Chapter 29: Brief Review of Infinite Series and Power Series

29.2 a. Using partial sum formula (29.2) on page 564 from the text,

$$\sum_{k=0}^{4}\left(\frac{1}{3}\right)^k = \frac{1-\left(\frac{1}{3}\right)^{4+1}}{1-\frac{1}{3}} = \frac{1-\left(\frac{1}{3}\right)^5}{\frac{2}{3}} = \frac{121}{81} \quad .$$

29.2 c. Since $\left|\frac{1}{3}\right| < 1$, theorem 29.1 on page 565 of the text applies, giving us

$$\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)^k = \frac{1}{1-\frac{1}{3}} = \frac{3}{2} \quad .$$

29.2 e. $$\sum_{k=0}^{\infty}\left(-\frac{2}{3}\right)^k = \frac{1}{1-\left(-\frac{2}{3}\right)} = \frac{3}{5} \quad .$$

29.2 g. This geometric series diverges since $\left|\frac{3}{2}\right| > 1$.

29.2 i. $$\begin{aligned}\sum_{k=0}^{\infty}\left[3\left(\frac{2}{5}\right)^k - 4\left(\frac{3}{5}\right)^k\right] &= 3\sum_{k=0}^{\infty}\left(\frac{2}{5}\right)^k - 4\sum_{k=0}^{\infty}\left(\frac{3}{5}\right)^k \\ &= 3\left[\frac{1}{1-\frac{2}{5}}\right] - 4\left[\frac{1}{1-\frac{3}{5}}\right] = 3\left[\frac{5}{3}\right] - 4\left[\frac{5}{2}\right] = -5 \quad .\end{aligned}$$

29.3 a. $$\begin{aligned}\sum_{k=0}^{\infty}\frac{1}{k+1}x^k + \sum_{k=2}^{\infty}\frac{1}{k-1}x^k &= \left[\frac{1}{0+1}x^0 + \frac{1}{1+1}x^1 + \sum_{k=2}^{\infty}\frac{1}{k+1}x^k\right] + \sum_{k=2}^{\infty}\frac{1}{k-1}x^k \\ &= 1 + \frac{1}{2}x + \sum_{k=2}^{\infty}\left[\frac{1}{k+1} + \frac{1}{k-1}\right]x^k \\ &= 1 + \frac{1}{2}x + \sum_{k=2}^{\infty}\left[\frac{[k-1]+[k+1]}{(k+1)(k-1)}\right]x^k \\ &= 1 + \frac{1}{2}x + \sum_{k=2}^{\infty}\frac{2k}{k^2-1}x^k \quad .\end{aligned}$$

29.3 c. Using the change of index $n = k-2$ (i.e., $k = n+2$),

$$\sum_{k=2}^{\infty}(k-1)x^{k-2} = \sum_{n+2=2}^{\infty}([n+2]-1)x^n = \sum_{n=0}^{\infty}(n+1)x^n \quad .$$

29.3 e. Using the change of variable $n = k + 1$ (i.e., $k = n - 1$) in the first summation, and $n = k - 1$ (i.e., $k = n + 1$) in the second,

$$\begin{aligned}
\sum_{k=0}^{\infty}(k+1)x^{k+1} - \sum_{k=4}^{\infty}(k-1)x^{k-1} &= \sum_{n-1=0}^{\infty} nx^n - \sum_{n+1=4}^{\infty} nx^n \\
&= \sum_{n=1}^{\infty} nx^n - \sum_{n=3}^{\infty} nx^n \\
&= \left[1x^1 + 2x^2 + \sum_{n=3}^{\infty} n\,x^n\right] - \sum_{n=3}^{\infty} nx^n \\
&= x + 2x^2 \quad .
\end{aligned}$$

29.3 g.

$$\begin{aligned}
\left(x^2+5\right)\sum_{k=0}^{\infty} a_k x^k &= x^2\sum_{k=0}^{\infty} a_k x^k + 5\sum_{k=0}^{\infty} a_k x^k \\
&= \underbrace{\sum_{k=0}^{\infty} a_k x^{k+2}}_{n=k+2} + \underbrace{\sum_{k=0}^{\infty} 5a_k x^k}_{n=k} \\
&= \sum_{n-2=0}^{\infty} a_{n-2}x^n + \sum_{n=0}^{\infty} 5a_n x^n \\
&= \sum_{n=2}^{\infty} a_{n-2}x^n + \left[5a_0x^0 + 5a_1x^1 + \sum_{n=2}^{\infty} 5a_n x^n\right] \\
&= 5a_0 + 5a_1 x + \sum_{n=2}^{\infty}\left[a_{n-2} + 5a_n\right]x^n \quad .
\end{aligned}$$

29.4 a.

$$\begin{aligned}
\sum_{k=0}^{\infty}\frac{1}{k+1}x^k - \sum_{k=1}^{\infty}\frac{1}{k}x^k &= \left[\frac{1}{0+1}x^0 + \sum_{k=1}^{\infty}\frac{1}{k+1}x^k\right] - \sum_{k=1}^{\infty}\frac{1}{k}x^k \\
&= 1 + \sum_{k=1}^{\infty}\left[\frac{1}{k+1} - \frac{1}{k}\right]x^k \\
&= 1 + \sum_{k=1}^{\infty}\frac{k-[k+1]}{k(k+1)}x^k = 1 + \sum_{k=1}^{\infty}\frac{-1}{k(k+1)}x^k \quad .
\end{aligned}$$

29.4 c. Using the change of index $n = k + 3$ (so $k = n - 3$),

$$\sum_{k=1}^{\infty} 3k^2(x-5)^{k+3} = \sum_{n-3=1}^{\infty} 3(n-3)^2(x-5)^n = \sum_{n=4}^{\infty} 3(n-3)^2(x-5)^n \quad .$$

29.4 e.

$$
\begin{aligned}
(x-3)\sum_{k=2}^{\infty}k(k-1)x^{k-2} &= x\sum_{k=2}^{\infty}k(k-1)x^{k-2} - 3\sum_{k=2}^{\infty}k(k-1)x^{k-2} \\
&= \underbrace{\sum_{k=2}^{\infty}k(k-1)x^{k-1}}_{n=k-1} - \underbrace{\sum_{k=2}^{\infty}3k(k-1)x^{k-2}}_{n=k-2} \\
&= \sum_{n+1=2}^{\infty}(n+1)nx^{n} - \sum_{n+2=2}^{\infty}3(n+2)([n+2]-1)x^{n} \\
&= \sum_{n=1}^{\infty}(n+1)nx^{n} - \sum_{n=0}^{\infty}3(n+2)(n+1)x^{n} \\
&= \sum_{n=1}^{\infty}(n+1)nx^{n} \\
&\qquad - \left[3(0+2)(0+1)x^{0} + \sum_{n=1}^{\infty}3(n+2)(n+1)x^{n}\right] \\
&= -6 + \sum_{n=1}^{\infty}[(n+1)n - 3(n+2)(n+1)]\,x^{n} \\
&= -6 + \sum_{n=1}^{\infty}[-2(n+1)(n+3)]\,x^{n} \quad .
\end{aligned}
$$

29.4 g. Using the change of index $n = k+3$,

$$
\sum_{k=1}^{\infty}k^{2}a_{k}x^{k+3} = \sum_{n-3=1}^{\infty}(n-3)^{2}a_{n-3}x^{n} = \sum_{n=4}^{\infty}(n-3)^{2}a_{n-3}x^{n} \quad .
$$

29.4 i.

$$
\begin{aligned}
&\sum_{k=0}^{\infty}(k+1)a_{k}x^{k+1} - \sum_{k=4}^{\infty}(k-1)a_{k}x^{k-1} \\
&\quad = \underbrace{\sum_{k=0}^{\infty}(k+1)a_{k}x^{k+1}}_{n=k+1} - \underbrace{\sum_{k=4}^{\infty}(k-1)a_{k}x^{k-1}}_{n=k-1} \\
&\quad = \sum_{n-1=0}^{\infty}na_{n-1}x^{n} - \sum_{n+1=4}^{\infty}na_{n+1}x^{n} \\
&\quad = \sum_{n=1}^{\infty}na_{n-1}x^{n} - \sum_{n=3}^{\infty}na_{n+1}x^{n} \\
&\quad = 1a_{1-1}x^{1} + 2a_{2-1}x^{2} - \sum_{n=3}^{\infty}na_{n-1}x^{n} + \sum_{n=3}^{\infty}na_{n+1}x^{n} \\
&\quad = a_{0}x + 2a_{1}x^{2} - \sum_{n=3}^{\infty}n\left[a_{n-1} - a_{n+1}\right]x^{n} \quad .
\end{aligned}
$$

29.4 k.

$$x^2\sum_{k=2}^{\infty}k(k-1)a_kx^{k-2} \;-\; 4\sum_{k=0}^{\infty}a_kx^k$$

$$= \sum_{k=2}^{\infty}k(k-1)a_kx^k \;-\; \sum_{k=0}^{\infty}4a_kx^k$$

$$= \sum_{k=2}^{\infty}k(k-1)a_kx^k \;-\; \left[4a_0x^0 + 4a_1x^1 + \sum_{k=2}^{\infty}4a_kx^k\right]$$

$$= -4a_0 \;-\; 4a_1x \;+\; \sum_{k=2}^{\infty}[k(k-1)a_k - 4a_k]\,x^k$$

$$= -4a_0 \;-\; 4a_1x \;+\; \sum_{k=2}^{\infty}\left[k^2 - k - 4\right]a_kx^k \quad .$$

29.5 a. Note that

$$\frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{-(-1)}{(1-x)^2} = \frac{1}{(1-x)^2} \quad .$$

So, for $|x| < 1$ (since the given power series converges for $|x| < 1$),

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{d}{dx}\left[\sum_{k=0}^{\infty}x^k\right]$$

$$= \sum_{k=0}^{\infty}\frac{d}{dx}\left[x^k\right]$$

$$= \underbrace{\sum_{k=1}^{\infty}kx^{k-1}}_{n=k-1}$$

$$= \underbrace{\sum_{n=0}^{\infty}(n+1)x^n}_{k=n} = \sum_{k=0}^{\infty}(k+1)x^k \quad .$$

29.6 a. Let $f(x) = e^x$. Since the derivative of e^x is simply e^x, it is clear that, for every nonnegative integer k,

$$f^{(k)}(x) = e^x \qquad \text{and} \qquad f^{(k)}(0) = e^0 = 1 \quad .$$

So, the Taylor series about $x_0 = 0$ for e^x is

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k = \sum_{k=0}^{\infty}\frac{1}{k!}x^k \quad .$$

29.6 c. Letting $f(x) = \sin(x)$, we have

$$f(x) = \sin(x) \qquad \text{and} \qquad f(0) = \sin(0) = 0 \quad ,$$

$$f'(x) = \cos(x) \qquad \text{and} \qquad f'(0) = \cos(0) = 1 \quad ,$$

$$\begin{aligned}
f''(x) &= -\sin(x) &\quad\text{and}\quad& f''(0) = -\sin(0) = 0 \quad,\\
f^{(3)}(x) &= -\cos(x) &\quad\text{and}\quad& f^{(3)}(0) = -\cos(0) = -1 \quad,\\
f^{(4)}(x) &= \sin(x) &\quad\text{and}\quad& f^{(4)}(0) = \sin(0) = 0 \quad,\\
f^{(5)}(x) &= \cos(x) &\quad\text{and}\quad& f^{(5)}(0) = \cos(0) = 1 \quad,\\
f^{(6)}(x) &= -\sin(x) &\quad\text{and}\quad& f^{(6)}(0) = -\sin(0) = 0 \quad,\\
f^{(7)}(x) &= -\cos(x) &\quad\text{and}\quad& f^{(7)}(0) = -\cos(0) = -1 \quad,\\
&&\vdots&
\end{aligned}$$

with the pattern obviously continuing. So, the Taylor series about $x_0 = 0$ for $\sin(x)$ is

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k &= \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3\\
&\quad + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \frac{f^{(6)}(0)}{6!}x^6 + \frac{f^{(7)}(0)}{7!}x^6 + \cdots\\
&= \frac{0}{0!} + \frac{1}{1!}x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3\\
&\quad + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \frac{0}{6!}x^6 + \frac{-1}{7!}x^7 + \cdots\\
&= \frac{1}{1!}x + \frac{-1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{-1}{7!}x^7 + \frac{1}{9!}x^9 + \frac{-1}{11!}x^{11} + \cdots\\
&= \frac{(-1)^0}{(2\cdot 0+1)!}x^{2\cdot 0+1} + \frac{(-1)^1}{(2\cdot 1+1)!}x^{2\cdot 1+1} + \frac{(-1)^2}{(2\cdot 2+1)!}x^{2\cdot 2+1}\\
&\quad + \frac{(-1)^3}{(2\cdot 3+1)!}x^{2\cdot 3+1} + \cdots + \frac{(-1)^k}{(2\cdot k+1)!}x^{2\cdot k+1} + \cdots\\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1} \quad .
\end{aligned}$$

29.7 a. Letting $f(x) = \sqrt{1+x}$, we have

$$\begin{aligned}
f(x) &= (1+x)^{1/2} &\quad\text{and}\quad& f(0) = (1+0)^{1/2} = 1 \quad,\\
f'(x) &= \tfrac{1}{2}(1+x)^{-1/2} &\quad\text{and}\quad& f'(0) = \tfrac{1}{2}(1+x)^{-1/2} = \tfrac{1}{2} \quad,\\
f''(x) &= \tfrac{-1}{4}(1+x)^{-3/2} &\quad\text{and}\quad& f''(0) = \tfrac{-1}{4}(1+x)^{-3/2} = \tfrac{-1}{4} \quad,\\
f^{(3)}(x) &= \tfrac{3}{8}(1+x)^{-5/2} &\quad\text{and}\quad& f^{(3)}(0) = \tfrac{3}{8}(1+x)^{-5/2} = \tfrac{3}{8} \quad,\\
f^{(4)}(x) &= \tfrac{-15}{16}(1+x)^{-7/2} &\quad\text{and}\quad& f^{(4)}(0) = \tfrac{-15}{16}(1+x)^{-7/2} = \tfrac{-15}{16} \quad.
\end{aligned}$$

So, the the fourth partial sum of the power series about 0 for $f(x) = \sqrt{1+x}$ is

$$\frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$
$$= \frac{1}{1} + \frac{1/2}{1}x + \frac{-1/4}{2}x^2 + \frac{3/8}{6}x^3 + \frac{-15/16}{24}x^4$$
$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 \quad .$$

29.7 b. Note that $g(x)$ is simply the $f(x)$ from the previous part with $-x^2$ replacing x. So, the first five terms of the power series about 0 for g(x) is simply the formula derived in the previous part with $-x^2$ replacing x:

$$1 + \frac{1}{2}\left(-x^2\right) - \frac{1}{8}\left(-x^2\right)^2 + \frac{1}{16}\left(-x^2\right)^3 - \frac{5}{128}\left(-x^2\right)^4$$
$$= 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \frac{5}{128}x^8 \quad .$$

29.7 c. Differentiating g we get

$$g'(x) = \frac{d}{dx}\left(1-x^2\right)^{1/2} = \cdots = -x\left(1-x^2\right)^{-1/2} = -xh(x) \quad .$$

Solving for $h(x)$ then yields

$$h(x) = -\frac{g'(x)}{x} \quad .$$

Consequently, the first four terms of the power series about 0 for $h(x)$ is simply $-\frac{1}{x}$ times the derivative of the first four nonconstant terms of the power series about 0 for $g(x)$ (found in the previous part):

$$-\frac{1}{x}\cdot\frac{d}{dx}\left[1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \frac{5}{128}x^8\right]$$
$$= -\frac{1}{x}\left[0 - \frac{2}{2}x - \frac{4}{8}x^3 - \frac{6}{16}x^5 - \frac{5\cdot 8}{128}x^7\right]$$
$$= 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 \quad .$$

29.9 a. We know

$$\frac{1}{1-X} = \sum_{k=0}^{\infty} X^k \qquad \text{for} \quad |X| < 1 \quad . \qquad (\star)$$

Letting $X = 2x$, we get

$$\frac{1}{1-(2x)} = \sum_{k=0}^{\infty}(2x)^k \qquad \text{for} \quad |2x| < 1$$
$$= \sum_{k=0}^{\infty} 2^k x^k \qquad \text{for} \quad |x| < \frac{1}{2} \quad .$$

And since geometric series ($\star$) diverges whenever $|2x| = |X| > 1$, the radius of convergence is $1/2$.

29.9 c. Using the known fact that

$$\frac{1}{1-X} = \sum_{k=0}^{\infty} X^k \qquad \text{for} \quad |X| < 1$$

along with a little algebra, we have

$$\frac{2}{2-x} = \frac{2}{2\left(1-\frac{x}{2}\right)} = \frac{1}{1-\frac{x}{2}} = \sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^k \qquad \text{for} \quad \left|\frac{x}{2}\right| < 1$$

$$= \sum_{k=0}^{\infty} \frac{1}{2^k} x^k \qquad \text{for} \quad |x| < 2 \quad .$$

And since the above geometric series diverges whenever $\left|\frac{x}{2}\right| = |X| > 1$, the radius of convergence is 2.

29.9 e. Using the known power series for the exponential,

$$e^X = \sum_{k=0}^{\infty} \frac{1}{k!} X^k \qquad \text{for} \quad |X| < \infty \quad ,$$

we have

$$e^{-x^2} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(-x^2\right)^k \qquad \text{for} \quad \left|-x^2\right| < \infty$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^{2k} \qquad \text{for} \quad |x| < \infty \quad .$$

And so the radius of convergence is ∞.

29.10 a. For $x \neq 0$,

$$f'(x) = \frac{d}{dx}\left[e^{-1/x^2}\right] = e^{-1/x^2}\frac{d}{dx}\left[-\frac{1}{x^2}\right] = e^{-1/x^2}\,\frac{2}{x^3} \quad .$$

For $x = 0$,

$$f'(0) = \lim_{\Delta x \to 0} \frac{f(0+\Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{e^{-1/(\Delta x)^2} - 0}{\Delta x} = \lim_{\Delta x \to 0} \frac{e^{-1/(\Delta x)^2}}{\Delta x} \quad ,$$

which, naively computed, yields the indeterminant form ${}^0\!/_0$. So we will try using L'Hôpital's rule to compute the limit. In this case, though, it helps to rewrite the quotient slightly:

$$f'(0) = \lim_{\Delta x \to 0} \frac{e^{-1/(\Delta x)^2}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\left(\frac{1}{\Delta x}\right)}{e^{1/(\Delta x)^2}}$$

$$= \lim_{\Delta x \to 0} \frac{\left(\frac{1}{\Delta x}\right)'}{\left(e^{1/(\Delta x)^2}\right)'}$$

$$= \lim_{\Delta x \to 0} \frac{-\dfrac{1}{(\Delta x)^2}}{e^{1/(\Delta x)^2} \dfrac{2}{(\Delta x)^3}}$$

$$= -\frac{1}{2} \lim_{\Delta x \to 0} (\Delta x) e^{-1/(\Delta x)^2} = 0 \quad .$$

In summary,

$$f'(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ e^{-1/x^2} \dfrac{2}{x^3} & \text{if} \quad x \neq 0 \end{cases} \quad .$$

29.10 c. The Taylor series is $\displaystyle \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{0}{k!} x^k = 0 \quad .$