Chapter 31: Power Series Solutions II: Generalizations and Theory

31.4 a. Suppose that \( e^z = 0 \) for some \( z = x + iy \). Then both the real and imaginary parts of \( e^z \) must be zero,

\[
\begin{align*}
  e^x \cos(y) &= 0 \\
  e^x \sin(y) &= 0
\end{align*}
\]

and since \( e^x \neq 0 \) for every real value \( x \), we must have

\[
\begin{align*}
  \cos(y) &= 0 \\
  \sin(y) &= 0
\end{align*}
\]

But this is impossible. After all, if \( \sin(y) = 0 \) for some real value \( y \), then \( y = n\pi \) for some integer \( n \), which then means that

\[
\cos(y) = \cos(n\pi) = (-1)^n \neq 0 .
\]

So it is impossible for \( e^z = 0 \). In other words, \( e^z \neq 0 \) for every \( z \) in the complex plane.

31.4 b. We will just verify the claims concerning \( \sin(z) \).

First of all,

\[
\sin(x + iy) = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i}
\]

\[
= \frac{1}{2i} \left[ e^{ix-y} - e^{-ix+y} \right]
\]

\[
= \frac{1}{2i} \left[ e^{-y}e^{ix} - e^y e^{-ix} \right]
\]

\[
= \frac{1}{2i} \left[ e^{-y} \cos(x) + i \sin(x) \right] - e^y \left[ \cos(x) - i \sin(x) \right]
\]

\[
= \frac{1}{2i} \left[ i \left[ e^y + e^{-y} \right] \sin(x) - \left[ e^y - e^{-y} \right] \cos(x) \right]
\]

\[
= \frac{1}{2} \cdot e^{y} + e^{-y} \sin(x) - \frac{1}{2} \cdot e^{y} - e^{-y} \cos(x)
\]

\[
= \frac{1}{2} e^{y} \sin(x) + \frac{1}{2} \cdot e^{y} - e^{-y} \cos(x)
\]

Next, assume \( \sin(z) = 0 \) for some \( z = x + iy \). Then both the real part and imaginary parts of \( \sin(x + iy) \) must equal 0, which, from above, means

\[
\frac{e^y + e^{-y}}{2} \sin(x) = 0 \quad \text{and} \quad \frac{e^y - e^{-y}}{2} \cos(x) = 0 .
\]

Note that since \( e^{y} + e^{-y} > 0 \),

\[
\frac{e^y + e^{-y}}{2} \sin(x) = 0 \quad \Rightarrow \quad \sin(x) = 0
\]

\[
\Rightarrow \quad x = n\pi \quad \text{for} \quad n = 0, \pm 1, \pm 2, \ldots .
\]

But then

\[
0 = \frac{e^y - e^{-y}}{2} \cos(x) = \frac{e^y - e^{-y}}{2} \cos(n\pi) = \frac{e^y - e^{-y}}{2} (-1)^n ,
\]

which means that \( e^{y} - e^{-y} = 0 \). Solving this for \( y \) yields \( y = 0 \). So, if

\[
\sin(z) = 0 .
\]
then
\[ z = x + iy = n\pi + i0 = n\pi \quad \text{with } n = 0, \pm 1, \pm 2, \ldots . \]

31.4 c. This is done in much the same way as the previous. Start by showing

\[
\sinh(x + iy) = \frac{e^{x+iy} - e^{-(x+iy)}}{2} = \frac{1}{2} [e^x [\cos(y) + i \sin(y)] - e^{-x} [\cos(y) - i \sin(y)]] \\
= \ldots \\
= \frac{e^x - e^{-x}}{2} \cos(y) + i \frac{e^x + e^{-x}}{2} \sin(y) .
\]

Next, assume \( \sinh(z) = 0 \) for some \( z = x + iy \). Then both the real part and imaginary parts of \( \sinh(x + iy) \) must equal 0, which, from above, means

\[
\frac{e^x - e^{-x}}{2} \cos(y) = 0 \quad \text{and} \quad \frac{e^x + e^{-x}}{2} \sin(y) = 0 .
\]

Solving this pair of equations then yields

\[ z = x + iy = i \left[ n + \frac{1}{2} \right] \pi \quad \text{with } n = 0, \pm 1, \pm 2, \ldots . \]

31.5 a. Both coefficients — 1 and \( -e^x \) — are analytic, and, from exercise 31.4, we know neither is ever zero. So, lemma 31.4 assures us that there are no singular points. Hence, also, the radius of analyticity \( R \) is \( \infty \).

31.5 c. All three coefficients — \( \sin(x) \), \( x^2 \) and \( -e^x \) — are analytic everywhere, and (from exercise 31.4) we know \( -e^z \neq 0 \) for every \( z \) in the complex plane, while

\[ \sin(z) = 0 \implies z = 0, \pm \pi, \pm 2\pi, \pm 3\pi, \ldots . \]

Lemma 31.5 then assures us that the singular points are all given by

\[ z_s = 0, \pm \pi, \pm 2\pi, \pm 3\pi, \ldots . \]

and since \( z_s = \pi \) is the singular point closest to \( x_0 = 2 \),

\[ R = |\pi - 2| = \pi - 2 . \]

31.5 e. The coefficients are all analytic on the entire complex plane, and (from exercise 31.4) we know

\[ \sinh(z) = 0 \Rightarrow z = in\pi \quad \text{for } n = 0, \pm 1, \pm 2, \pm 3, \ldots . \] \( (\ast) \)

The other coefficients — \( z^2 \) and \( -\sin(x) \) — are clearly nonzero at all these points except \( z = 0 \). So we need to check the point \( z = 0 \) more carefully, by computing

\[ \lim_{x \to 0} \frac{x^2}{\sinh(x)} \quad \text{and} \quad \lim_{x \to 0} \frac{-\sin(x)}{\sinh(x)} . \]
Using L'Hôpital's rule, we have

\[
\lim_{x \to 0} \frac{x^2}{\sinh(x)} = \lim_{x \to 0} \frac{2x}{\cosh(x)} = \frac{0}{1} = 0
\]

and

\[
\lim_{x \to 0} \frac{-\sin(x)}{\sinh(x)} = \lim_{x \to 0} \frac{-\cos(x)}{\cosh(x)} = \frac{-1}{1} = -1 .
\]

Since both of these limits are finite, lemma 31.5 tells us that the point \( x = z = 0 \) is an ordinary point. Hence, all the values of \( z \) given in line (⋆) are singular points except \( z = 0 \).

So,

\[ z_s = i n \pi \text{ with } n = \pm 1, \pm 2, \ldots . \]

Since the singular points closest to \( x_0 = 2 \) are \( \pm i \pi \),

\[ R = |2 - (\pm i \pi)| = \sqrt{2^2 + \pi^2} = \sqrt{4 + \pi^2} . \]

31.5 g. Multiplying through by \( 1 - e^x \), we get

\[
[1 - e^x] y'' + [1 + e^x] y = 0 .
\]

Both coefficients are analytic on the entire complex plane, and, letting \( z = x + iy \),

\[
1 - e^z = 0 \implies 1 = e^z
\]

\[
\iff 1 + i0 = e^{x+iy} = e^x[\cos(y) + i \sin(y)]
\]

\[
\iff 1 = e^x \cos(y) \quad \text{and} \quad 0 = e^x \sin(y)
\]

\[
\iff 1 = e^x \cos(n\pi) \quad \text{and} \quad y = n\pi \quad \text{for} \quad n = 0, \pm 1, \pm 2, \ldots
\]

\[
\iff 1 = e^x \cos(n\pi) = e^x(-1)^n \quad \text{for} \quad n = 0, \pm 1, \pm 2, \ldots
\]

\[
\iff x = 0 \quad \text{and} \quad n \text{ is an even integer}
\]

\[
\iff z = x + iy = 0 + i2k\pi \quad \text{for} \quad k = 0, \pm 1, \pm 2, \pm 3, \ldots \quad (⋆)
\]

Checking, we see that if \( z = i2k\pi \) for any integer \( k \),

\[
1 + e^z = 1 + e^{i2k\pi} = 1 + \cos(2k\pi) + i \sin(2k\pi) = 2 \neq 0 .
\]

Hence, all the points given in line (⋆) are singular points,

\[ z_s = i2k\pi \quad \text{for} \quad k = 0, \pm 1, \pm 2, \pm 3, \ldots , \]

and

\[ R = |x_0 - \text{closest singular point}| = |3 - 0i| = 3 . \]
31.5 i. All the coefficients are analytic on the entire complex plane, with the first being zero if and only if \( x = z = 0 \). But we also have \( 1 - e^z = 0 \) if \( z = 0 \). So we need to see if
\[
\lim_{x \to 0} \frac{1 - e^x}{x}
\]
is finite. Using L’Hôpital’s rule, we see that
\[
\lim_{x \to 0} \frac{1 - e^x}{x} = \lim_{x \to 0} \frac{-e^x}{1} = -1,
\]
which is finite, telling us that \( z = 0 \) is an ordinary point. Hence, there are no singular points, and \( R = \infty \).

31.6 a. The coefficients of this equation — 1 and \(-e^x\) — are analytic everywhere, and the first is simply a nonzero constant. So there are no singular points and the power series solution will be valid on \( I = (-\infty, \infty) \).

The solution is \( \sum_{k=0}^{\infty} a_k x^k \) with \( a_0 \) being arbitrary and
\[
a_k = -\frac{1}{k} \sum_{j=0}^{k-1} a_j p_{k-1-j} \quad \text{for} \quad k > 0
\]
where, in this case,
\[
\sum_{n=0}^{\infty} p_n x^n = P(x) = -e^x = -\sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{-1}{n!} x^n.
\]
So,
\[
p_n = \frac{-1}{n!} \quad \text{for} \quad n = 0, 1, 2, 3, \ldots,
\]
and
\[
a_k = -\frac{1}{k} \sum_{j=0}^{k-1} a_j p_{k-1-j} = -\frac{1}{k} \sum_{j=0}^{k-1} a_j \frac{-1}{(k - 1 - j)!} = \frac{1}{k} \sum_{j=0}^{k-1} a_j \frac{1}{(k - 1 - j)!}.
\]
In particular,
\[
a_1 = \frac{1}{1} \sum_{j=0}^{1-1} a_j \frac{1}{(1 - 1 - j)!} = \sum_{j=0}^{0} a_j \frac{1}{(0 - j)!} = a_0 \frac{1}{(0 - 0)!} = a_0,
\]
\[
a_2 = \frac{1}{2} \sum_{j=0}^{2-1} a_j \frac{1}{(2 - 1 - j)!} = \frac{1}{2} \sum_{j=0}^{1} a_j \frac{1}{(1 - j)!} = \frac{1}{2} \left[ a_0 \frac{1}{(1 - 0)!} + a_1 \frac{1}{(1 - 1)!} \right] = \frac{1}{2} [a_0 + a_0] = a_0,
\]
\[ a_3 = \frac{1}{3} \sum_{j=0}^{3-1} \frac{a_j}{(3-1-j)!} \]
\[ = \frac{1}{3} \sum_{j=0}^{2} \frac{a_j}{(2-j)!} \]
\[ = \frac{1}{3} \left[ \frac{a_0}{(2-0)!} + \frac{a_1}{(2-1)!} + \frac{a_2}{(2-2)!} \right] = \frac{1}{3} \left[ \frac{a_0}{2} + a_0 \frac{1}{1} + a_0 \frac{1}{1} \right] = \frac{5}{6} a_0 \]
and
\[ a_4 = \frac{1}{4} \sum_{j=0}^{4-1} \frac{a_j}{(4-1-j)!} \]
\[ = \frac{1}{4} \sum_{j=0}^{3} \frac{a_j}{(3-j)!} \]
\[ = \frac{1}{4} \left[ \frac{a_0}{(3-0)!} + \frac{a_1}{(3-1)!} + \frac{a_2}{(3-2)!} + \frac{a_3}{(3-3)!} \right] \]
\[ = \frac{1}{4} \left[ a_0 \frac{1}{6} + a_0 \frac{1}{2} + a_0 \frac{1}{1} + \left( \frac{5}{6} a_0 \right) \frac{1}{1} \right] = \frac{5}{8} a_0 \]

Thus, the 4th-degree partial sum of the general power series solution is
\[ \sum_{k=0}^{4} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \]
\[ = a_0 + a_0 x + a_0 x^2 + \frac{5}{6} a_0 x^3 + \frac{5}{8} a_0 x^4 \]
\[ = a_0 \left[ 1 + x + x^2 + \frac{5}{6} x^3 + \frac{5}{8} x^4 \right] . \]

31.6c. The coefficients of this equation — 1 and \( \cos(x) \) — are analytic everywhere, and the first is simply a nonzero constant. So there are no singular points and the power series solution will be valid on \( I = (-\infty, \infty) \).

The solution is \( \sum_{k=0}^{\infty} a_k x^k \) with \( a_0 \) being arbitrary and
\[ a_k = -\frac{1}{k} \sum_{j=0}^{k-1} a_j p_{k-1-j} \quad \text{for} \quad k > 0 \]

where, in this case,
\[ \sum_{n=0}^{\infty} p_n x^n = P(x) = \cos(x) \]
\[ = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m} \]
\[ = 1 x^0 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \ldots \]
\[
= (-1)^0 x^0 + 0 x^1 + (-1)^{2/2} \frac{1}{2!} x^2 + 0 x^3 \\
+ (-1)^{4/2} \frac{1}{4!} x^4 + 0 x^5 + (-1)^{6/2} \frac{1}{6!} x^6 + 0 x^7 + \cdots .
\]

So,
\[
p_0 = 1 , \quad p_1 = 0 , \quad p_2 = -\frac{1}{2!} = -\frac{1}{2} , \quad p_3 = 0
\]

and, in general,
\[
p_n = \begin{cases} (-1)^{n/2} \frac{1}{n!} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}
\]

Applying the above formulas for \(a_k\) and \(p_k\), we get:

\[
a_1 = -\frac{1}{1} \sum_{j=0}^{1-1} a_j p_{1-j}
\]
\[
= -\sum_{j=0}^{0} a_j p_{0-j} = -a_0 p_0 = -a_0 \cdot 1 = -a_0
\]

\[
a_2 = -\frac{1}{2} \sum_{j=0}^{2-1} a_j p_{2-j}
\]
\[
= -\frac{1}{2} \sum_{j=0}^{1} a_j p_{1-j}
\]
\[
= -\frac{1}{2} \left[ a_0 p_1 + a_1 p_0 \right] = -\frac{1}{2} \left[ a_0 \cdot 0 - a_0 \cdot 1 \right] = \frac{1}{2} a_0
\]

\[
a_3 = -\frac{1}{3} \sum_{j=0}^{3-1} a_j p_{3-j}
\]
\[
= -\frac{1}{3} \left[ a_0 p_2 + a_1 p_1 + a_2 p_0 \right]
\]
\[
= -\frac{1}{3} \left[ a_0 \left( -\frac{1}{2} \right) - a_0 \cdot 0 + \frac{1}{2} a_0 \cdot 1 \right] = 0
\]

and
\[
a_4 = -\frac{1}{4} \sum_{j=0}^{4-1} a_j p_{4-j}
\]
\[
= -\frac{1}{4} \left[ a_0 p_3 + a_1 p_2 + a_2 p_1 + a_3 p_0 \right]
\]
\[
= -\frac{1}{4} \left[ a_0 \cdot 0 + (-a_0) \left( -\frac{1}{2} \right) + \frac{1}{2} a_0 \cdot 0 + 0 \cdot 1 \right] = -\frac{1}{8} a_0
\]
Thus, the 4\textsuperscript{th}-degree partial sum of the general power series solution is
\[ \sum_{k=0}^{4} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 = a_0 - a_0 x + \frac{1}{2} a_0 x^2 + 0 a_0 x^3 - \frac{1}{8} a_0 a_0 x^4 = a_0 \left[ 1 - x + \frac{1}{2} x^2 - \frac{1}{8} x^4 \right] . \]

31.7 a. The equation is in reduced form
\[ y'' + Py' + Qy = 0 \]
with
\[ P(x) = 0 = \sum_{n=0}^{\infty} 0 x^n \quad \text{for} \quad |x| < \infty \]
and
\[ Q(x) = -e^x = \sum_{n=0}^{\infty} -\frac{1}{n!} x^n \quad \text{for} \quad |x| < \infty . \]

Since the radii of convergence for each of these two series is \( \infty \), so is the radius of convergence for the power series solution. Thus, the power series solution will be valid on \( I = (-\infty, \infty) \). Moreover, by the above,
\[ \sum_{n=0}^{\infty} p_n x^n = \sum_{n=0}^{\infty} 0 x^n \quad \text{and} \quad \sum_{n=0}^{\infty} q_n x^n = \sum_{n=0}^{\infty} -\frac{1}{n!} x^n , \]
which means
\[ p_n = 0 \quad \text{for} \quad n = 0, 1, 2, 3, \ldots \]
and
\[ q_n = -\frac{1}{n!} = 0, 1, 2, 3, \ldots . \]

In particular, let’s note that
\[ q_0 = \frac{-1}{0!} = -1 , \quad q_1 = \frac{-1}{1!} = -1 \quad \text{and} \quad q_2 = \frac{-1}{2!} = \frac{-1}{2} . \]

The power series solution is given by \( \sum_{k=0}^{\infty} a_k x^k \) where \( a_0 \) and \( a_1 \) are arbitrary, and
\[ a_k = -\frac{1}{k(k-1)} \sum_{j=0}^{k-2} \left[ (j + 1)a_{j+1}p_{k-2-j} + a_{j}q_{j-2} \right] \quad \text{for} \quad k > 2 . \]

In this case, since every \( p_n \) is 0, the above recursion formula simplifies to
\[ a_k = -\frac{1}{k(k-1)} \sum_{j=0}^{k-2} a_{j}q_{k-2-j} \quad \text{for} \quad k > 2 . \]
Applying this recursion formula, we get

\[ a_2 = -\frac{1}{2(2 - 1)} \sum_{j=0}^{2-2} a_j q^{2-j} = -\frac{1}{2} \sum_{j=0}^{0} a_j q^{-j} = -\frac{1}{2} a_0 q^0 = -\frac{1}{2} a_0(-1) = \frac{1}{2} a_0 , \]

\[ a_3 = -\frac{1}{3(3 - 1)} \sum_{j=0}^{3-2} a_j q^{3-j} = -\frac{1}{6} \sum_{j=0}^{1} a_j q^{1-j} = -\frac{1}{6} [a_0 q_1 + a_1 q_0] = -\frac{1}{6} [a_0(-1) + a_1(-1)] = \frac{1}{6} [a_0 + a_1] \]

and

\[ a_4 = -\frac{1}{4(4 - 1)} \sum_{j=0}^{4-2} a_j q^{4-j} = -\frac{1}{12} \sum_{j=0}^{2} a_j q^{2-j} = -\frac{1}{12} [a_0 q_2 + a_1 q_1 + a_2 q_0] = -\frac{1}{12} [a_0 (-\frac{1}{2}) + a_1(-1) + \frac{1}{2} a_0(-1)] = \frac{1}{12} [a_0 + a_1] . \]

Thus, the 4th-degree partial sum of the general power series solution is

\[ \sum_{k=0}^{4} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \]

\[ = a_0 + a_1 x + \frac{1}{2} a_0 x^2 + \frac{1}{6} [a_0 + a_1] x^3 + \frac{1}{12} [a_0 + a_1] x^4 \]

\[ = a_0 \left[ 1 + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \frac{1}{12} x^4 \right] + a_1 \left[ x + \frac{1}{6} x^3 + \frac{1}{12} x^4 \right] . \]

31.7 c. Dividing through by \( x \), we get the equation into reduced form,

\[ y'' - 3y' + \frac{\sin(x)}{x} y = 0 , \]

which is

\[ y'' + Py' + Qy = 0 \quad \text{with} \quad P(x) = -3 \quad \text{and} \quad Q(x) = \frac{\sin(x)}{x} . \]

Obviously, \( P \) and \( Q \) are analytic everywhere except, possibly, for a singularity at \( x = 0 \) for \( Q(x) \) (because of the \( x \) in the denominator). But, using L'Hôpital's rule, we see that

\[ \lim_{x \to 0} Q(x) = \lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1} = 1 , \]
which is finite. So $Q(x)$ is analytic at $x = 0$, as well as everywhere else. Hence, there are no singular points, and $I = (-\infty, \infty)$.

The power series for $P$ and $Q$ about $x_0 = 0$ are easily found. For $P$, we trivially have

$$\sum_{n=0}^{\infty} p_n x^n = P(x) = -3 = -3 + 0x + 0x^2 + 0x^3 + \cdots$$

which means

$$p_0 = -3 \quad \text{while} \quad p_n = 0 \quad \text{for} \quad n \geq 1.$$ 

For $Q$, we use the well-known series for the sine as follows:

$$\sum_{n=0}^{\infty} q_n x^n = Q(x) = \frac{1}{x} \sin(x)$$

$$= x^{-1} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1}$$

$$= x^{-1} \left[ x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \cdots \right]$$

$$= x^0 - \frac{1}{3!} x^2 + \frac{1}{5!} x^4 - \frac{1}{7!} x^6 + \cdots$$

$$= 1x^0 + 0x^1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 + \frac{1}{5!}x^5 - \frac{1}{7!}x^6 + \cdots .$$

Hence,

$$q_0 = 1 \quad , \quad q_1 = 0 \quad , \quad q_2 = -\frac{1}{3!} = -\frac{1}{6} \quad , \quad q_3 = 0 \quad , \quad \ldots.$$

In general,

$$q_n(x) = \begin{cases} (-1)^{n/2} \frac{1}{n+1} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}.$$ 

Using this with recursion formula (31.4), we get

$$a_2 = -\frac{1}{2(2-1)} \sum_{j=0}^{2-2} [(j+1)a_{j+1}p_{2-j} + a_jq_{2-j} - j]$$

$$= -\frac{1}{2} \sum_{j=0}^{0} [(j+1)a_{j+1}p_{2-j} + a_jq_{2-j} - j]$$

$$= -\frac{1}{2} [a_0 p_0 - a_0 q_0]$$

$$= -\frac{1}{2} [a_1(-3) + a_0 \cdot 1] = \frac{1}{2} [3a_1 - a_0] \quad ,$$

$$a_3 = -\frac{1}{3(3-1)} \sum_{j=0}^{3-2} [(j+1)a_{j+1}p_{3-j} + a_jq_{3-j} - j]$$

$$= -\frac{1}{6} \sum_{j=0}^{1} [(j+1)a_{j+1}p_{1-j} + a_jq_{1-j} - j]$$
\[ a_4 = \frac{-1}{4(4-1)} \sum_{j=0}^{4-2} [(j+1)a_{j+1}p_{4-2-j} + a_jq_{4-2-j}] \]

\[ = \frac{-1}{12} \sum_{j=0}^2 [(j+1)a_{j+1}p_{2-j} + a_jq_{2-j}] \]

\[ = \frac{-1}{12} \left( [a_1 p_2 + a_0 q_2] + [2a_2 p_1 + a_1 q_1] + [3a_3 p_0 + a_2 q_0] \right) \]

\[ = \frac{-1}{12} \left( \left[a_1 \cdot 0 + a_0 \left(-\frac{1}{6}\right)\right] + [2a_2 \cdot 0 + a_1 \cdot 0] \right) \]

\[ + \left[ 3 \left(-\frac{1}{2} a_0 + \frac{4}{3} a_1 \right) (3) + \frac{3}{2} [3a_1 - a_0] \cdot 1 \right) \]

\[ = \frac{-23}{72} a_0 + \frac{7}{8} a_1 . \]

Thus, the 4th-degree partial sum of the general power series solution is

\[ \sum_{k=0}^{4} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \]

\[ = a_0 + a_1 x + \frac{1}{2} [3a_1 - a_0] x^2 + \left[ -\frac{1}{2} a_0 + \frac{4}{3} a_1 \right] x^3 \]

\[ + \left[ -\frac{23}{72} a_0 + \frac{7}{8} a_1 \right] x^4 \]

\[ = a_0 \left[ 1 - \frac{1}{2} x^2 - \frac{1}{2} x^3 - \frac{23}{72} x^4 \right] + a_1 \left[ x + \frac{3}{2} x^2 + \frac{4}{3} x^3 + \frac{7}{8} x^4 \right] . \]

31.7 e. The first coefficient is 0 if and only if \( x = 0 \), while the coefficient on \( y \) is never 0. From that it follows that \( z_s = 0 \) is the only singular point,

\[ R = |z_s - x_0| = |0 - 1| = 1 \]

and

\[ I = (x_0 - R, x_0 + R) = (0, 2) . \]

Dividing the equation by the first coefficient and letting \( Y(X) = y(x) \) with \( X = x - 1 \) converts the differential equation to

\[ Y'' + \frac{1}{\sqrt{X+1}} Y = 0 \quad \text{with} \quad X_0 = 0 , \]

which is

\[ Y'' + P Y' + Q Y = 0 \quad \text{with} \quad P(X) = 0 \quad \text{and} \quad Q(X) = (X + 1)^{-1/2} . \]
Obviously,\[ P(X) = \sum_{n=0}^{\infty} p_n X^n \quad \text{with} \quad p_n = 0 \quad \text{for all} \quad n . \]

Thus, the recursion formula for the power series of $Y$ about $X_0$ reduces a bit,

\[
  a_k = -\frac{1}{k(k-1)} \sum_{j=0}^{k-2} [(j + 1)a_{j+1}p_{k-2-j} + a_jq_{k-2-j}] \\
  = -\frac{1}{k(k-1)} \sum_{j=0}^{k-2} a_jq_{k-2-j} .
\]

For $Q$, we use the Taylor series,\[
  \sum_{n=0}^{\infty} q_n X^n = Q(x) = \sum_{n=0}^{\infty} \frac{Q^{(n)}(0)}{n!} X^n .
\]

So,
\[
  q_0 = \frac{Q^{(0)}(0)}{0!} = Q(0) = (0 + 1)^{-\frac{1}{2}} = 1 ,
\]
\[
  q_1 = \frac{Q^{(1)}(0)}{1!} = Q'(0) = -\frac{1}{2} (0 + 1)^{-\frac{3}{2}} = -\frac{1}{2} ,
\]
\[
  q_2 = \frac{Q^{(2)}(0)}{2!} = \frac{1}{2!} Q''(0) = \frac{1}{2!} \frac{3}{2^2} (0 + 1)^{-\frac{5}{2}} = \frac{3}{8} ,
\]
\[
  q_3 = \frac{Q^{(3)}(0)}{3!} = \frac{1}{3!} Q^{(3)}(0) = \frac{1}{3!} \frac{\cdot 3 \cdot 5}{2^3} (0 + 1)^{-\frac{7}{2}} = \frac{5}{16} ,
\]

\vdots

Plugging the above into our recursion formula, we get

\[
  a_2 = -\frac{1}{2(2-1)} \sum_{j=0}^{2-2} a_j q_{2-2-j} = -\frac{1}{2} a_0 q_0 = -\frac{1}{2} a_0 \cdot 1 = -\frac{1}{2} a_0 ,
\]
\[
  a_3 = -\frac{1}{3(3-1)} \sum_{j=0}^{3-2} a_j q_{3-2-j} \\
  = -\frac{1}{6} [a_0 q_1 + a_1 q_0] = -\frac{1}{6} \left[ a_0 \left( -\frac{1}{2} \right) + a_1 \cdot 1 \right] = \frac{1}{12} a_0 - \frac{1}{6} a_1 ,
\]
\[
  a_4 = -\frac{1}{4(4-1)} \sum_{j=0}^{4-2} a_j q_{4-2-j} \\
  = -\frac{1}{12} [a_0 q_2 + a_1 q_1 + a_2 q_0] \\
  = -\frac{1}{12} \left[ a_0 \left( \frac{3}{8} \right) + a_1 \left( -\frac{1}{2} \right) + a_2 q_0 \right] = \frac{1}{96} a_0 + \frac{1}{24} a_1 ,
\]
\vdots
Thus, the 4\textsuperscript{th}-degree partial sum of the general power series for $Y(X)$ about $X_0 = 0$ is

$$\sum_{k=0}^{4} a_k X^k = a_0 + a_1 X + a_2 X^2 + a_3 X^3 + a_4 X^4$$

$$= a_0 + a_1 X - \frac{1}{2} a_0 X^2 + \left[ \frac{1}{12} a_0 - \frac{1}{6} a_1 \right] X^3 + \left[ \frac{1}{96} a_0 + \frac{1}{24} a_1 \right] X^4$$

$$= a_0 \left[ 1 - \frac{1}{2} X^2 + \frac{1}{12} X^3 + \frac{1}{96} X^4 \right] + a_1 \left[ X - \frac{1}{6} X^3 + \frac{1}{24} X^4 \right].$$

Replacing $X$ with $x - 1$ then gives us the 4\textsuperscript{th}-degree partial sum of the general power series solution about $x_0 = 1$ to the original differential equation:

$$\sum_{k=0}^{4} a_k (x - 1)^k = a_0 \left[ 1 - \frac{1}{2} (x - 1)^2 + \frac{1}{12} (x - 1)^3 + \frac{1}{96} (x - 1)^4 \right]$$

$$+ a_1 \left[ (x - 1) - \frac{1}{6} (x - 1)^3 + \frac{1}{24} (x - 1)^4 \right].$$