-1)^4}{(4!)^2} a_0 \quad , \\ &\vdots \end{aligned}$$

It's easy to see that, in fact,

$$a_k = \frac{(-1)^k}{(k!)^2} a_0 \qquad \text{for} \quad k = 0, 1, 2, 3, \ldots \quad .$$

So,

$$Y(X) = X^{r_1} \sum_{k=0}^{\infty} a_k X^k = X^{1/2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} a_0 X^k = a_0 Y_1(X)$$

where

$$\begin{aligned} Y_1(X) &= X^{1/2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} X^k \\ &= X^{1/2} \sum_{k=0}^{\infty} \left[1 - X + \frac{1}{2^2} X^2 - \frac{1}{(3!)^2} X^3 - \frac{1}{(4!)^2} X^4 + \cdots \right] \quad . \end{aligned}$$

But the corresponding solutions to the original differential equation are given by $y(x) = Y(X)$ where $X = x - 1$. Hence, (at least for $x > 1$), $y(x) = a_0 y_1(x)$ where

$$\begin{aligned} y_1(x) &= Y_1(x-1) \\ &= \sqrt{x-1}\sum_{k=0}^{\infty}\frac{(-1)^k}{(k!)^2}(x-1)^k \\ &= \sqrt{x-1}\left[1-(x-1)+\frac{1}{2^2}(x-1)^2-\frac{1}{(3!)^2}(x-1)^3-\frac{1}{(4!)^2}(x-1)^4+\cdots\right] \quad . \end{aligned}$$

Since $r_1 = r_2$, no other solutions can be found by the basic method of Frobenius.

32.6 a. Applying the product rule, theorem 29.6 on page 572 on the differentiation of power series, and basic power series algebra:

$$\begin{aligned} \frac{d}{dx}\left[x^r\sum_{k=0}^{\infty}a_k x^k\right] &= \frac{dx^r}{dx}\sum_{k=0}^{\infty}a_k x^k \;+\; x^r\frac{d}{dx}\sum_{k=0}^{\infty}a_k x^k \\ &= rx^{r-1}\sum_{k=0}^{\infty}a_k x^k \;+\; x^r\sum_{k=0}^{\infty}a_k k x^{k-1} \\ &= \sum_{k=0}^{\infty}a_k r x^{k+r-1} \;+\; \sum_{k=0}^{\infty}a_k k x^{k+r-1} \\ &= \sum_{k=0}^{\infty}[a_k r + a_k k]\, x^{k+r-1} \\ &= \sum_{k=0}^{\infty}[a_k(k+r)x^{k+r-1} \quad . \end{aligned}$$

32.6 b. The formula rigorously obtained above for

$$\frac{d}{dx}\left[x^r\sum_{k=0}^{\infty}a_k x^k\right]$$

is the same as the one naively obtained by differentiating term-by-term.

32.7 a. For the first equation:

$$\frac{dy^*}{dx} = \frac{d}{dx}\big[(u+iv)^*\big] = \frac{d}{dx}\big[u-iv\big] = \frac{du}{dx} - i\frac{dv}{dx} = \left(\frac{du}{dx}+i\frac{dv}{dx}\right)^* = \left(\frac{dy}{dx}\right)^* \quad .$$

The second equation is verified in the same way, replacing the first derivatives with second derivatives.

32.7 b. Taking the complex conjugate of

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = 0 \quad ,$$

and using what was just derived along with basic complex algebra and the fact that a , b and c are all real-valued functions (hence, $a^* = a$, $b^* = b$ and $c^* = c$):

$$\left(a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y\right)^* = 0^*$$

$$\hookrightarrow \qquad \left(a(x)\frac{d^2y}{dx^2}\right)^* + \left(b(x)\frac{dy}{dx}\right)^* + (c(x)y)^* = 0$$

$$\hookrightarrow \qquad a^*(x)\left(\frac{d^2y}{dx^2}\right)^* + b^*(x)\left(\frac{dy}{dx}\right)^* + c^*(x)\,(y)^* = 0$$

$$\hookrightarrow \qquad a(x)\frac{d^2y^*}{dx^2} + b(x)\frac{dy^*}{dx} + c(x)y^* = 0 \quad .$$

32.8 a. Suppose that, contrary to the claim, y_1 and y_2 are constant multiples of each other. Since both are nonzero functions, there is then a nonzero constant c such that $y_2 = cy_1$. But then

$$x^{-r_2}y_2(x) = x^{-r_2}cy_1(x)$$

$$\hookrightarrow \qquad x^{-r_2}x^{r_2}\sum_{k=0}^{\infty}\beta_k x^k = cx^{-r_2}x^{r_1}\sum_{k=0}^{\infty}\alpha_k x^k$$

$$\hookrightarrow \qquad \sum_{k=0}^{\infty}\beta_k x^k = cx^{r_1-r_2}\sum_{k=0}^{\infty}\alpha_k x^k$$

Taking the limit as $x \to 0$ of both sides of the last equation, and keeping in mind that $r_2 > r_1$ (so $r_1 - r_2 > 0$), then gives us

$$\begin{aligned} 1 = \beta_0 &= \lim_{x\to 0}\sum_{k=0}^{\infty}\beta_k x^k \\ &= c\lim_{x\to 0}x^{r_1-r_2}\sum_{k=0}^{\infty}\alpha_k x^k \\ &= c\lim_{x\to 0}x^{r_1-r_2}\cdot\lim_{x\to 0}\sum_{k=0}^{\infty}\alpha_k x^k \\ &= c\cdot 0\cdot\alpha_0 = 0 \quad , \end{aligned}$$

which is nonsense. Hence, our supposition that y_1 and y_2 could be constant multiples of each other was wrong. So they cannot be constant multiples of each other, and, as noted in chapter 13, this means the two solutions y_1 and y_2 make up a fundamental set of solutions for the differential equation.

32.8 b. Since $\{y_1, y_2\}$ is a fundamental set of solutions to the differential equation, solution y_3 must be a linear combination of y_1 and y_2 .

32.8 c. As just noted, y_3 must be a linear combination of y_1 and y_2. So there are constants A and B, not both zero, such that

$$y_3 = Ay_1 + By_2 \quad .$$

Note, also, that

$$\begin{aligned} y_3(x) &= x^{r_2} \sum_{k=M}^{\infty} \gamma_k x^k \\ &= x^{r_2} \left[\gamma_M x^M + \gamma_{M+1} x^{M+1} + \gamma_{M+2} x^{M+2} \cdots \right] = x^{M+r_2} \sum_{k=0}^{\infty} \gamma_{M+k} x^k \quad . \end{aligned}$$

So, going through computations similar to that done above,

$$\begin{aligned} \gamma_M &= \lim_{x \to 0} \sum_{k=0}^{\infty} \gamma_{M+k} x^k \\ &= \lim_{x \to 0} x^{-(M+r_2)} y_3(x) \\ &= \lim_{x \to 0} x^{-(M+r_2)} \left[Ay_1(x) + By_2(x) \right] \\ &= A \lim_{x \to 0} x^{-(M+r_2)} y_1(x) + B \lim_{x \to 0} x^{-(M+r_2)} y_2(x) \\ &= A \lim_{x \to 0} x^{-(M+r_2)} x^{r_1} \sum_{k=0}^{\infty} \alpha_k x^k + B \lim_{x \to 0} x^{-(M+r_2)} x^{r_2} \sum_{k=0}^{\infty} \beta_k x^k \\ &= A \lim_{x \to 0} x^{r_1 - (M+r_2)} \alpha_0 + B \lim_{x \to 0} x^{-M} \beta_0 \quad . \end{aligned}$$

Since $\alpha_0 = \beta_0 = \gamma_0 = 1$, the above reduces to

$$1 = A \lim_{x \to 0} x^{(r_1 - r_2) - M} + B \lim_{x \to 0} x^{-M}$$

Remember, both $r_1 - r_2$ and M are positive, and at least A or B is nonzero. Now, observe that, if $M > r_1 - r_2$, then both of the above limits are infinite, which, clearly, is impossible. If, on the other hand, $M < r_1 - r_2$, then

$$1 = A \lim_{x \to 0} x^{(r_1 - r_2) - M} + B \lim_{x \to 0} x^{-M} = A \cdot 0 + B \cdot (\pm\infty)$$

which is also clearly impossible. This leaves $M = r_1 - r_2$ as the only possibility. But then

$$1 = A \lim_{x \to 0} x^{M-M} + B \lim_{x \to 0} x^{-M} = A \cdot 1 + B \cdot (\pm\infty)$$

which obviously means that $A = 1$ and $B = 0$. So, $M = r_1 - r_2$ and

$$y_3 = Ay_1 + By_2 = 1 \cdot y_1 + 0 \cdot y_2 = y_1 \quad .$$

32.8 d. First of all, it follows from part **a** that if r_1 and r_2 are the equation's exponents and $r_1 > r > 2$, then any two solutions of the form

$$y_1(x) = x^{r_1} \sum_{k=0}^{\infty} \alpha_k x^k \qquad \text{with} \quad \alpha_0 = 1$$

and

$$y_2(x) = x^{r_2} \sum_{k=0}^{\infty} \beta_k x^k \qquad \text{with} \quad \beta_0 = 1$$

make up a fundamental set of solutions. Hence, for y_2, we can choose any such modified series solution. If, in our first choice, $\beta_M \neq 0$ with $M = r_1 - r_2$ (equivalently, $r_1 = r_2 + M$), then another valid solution is

$$y_4(x) = y_2(x) - \beta_M y_1(x) = \cdots = x^{r_2} \sum_{k=0}^{\infty} a_k x^k \quad ,$$

where

$$a_k = \left\{ \begin{array}{cl} \beta_k - 0 & \text{if} \quad k < M \\ \beta_k - \beta_M \alpha_{M-k} & \text{if} \quad k \geq M \end{array} \right\} = \left\{ \begin{array}{cl} \beta_k & \text{if} \quad k < M \\ 0 & \text{if} \quad k = M \\ \beta_k - \beta_M \alpha_{M-k} & \text{if} \quad k > M \end{array} \right. \quad ,$$

which is simply the second solution obtained in the basic Frobenius method when $M = r_1 - r_2$ is a positive integer and the method actually does yield a solution corresponding to r_2.