Chapter 3: Some Basics about First-Order Equations

3.4 a. Rewriting the differential equation in derivative formula form:

$$\frac{dy}{dx} + 3xy = 6x \quad \rightarrowtail \quad \frac{dy}{dx} = 6x - 3xy \quad .$$

To find the constant solutions, replace the derivative with zero and solve for y:

$$0 = 6x - 3xy \quad \rightarrowtail \quad 0 = 3x(2 - y) \quad \rightarrowtail \quad y = 2 \quad .$$

So $y = 2$ is the only constant solution.

3.4 c. Rewriting the differential equation in derivative formula form:

$$\frac{dy}{dx} - y^3 = 8 \quad \rightarrowtail \quad \frac{dy}{dx} = 8 + y^3 \quad .$$

To find the constant solutions, replace the derivative with zero and solve for y:

$$0 = 8 + y^3 \quad \rightarrowtail \quad y^3 = -8 \quad \rightarrowtail \quad y = \sqrt[3]{-8} = -2 \quad .$$

So $y = -2$ is the only constant solution.

3.4 e. Derivative formula form: $\frac{dy}{dx} - y^2 = x \quad \rightarrowtail \quad \frac{dy}{dx} = x + y^2 \quad .$
Constant solutions: $0 = x + y^2 \quad \rightarrowtail \quad y^2 = -x \quad .$
But x is not a constant. Hence, there are no constant solutions.

3.4 g. Derivative formula form: $(x - 2)\frac{dy}{dx} = y + 3 \quad \rightarrowtail \quad \frac{dy}{dx} = \frac{y + 3}{x - 2} \quad .$
Constant solutions: $0 = \frac{y + 3}{x - 2} \quad \rightarrowtail \quad 0 = y + 3 \quad \rightarrowtail \quad y = -3 \quad .$

3.4 i. Derivative formula form: $\frac{dy}{dx} + 2y - y^2 = -2 \quad \rightarrowtail \quad \frac{dy}{dx} = y^2 - 2y - 2 \quad .$
Constant solutions:

$$0 = y^2 - 2y - 2 \quad \rightarrowtail \quad y = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(-2)}}{2} = 1 \pm \sqrt{3} \quad .$$

So there are two constant solutions: $y = 1 + \sqrt{3}$ and $y = 1 - \sqrt{3}$.

3.7 a. Here, $F(x, y) = xy$. Thus, $F(s, \psi(s)) = s\psi(s)$. Using this and the formula for Picard's method, we then have

$$\begin{aligned}\psi_1(x) &= y_0 + \int_0^x F(s, \psi_0(s))\, ds \\ &= 2 + \int_0^x s\psi_0(s)\, ds = 2 + \int_0^x s2\, ds = 2 + x^2 \quad .\end{aligned}$$

3.7 b.

$$\psi_2(x) = 2 + \int_0^x s\psi_1(s)\,ds = 2 + \int_0^x s\left(2 + s^2\right) ds$$
$$= 2 + \int_0^x \left(2s + s^3\right) ds = 2 + x^2 + \tfrac{1}{4}x^4 \quad .$$

3.7 c.

$$\psi_3(x) = 2 + \int_0^x s\psi_2(s)\,ds$$
$$= 2 + \int_0^x s\left(2 + s^2 + \frac{1}{4}s^4\right) ds$$
$$= 2 + \int_0^x \left(2s + s^3 + \frac{1}{4}s^5\right) ds = 2 + x^2 + \tfrac{1}{4}x^4 + \frac{1}{24}x^6 \quad .$$