Chapter 31: Power Series Solutions II: Generalizations and Theory

31.4 a. Suppose that $e^z = 0$ for some $z = x + iy$. Then both the real and imaginary parts of e^z must be zero,

$$e^x \cos(y) = 0 \quad \text{and} \quad e^x \sin(y) = 0 \quad ,$$

and since $e^x \neq 0$ for every real value x, we must have

$$\cos(y) = 0 \quad \text{and} \quad \sin(y) = 0 \quad .$$

But this is impossible. After all, if $\sin(y) = 0$ for some real value y, then $y = n\pi$ for some integer n, which then means that

$$\cos(y) = \cos(n\pi) = (-1)^n \neq 0 \quad .$$

So it is impossible for $e^z = 0$. In other words, $e^z \neq 0$ for every z in the complex plane.

31.4 b. We will just verify the claims concerning $\sin(z)$.

First of all,

$$\begin{aligned}
\sin(x+iy) &= \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} \\
&= \frac{1}{2i}\left[e^{ix-y} - e^{-ix+y}\right] \\
&= \frac{1}{2i}\left[e^{-y}e^{ix} - e^{y}e^{-ix}\right] \\
&= \frac{1}{2i}\left[e^{-y}[\cos(x) + i\sin(x)] - e^{y}[\cos(x) - i\sin(x)]\right] \\
&= \frac{1}{2i}\left[i\left[e^{y} + e^{-y}\right]\sin(x) - \left[e^{y} - e^{-y}\right]\cos(x)\right] \\
&= \frac{i}{i}\cdot\frac{e^{y}+e^{-y}}{2}\sin(x) - \frac{1}{i}\cdot\frac{e^{y}-e^{-y}}{2}\cos(x) \\
&= \frac{e^{y}+e^{-y}}{2}\sin(x) - \frac{1}{i}\cdot\frac{i}{i}\cdot\frac{e^{y}-e^{-y}}{2}\cos(x) \\
&= \frac{e^{y}+e^{-y}}{2}\sin(x) + \frac{e^{y}-e^{-y}}{2}\cos(x) \quad .
\end{aligned}$$

Next, assume $\sin(z) = 0$ for some $z = x + iy$. Then both the real part and imaginary parts of $\sin(x+iy)$ must equal 0, which, from above, means

$$\frac{e^{y}+e^{-y}}{2}\sin(x) = 0 \quad \text{and} \quad \frac{e^{y}-e^{-y}}{2}\cos(x) = 0 \quad .$$

Note that since $e^y + e^{-y} > 0$,

$$\begin{aligned}
\frac{e^{y}+e^{-y}}{2}\sin(x) = 0 \quad &\Longrightarrow \quad \sin(x) = 0 \\
&\Longrightarrow \quad x = n\pi \quad \text{for} \quad n = 0, \pm 1, \pm 2, \dots \quad .
\end{aligned}$$

But then

$$0 = \frac{e^{y}-e^{-y}}{2}\cos(x) = \frac{e^{y}-e^{-y}}{2}\cos(n\pi) = \frac{e^{y}-e^{-y}}{2}(-1)^n \quad ,$$

which means that $e^y - e^{-y} = 0$. Solving this for y yields $y = 0$. So, if

$$\sin(z) = 0 \quad ,$$

then

$$z = x + iy = n\pi + i0 = n\pi \qquad \text{with} \quad n = 0, \pm 1, \pm 2, \ldots \quad .$$

31.4 c. This is done in much the same way as the previous. Start by showing

$$\begin{aligned}
\sinh(x+iy) &= \frac{e^{x+iy} - e^{-(x+iy)}}{2} \\
&= \frac{1}{2}\left[e^x[\cos(y) + i\sin(y)] - e^{-x}[\cos(y) - i\sin(y)]\right] \\
&= \cdots \\
&= \frac{e^x - e^{-x}}{2}\cos(y) + i\frac{e^x + e^{-x}}{2}\sin(y) \quad .
\end{aligned}$$

Next, assume $\sinh(z) = 0$ for some $z = x + iy$. Then both the real part and imaginary parts of $\sinh(x+iy)$ must equal 0, which, from above, means

$$\frac{e^x - e^{-x}}{2}\cos(y) = 0 \qquad \text{and} \qquad \frac{e^x + e^{-x}}{2}\sin(y) = 0 \quad .$$

Solving this pair of equations then yields

$$z = x + iy = i\left[n + \frac{1}{2}\right]\pi \qquad \text{with} \quad n = 0, \pm 1, \pm 2, \ldots \quad .$$

31.5 a. Both coefficients — 1 and $-e^x$ — are analytic, and, from exercise 31.4, we know neither is ever zero. So, lemma 31.4 assures us that there are no singular points. Hence, also, the radius of analyticity R is ∞.

31.5 c. All three coefficients — $\sin(x)$, x^2 and $-e^x$ — are analytic everywhere, and (from exercise 31.4) we know $-e^z \neq 0$ for every z in the complex plane, while

$$\sin(z) = 0 \quad \Longrightarrow \quad z = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots \quad .$$

Lemma 31.5 then assures us that the singular points are all given by

$$z_s = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots \quad ,$$

and since $z_s = \pi$ is the singular point closest to $x_0 = 2$,

$$, R = |\pi - 2| = \pi - 2 \quad .$$

31.5 e. The coefficients are all analytic on the entire complex plane, and (from exercise 31.4) we know

$$\sinh(z) = 0 \Rightarrow z = in\pi \qquad \text{for} \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots \quad . \qquad (\star)$$

The other coefficients — z^2 and $-\sin(x)$ — are clearly nonzero at all these points except $z = 0$. So we need to check the point $z = 0$ more carefully, by computing

$$\lim_{x \to 0} \frac{x^2}{\sinh(x)} \qquad \text{and} \qquad \lim_{x \to 0} \frac{-\sin(x)}{\sinh(x)} \quad .$$

Using L'Hôpital's rule, we have

$$\lim_{x\to 0} \frac{x^2}{\sinh(x)} = \lim_{x\to 0} \frac{2x}{\cosh(x)} = \frac{0}{1} = 0$$

and

$$\lim_{x\to 0} \frac{-\sin(x)}{\sinh(x)} = \lim_{x\to 0} \frac{-\cos(x)}{\cosh(x)} = \frac{-1}{1} = -1 \quad .$$

Since both of these limits are finite, lemma 31.5 tells us that the point $x = z = 0$ is an ordinary point. Hence, all the values of z given in line ($\star$) are singular points except $z = 0$. So,

$$z_s = in\pi \quad \text{with } n = \pm 1, \pm 2, \ldots \quad .$$

Since the singular points closest to $x_0 = 2$ are $\pm i\pi$,

$$R = |2 - (\pm i\pi)| = \sqrt{2^2 + \pi^2} = \sqrt{4 + \pi^2} \quad .$$

31.5 g. Multiplying through by $1 - e^x$, we get

$$\left[1 - e^x\right] y'' + \left[1 + e^x\right] y = 0 \quad .$$

Both coefficients are analytic on the entire complex plane, and, letting $z = x + iy$,

$$1 - e^z = 0 \quad \rightarrowtail \quad 1 = e^z$$

$$\hookrightarrow \quad 1 + i0 = e^{x+iy} = e^x[\cos(y) + i\sin(y)]$$

$$\hookrightarrow \quad 1 = e^x \cos(y) \quad \text{and} \quad 0 = e^x \sin(y)$$

$$\hookrightarrow \quad 1 = e^x \cos(y) \quad \text{and} \quad y = n\pi \quad \text{for} \quad n = 0, \pm 1, \pm 2, \ldots$$

$$\hookrightarrow \quad 1 = e^x \cos(n\pi) = e^x(-1)^n \quad \text{for} \quad n = 0, \pm 1, \pm 2, \ldots$$

$$\hookrightarrow \quad x = 0 \quad \text{and} \quad n \text{ is an even integer}$$

$$\hookrightarrow \quad z = x + iy = 0 + i2k\pi \quad \text{for} \quad k = 0, \pm 1, \pm 2, \pm 3, \ldots \quad . \qquad (\star)$$

Checking, we see that, if $z = i2k\pi$ for any integer k,

$$1 + e^z = 1 + e^{i2k\pi} = 1 + \cos(2k\pi) + i\sin(2k\pi) = 2 \neq 0 \quad .$$

Hence, all the points given in line ($\star$) are singular points,

$$z_s = i2k\pi \quad \text{for} \quad k = 0, \pm 1, \pm 2, \pm 3, \ldots \quad ,$$

and

$$R = |x_0 - \text{closest singular point}| = |3 - 0i| = 3 \quad .$$

31.5 i. All the coefficients are analytic on the entire complex plane, with the first being zero if and only if $x = z = 0$. But we also have $1 - e^z = 0$ if $z = 0$. So we need to see if

$$\lim_{x \to 0} \frac{1 - e^x}{x}$$

is finite. Using L'Hôpital's rule, we see that

$$\lim_{x \to 0} \frac{1 - e^x}{x} = \lim_{x \to 0} \frac{-e^x}{1} = -1 \quad ,$$

which is finite, telling us that $z = 0$ is an ordinary point. Hence, there are no singular points, and $R = \infty$.

31.6 a. The coefficients of this equation — 1 and $-e^x$ — are analytic everywhere, and the first is simply a nonzero constant. So there are no singular points and the power series solution will be valid on $I = (-\infty, \infty)$.

The solution is $\sum_{k=0}^{\infty} a_k x^k$ with a_0 being arbitrary and

$$a_k = -\frac{1}{k} \sum_{j=0}^{k-1} a_j p_{k-1-j} \qquad \text{for} \quad k > 0$$

where, in this case,

$$\sum_{n=0}^{\infty} p_n x^n = P(x) = -e^x = -\sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{-1}{n!} x^n \quad .$$

So,

$$p_n = \frac{-1}{n!} \qquad \text{for} \quad n = 0, 1, 2, 3, \ldots \quad ,$$

and

$$a_k = -\frac{1}{k} \sum_{j=0}^{k-1} a_j p_{k-1-j} = -\frac{1}{k} \sum_{j=0}^{k-1} a_j \frac{-1}{(k-1-j)!} = \frac{1}{k} \sum_{j=0}^{k-1} a_j \frac{1}{(k-1-j)!} \quad .$$

In particular,

$$\begin{aligned} a_1 &= \frac{1}{1} \sum_{j=0}^{1-1} a_j \frac{1}{(1-1-j)!} \\ &= \sum_{j=0}^{0} a_j \frac{1}{(0-j)!} = a_0 \frac{1}{(0-0)!} = a_0 \quad , \end{aligned}$$

$$\begin{aligned} a_2 &= \frac{1}{2} \sum_{j=0}^{2-1} a_j \frac{1}{(2-1-j)!} \\ &= \frac{1}{2} \sum_{j=0}^{1} a_j \frac{1}{(1-j)!} \\ &= \frac{1}{2} \left[a_0 \frac{1}{(1-0)!} + a_1 \frac{1}{(1-1)!} \right] = \frac{1}{2} [a_0 + a_0] = a_0 \quad , \end{aligned}$$

$$
\begin{aligned}
a_3 &= \frac{1}{3}\sum_{j=0}^{3-1} a_j \frac{1}{(3-1-j)!} \\
&= \frac{1}{3}\sum_{j=0}^{2} a_j \frac{1}{(2-j)!} \\
&= \frac{1}{3}\left[a_0\frac{1}{(2-0)!} + a_1\frac{1}{(2-1)!} + a_2\frac{1}{(2-2)!}\right] = \frac{1}{3}\left[a_0\frac{1}{2} + a_0\frac{1}{1} + a_0\frac{1}{1}\right] = \frac{5}{6}a_0
\end{aligned}
$$

and

$$
\begin{aligned}
a_4 &= \frac{1}{4}\sum_{j=0}^{4-1} a_j \frac{1}{(4-1-j)!} \\
&= \frac{1}{4}\sum_{j=0}^{3} a_j \frac{1}{(3-j)!} \\
&= \frac{1}{4}\left[a_0\frac{1}{(3-0)!} + a_1\frac{1}{(3-1)!} + a_2\frac{1}{(3-2)!} + a_3\frac{1}{(3-3)!}\right] \\
&= \frac{1}{4}\left[a_0\frac{1}{6} + a_0\frac{1}{2} + a_0\frac{1}{1} + \left(\frac{5}{6}a_0\right)\frac{1}{1}\right] = \frac{5}{8}a_0
\end{aligned}
$$

Thus, the 4th-degree partial sum of the general power series solution is

$$
\begin{aligned}
\sum_{k=0}^{4} a_k x^k &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \\
&= a_0 + a_0 x + a_0 x^2 + \frac{5}{6}a_0 x^3 + \frac{5}{8}a_0 x^4 \\
&= a_0\left[1 + x + x^2 + \frac{5}{6}x^3 + \frac{5}{8}x^4\right] .
\end{aligned}
$$

31.6 c. The coefficients of this equation — 1 and $\cos(x)$ — are analytic everywhere, and the first is simply a nonzero constant. So there are no singular points and the power series solution will be valid on $I = (-\infty, \infty)$.

The solution is $\sum_{k=0}^{\infty} a_k x^k$ with a_0 being arbitrary and

$$
a_k = -\frac{1}{k}\sum_{j=0}^{k-1} a_j p_{k-1-j} \qquad \text{for} \quad k > 0
$$

where, in this case,

$$
\begin{aligned}
\sum_{n=0}^{\infty} p_n x^n &= P(x) = \cos(x) \\
&= \sum_{m=0}^{\infty} (-1)^m \frac{1}{(2m)!} x^{2m} \\
&= 1x^0 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots
\end{aligned}
$$

$$= (-1)^{0/2}x^0 + 0x^1 + (-1)^{2/2}\frac{1}{2!}x^2 + 0x^3$$
$$+ (-1)^{4/2}\frac{1}{4!}x^4 + 0x^5 + (-1)^{6/2}\frac{1}{6!}x^6 + 0x^7 + \cdots \quad .$$

So,

$$p_0 = 1 \quad , \quad p_1 = 0 \quad , \quad p_2 = -\frac{1}{2!} = \frac{-1}{2} \quad , \quad p_3 = 0$$

and, in general,

$$p_n = \begin{cases} (-1)^{n/2}\frac{1}{n!} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \quad .$$

Applying the above formulas for a_k and p_k, we get:

$$a_1 = -\frac{1}{1}\sum_{j=0}^{1-1} a_j p_{1-1-j}$$
$$= -\sum_{j=0}^{0} a_j p_{0-j} = -a_0 p_0 = -a_0 \cdot 1 = -a_0 \quad ,$$

$$a_2 = -\frac{1}{2}\sum_{j=0}^{2-1} a_j p_{2-1-j}$$
$$= -\frac{1}{2}\sum_{j=0}^{1} a_j p_{1-j}$$
$$= -\frac{1}{2}[a_0 p_1 + a_1 p_0] = -\frac{1}{2}[a_0 \cdot 0 - a_0 \cdot 1] = \frac{1}{2}a_0 \quad ,$$

$$a_3 = -\frac{1}{3}\sum_{j=0}^{3-1} a_j p_{3-1-j}$$
$$= -\frac{1}{3}[a_0 p_2 + a_1 p_1 + a_2 p_0]$$
$$= -\frac{1}{3}\left[a_0\left(\frac{-1}{2}\right) - a_0 \cdot 0 + \frac{1}{2}a_0 \cdot 1\right] = 0$$

and

$$a_4 = -\frac{1}{4}\sum_{j=0}^{4-1} a_j p_{4-1-j}$$
$$= -\frac{1}{4}[a_0 p_3 + a_1 p_2 + a_2 p_1 + a_3 p_0]$$
$$= -\frac{1}{4}\left[a_0 \cdot 0 + (-a_0)\left(\frac{-1}{2}\right) + \frac{1}{2}a_0 \cdot 0 + 0 \cdot 1\right] = -\frac{1}{8}a_0 \quad .$$

Thus, the 4th-degree partial sum of the general power series solution is

$$\begin{aligned}\sum_{k=0}^{4} a_k x^k &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \\ &= a_0 - a_0 x + \frac{1}{2} a_0 x^2 + 0 a_0 x^3 - \frac{1}{8} a_0 a_0 x^4 \\ &= a_0 \left[1 - x + \frac{1}{2} x^2 - \frac{1}{8} x^4 \right] \quad .\end{aligned}$$

31.7 a. The equation is in reduced form

$$y'' + Py' + Qy = 0$$

with

$$P(x) = 0 = \sum_{n=0}^{\infty} 0x^n \qquad \text{for} \quad |x| < \infty$$

and

$$Q(x) = -e^x = \sum_{n=0}^{\infty} \frac{-1}{n!} x^n \qquad \text{for} \quad |x| < \infty \quad .$$

Since the radii of convergence for each of these two series is ∞, so is the radius of convergence for the power series solution. Thus, the power series solution will be valid on $I = (-\infty, \infty)$. Moreover, by the above,

$$\sum_{n=0}^{\infty} p_k x^k = \sum_{n=0}^{\infty} 0x^n \qquad \text{and} \qquad \sum_{n=0}^{\infty} q_n x^n = \sum_{n=0}^{\infty} \frac{-1}{n!} x^n \quad ,$$

which means

$$p_n = 0 \qquad \text{for} \quad n = 0, 1, 2, 3, \ldots$$

and

$$q_n = \frac{-1}{n!} = 0, 1, 2, 3, \ldots \quad .$$

In particular, let's note that

$$q_0 = \frac{-1}{0!} = -1 \quad , \qquad q_1 = \frac{-1}{1!} = -1 \qquad \text{and} \qquad q_2 = \frac{-1}{2!} = \frac{-1}{2} \quad .$$

The power series solution is given by $\sum_{k=0}^{\infty} a_k x^k$ where a_0 and a_1 are arbitrary, and

$$a_k = -\frac{1}{k(k-1)} \sum_{j=0}^{k-2} \left[(j+1) a_{j+1} p_{k-2-j} + a_j q_{k-2-j} \right] \qquad \text{for} \quad k > 2 \quad .$$

In this case, since every p_n is 0, the above recursion formula simplifies to

$$a_k = -\frac{1}{k(k-1)} \sum_{j=0}^{k-2} a_j q_{k-2-j} \qquad \text{for} \quad k > 2 \quad .$$

Applying this recursion formula, we get

$$\begin{aligned} a_2 &= -\frac{1}{2(2-1)}\sum_{j=0}^{2-2} a_j q_{2-2-j} \\ &= -\frac{1}{2}\sum_{j=0}^{0} a_j q_{-j} = -\frac{1}{2}a_0 q_0 = -\frac{1}{2}a_0(-1) = \frac{1}{2}a_0 \quad , \end{aligned}$$

$$\begin{aligned} a_3 &= -\frac{1}{3(3-1)}\sum_{j=0}^{3-2} a_j q_{3-2-j} \\ &= -\frac{1}{6}\sum_{j=0}^{1} a_j q_{1-j} \\ &= -\frac{1}{6}\left[a_0 q_1 + a_1 q_0\right] = -\frac{1}{6}\left[a_0(-1) + a_1(-1)\right] = \frac{1}{6}\left[a_0 + a_1\right] \end{aligned}$$

and

$$\begin{aligned} a_4 &= -\frac{1}{4(4-1)}\sum_{j=0}^{4-2} a_j q_{4-2-j} \\ &= -\frac{1}{12}\sum_{j=0}^{2} a_j q_{2-j} \\ &= -\frac{1}{12}\left[a_0 q_2 + a_1 q_1 + a_2 q_0\right] \\ &= -\frac{1}{12}\left[a_0\left(\frac{-1}{2}\right) + a_1(-1) + \frac{1}{2}a_0(-1)\right] = \frac{1}{12}[a_0 + a_1] \quad . \end{aligned}$$

Thus, the 4th-degree partial sum of the general power series solution is

$$\begin{aligned} \sum_{k=0}^{4} a_k x^k &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \\ &= a_0 + a_1 x + \frac{1}{2}a_0 x^2 + \frac{1}{6}\left[a_0 + a_1\right]x^3 + \frac{1}{12}[a_0 + a_1]x^4 \\ &= a_0\left[1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{12}x^4\right] + a_1\left[x + \frac{1}{6}x^3 + \frac{1}{12}x^4\right] \quad . \end{aligned}$$

31.7 c. Dividing through by x, we get the equation into reduced form,

$$y'' - 3y' + \frac{\sin(x)}{x}y = 0 \quad ,$$

which is

$$y'' + Py' + Qy = 0 \qquad \text{with} \quad P(x) = -3 \quad \text{and} \quad Q(x) = \frac{\sin(x)}{x} \quad .$$

Obviously, P and Q are analytic everywhere except, possibly, for a singularity at $x = 0$ for $Q(x)$ (because of the x in the denominator). But, using L'Hôpital's rule, we see that

$$\lim_{x\to 0} Q(x) = \lim_{x\to 0}\frac{\sin(x)}{x} = \lim_{x\to 0}\frac{\cos(x)}{1} = 1 \quad ,$$

which is finite. So $Q(x)$ is analytic at $x = 0$, as well as everywhere else. Hence, there are no singular points, and $I = (-infty, \infty)$.

The power series for P and Q about $x_0 = 0$ are easily found. For P, we trivially have

$$\sum_{n=0}^{\infty} p_n x^n = P(x) = -3 = -3 + 0x + 0x^2 + 0x^3 + \cdots$$

which means

$$p_0 = -3 \quad \text{while} \quad p_n = 0 \quad \text{for} \quad n \geq 1 \quad .$$

For Q, we use the well-known series for the sine as follows:

$$\begin{aligned}
\sum_{n=0}^{\infty} q_n x^n = Q(x) &= \frac{1}{x}\sin(x) \\
&= x^{-1} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1} \\
&= x^{-1} \left[x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots \right] \\
&= x^0 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \frac{1}{7!}x^6 + \cdots \\
&= 1x^0 + 0x^1 - \frac{1}{3!}x^2 + 0x^3 \frac{1}{5!}x^4 + 0x^5 - \frac{1}{7!}x^6 + \cdots \quad .
\end{aligned}$$

Hence,

$$q_0 = 1 \quad , \quad q_1 = 0 \quad , \quad q_2 = \frac{-1}{3!} = \frac{-1}{6} \quad , \quad q_3 = 0 \quad , \quad \ldots \quad .$$

In general,

$$q_n(x) = \begin{cases} (-1)^{n/2} \dfrac{1}{n+1} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \quad .$$

Using this with recursion formula (31.4), we get

$$\begin{aligned}
a_2 &= -\frac{1}{2(2-1)} \sum_{j=0}^{2-2} \left[(j+1)a_{j+1} p_{2-2-j} + a_j q_{2-2-j} \right] \\
&= -\frac{1}{2} \sum_{j=0}^{0} \left[(j+1)a_{j+1} p_{0-j} + a_j q_{0-j} \right] \\
&= -\frac{1}{2} \left[a_{0+1} p_{0-0} + a_0 q_{0-0} \right] \\
&= -\frac{1}{2} [a_1 p_0 + a_0 q_0] \\
&= -\frac{1}{2} [a_1(-3) + a_0 \cdot 1] = \frac{1}{2} [3a_1 - a_0] \quad ,
\end{aligned}$$

$$\begin{aligned}
a_3 &= -\frac{1}{3(3-1)} \sum_{j=0}^{3-2} \left[(j+1)a_{j+1} p_{3-2-j} + a_j q_{3-2-j} \right] \\
&= -\frac{1}{6} \sum_{j=0}^{1} \left[(j+1)a_{j+1} p_{1-j} + a_j q_{1-j} \right]
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{6}\Big(\big[(0+1)a_{0+1}p_{1-0} + a_0 q_{1-0}\big] + \big[(1+1)a_{1+1}p_{1-1} + a_1 q_{1-1}\big]\Big) \\
&= -\frac{1}{6}\Big([a_1 p_1 + a_0 q_1] + [2a_2 p_0 + a_1 q_0]\Big) \\
&= -\frac{1}{6}\Big([a_1 \cdot 0 \ + \ a_0 \cdot 0] + \Big[2\cdot\frac{1}{2}[3a_1 - a_0](-3) + a_1 \cdot 1\Big]\Big) \\
&= -\frac{1}{2}a_0 \ + \ \frac{4}{3}a_1
\end{aligned}
$$

and

$$
\begin{aligned}
a_4 &= -\frac{1}{4(4-1)}\sum_{j=0}^{4-2}\big[(j+1)a_{j+1}p_{4-2-j} + a_j q_{4-2-j}\big] \\
&= -\frac{1}{12}\sum_{j=0}^{2}\big[(j+1)a_{j+1}p_{2-j} + a_j q_{2-j}\big] \\
&= -\frac{1}{12}\Big([1a_1 p_2 + a_0 q_2] + [2a_2 p_1 + a_1 q_1] + [3a_3 p_0 + a_2 q_0]\Big) \\
&= -\frac{1}{12}\Big(\Big[1a_1\cdot 0 + a_0\Big(\frac{-1}{6}\Big)\Big] + [2a_2\cdot 0 + a_1 \cdot 0] \\
&\qquad + \Big[3\Big(-\frac{1}{2}a_0 \ + \ \frac{4}{3}a_1\Big)(-3) + \frac{1}{2}[3a_1 - a_0]\cdot 1\Big]\Big) \\
&= -\frac{23}{72}a_0 \ + \ \frac{7}{8}a_1 \quad .
\end{aligned}
$$

Thus, the 4th-degree partial sum of the general power series solution is

$$
\begin{aligned}
\sum_{k=0}^{4} a_k x^k &= a_0 \ + \ a_1 x \ + \ a_2 x^2 \ + \ a_3 x^3 \ + \ a_4 x^4 \\
&= a_0 \ + \ a_1 x \ + \ \frac{1}{2}[3a_1 - a_0]x^2 \ + \ \Big[-\frac{1}{2}a_0 + \frac{4}{3}a_1\Big]x^3 \\
&\qquad + \ \Big[-\frac{23}{72}a_0 + \frac{7}{8}a_1\Big]x^4 \\
&= a_0\Big[1 - \frac{1}{2}x^2 - \frac{1}{2}x^3 - \frac{23}{72}x^4\Big] \ + \ a_1\Big[x + \frac{3}{2}x^2 + \frac{4}{3}x^3 + \frac{7}{8}x^4\Big] \quad .
\end{aligned}
$$

31.7 e. The first coefficient is 0 if and only if $x = 0$, while the coefficient on y is never 0. From that it follows that $z_s = 0$ is the only singular point,

$$R \ = \ |z_s - x_0| \ = \ |0 - 1| \ = \ 1$$

and

$$I \ = \ (x_0 - R, x_0 + R) \ = \ (0, 2) \quad .$$

Dividing the equation by the first coefficient and letting $Y(X) = y(x)$ with $X = x - 1$ converts the differential equation to

$$Y'' \ + \ \frac{1}{\sqrt{X+1}}Y \ = \ 0 \qquad \text{with} \quad X_0 \ = \ 0 \quad ,$$

which is

$$Y'' \ + \ PY' \ + \ QY \ = \ 0 \qquad \text{with} \quad P(X) \ = \ 0 \quad \text{and} \quad Q(X) \ = \ (X+1)^{-1/2} \quad .$$

Obviously,

$$P(X) \;=\; \sum_{n=0}^{\infty} p_n X^n \qquad \text{with} \quad p_n = 0 \qquad \text{for all} \quad n \quad .$$

Thus, the recursion formula for the power series of Y about X_0 reduces a bit,

$$\begin{aligned} a_k \;&=\; -\frac{1}{k(k-1)} \sum_{j=0}^{k-2} \big[(j+1) a_{j+1} p_{k-2-j} \;+\; a_j q_{k-2-j} \big] \\ &=\; -\frac{1}{k(k-1)} \sum_{j=0}^{k-2} a_j q_{k-2-j} \quad . \end{aligned}$$

For Q, we use the Taylor series,

$$\sum_{n=0}^{\infty} q_n X^n \;=\; Q(x) \;=\; \sum_{n=0}^{\infty} \frac{Q^{(n)}(0)}{n!} X^n \quad .$$

So,

$$\begin{aligned} q_0 \;&=\; \frac{Q^{(0)}(0)}{0!} \;=\; Q(0) \;=\; (0+1)^{-1/2} \;=\; 1 \quad , \\ q_1 \;&=\; \frac{Q'(0)}{1!} \;=\; Q'(0) \;=\; \frac{-1}{2}(0+1)^{-3/2} \;=\; -\frac{1}{2} \quad , \\ q_2 \;&=\; \frac{Q''(0)}{2!} \;=\; \frac{1}{2!} Q''(0) \;=\; \frac{1}{2!} \cdot \frac{3}{2^2}(0+1)^{-5/2} \;=\; \frac{3}{8} \quad , \\ q_3 \;&=\; \frac{Q''(0)}{3!} \;=\; \frac{1}{3!} Q^{(3)}(0) \;=\; \frac{1}{3!} \cdot \frac{-3 \cdot 5}{2^3}(0+1)^{-7/2} \;=\; \frac{-5}{16} \quad , \\ &\;\;\vdots \end{aligned}$$

Plugging the above into our recursion formula, we get

$$\begin{aligned} a_2 \;&=\; -\frac{1}{2(2-1)} \sum_{j=0}^{2-2} a_j q_{2-2-j} \;=\; -\frac{1}{2} a_0 q_0 \;=\; -\frac{1}{2} a_0 \cdot 1 \;=\; -\frac{1}{2} a_0 \quad , \\ a_3 \;&=\; -\frac{1}{3(3-1)} \sum_{j=0}^{3-2} a_j q_{3-2-j} \\ &=\; -\frac{1}{6} [a_0 q_1 + a_1 q_0] \;=\; -\frac{1}{6} \left[a_0 \left(-\frac{1}{2} \right) + a_1 \cdot 1 \right] \;=\; \frac{1}{12} a_0 \;-\; \frac{1}{6} a_1 \quad , \\ a_4 \;&=\; -\frac{1}{4(4-1)} \sum_{j=0}^{4-2} a_j q_{4-2-j} \\ &=\; -\frac{1}{12} [a_0 q_2 + a_1 q_1 + a_2 q_0] \\ &=\; -\frac{1}{12} \left[a_0 \left(\frac{3}{8} \right) + a_1 \left(-\frac{1}{2} \right) - \frac{1}{2} a_0 \cdot 1 \right] \;=\; \frac{1}{96} a_0 \;+\; \frac{1}{24} a_1 \\ &\;\;\vdots \end{aligned}$$

Thus, the 4^{th}-degree partial sum of the general power series for $Y(X)$ about $X_0 = 0$ is

$$\begin{aligned}\sum_{k=0}^{4} a_k X^k &= a_0 + a_1 X + a_2 X^2 + a_3 X^3 + a_4 X^4 \\ &= a_0 + a_1 X - \frac{1}{2} a_0 X^2 + \left[\frac{1}{12} a_0 - \frac{1}{6} a_1\right] X^3 + \left[\frac{1}{96} a_0 + \frac{1}{24} a_1\right] X^4 \\ &= a_0 \left[1 - \frac{1}{2} X^2 + \frac{1}{12} X^3 + \frac{1}{96} X^4\right] + a_1 \left[X - \frac{1}{6} X^3 + \frac{1}{24} X^4\right] \quad .\end{aligned}$$

Replacing X with $x - 1$ then gives us the 4^{th}-degree partial sum of the general power series solution about $x_0 = 1$ to the original differential equation:

$$\begin{aligned}\sum_{k=0}^{4} a_k (x-1)^k = a_0 &\left[1 - \frac{1}{2}(x-1)^2 + \frac{1}{12}(x-1)^3 + \frac{1}{96}(x-1)^4\right] \\ &+ a_1 \left[(x-1) - \frac{1}{6}(x-1)^3 + \frac{1}{24}(x-1)^4\right] \quad .\end{aligned}$$