Chapter 27: Piecewise-Defined Functions and Periodic Functions

27.2 a. Using the first translation identity, we have

$$\mathcal{L}\Big[e^{4t}\,\text{step}_6(t)\Big]\Big|_s \;=\; \mathcal{L}\big[e^{4t}\underbrace{\text{step}_6(t)}_{f(t)}\big]\big|_s \;=\; \mathcal{L}\big[e^{4t}f(t)\big]\big|_s \;=\; F(s-4)$$

with

$$F(s) \;=\; \mathcal{L}[f(t)]|_s \;=\; \mathcal{L}\big[\text{step}_6(t)\big]\big|_s \;=\; \frac{e^{-6s}}{s} \qquad \text{for} \quad s>0$$

$$\hookrightarrow \qquad F(X) \;=\; \frac{e^{-6X}}{X} \qquad \text{for} \quad X>0$$

$$\hookrightarrow \qquad F(s-4) \;=\; \frac{e^{-6(s-4)}}{s-4} \qquad \text{for} \quad s-4>0 \quad .$$

So the first line in our computations continues as

$$\mathcal{L}\Big[e^{4t}\,\text{step}_6(t)\Big]\Big|_s \;=\; \cdots \;=\; F(s-4) \;=\; \frac{e^{-6(s-4)}}{s-4} \qquad \text{for} \quad s>4 \quad .$$

27.3 a. $$\mathcal{L}^{-1}\left[\frac{e^{-4s}}{s^3}\right]\bigg|_t \;=\; \mathcal{L}^{-1}\left[e^{-4s}\frac{1}{s^3}\right]\bigg|_t \;=\; \mathcal{L}^{-1}\Big[e^{-4s}F(s)\Big]\Big|_t \;=\; f(t-4)\,\text{step}_4(t)$$

with

$$f(t) \;=\; \mathcal{L}^{-1}[F(s)]|_t \;=\; \mathcal{L}^{-1}\left[\frac{1}{s^3}\right]\bigg|_t \;=\; \frac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{s^{2+1}}\right]\bigg|_t \;=\; \frac{1}{2}t^2$$

$$\hookrightarrow \qquad f(X) \;=\; \frac{1}{2}X^2 \quad \rightarrowtail \quad f(t-4) \;=\; \frac{1}{2}(t-4)^2 \quad .$$

So the first line in our computations continues as

$$\mathcal{L}^{-1}\left[\frac{e^{-4s}}{s^3}\right]\bigg|_t \;=\; \cdots \;=\; f(t-4)\,\text{step}_4(t) \;=\; \frac{1}{2}(t-4)^2\,\text{step}_4(t) \quad .$$

27.3 c. $$\mathcal{L}^{-1}\Big[\sqrt{\pi}\,s^{-3/2}e^{-s}\Big]\Big|_t \;=\; \mathcal{L}^{-1}\left[e^{-1s}\frac{\sqrt{\pi}}{s^{3/2}}\right]\bigg|_t$$
$$\;=\; \mathcal{L}^{-1}\Big[e^{-1s}F(s)\Big]\Big|_t \;=\; f(t-1)\,\text{step}_1(t)$$

with

$$f(t) \;=\; \mathcal{L}^{-1}[F(s)]|_t \;=\; \mathcal{L}^{-1}\left[\frac{\sqrt{\pi}}{s^{3/2}}\right]\bigg|_t$$
$$\;=\; \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2}+1\right)}\mathcal{L}^{-1}\left[\frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{1/2+1}}\right]\Bigg|_t \;=\; \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2}+1\right)}t^{1/2} \quad .$$

Replacing t with X and recalling that

$$\Gamma\left(\frac{1}{2}+1\right) \;=\; \frac{1}{2}\Gamma\left(\frac{1}{2}\right) \;=\; \frac{1}{2}\sqrt{\pi} \quad ,$$

we see that

$$f(X) \;=\; 2X^{1/2} \;=\; 2\sqrt{X} \quad ,$$

$$f(t-1) = 2\sqrt{t-1}$$

and

$$\mathcal{L}^{-1}\left[\sqrt{\pi}s^{-3/2}e^{-s}\right]\Big|_t = \cdots = f(t-1)\,\text{step}_1(t) = 2\sqrt{t-1}\,\text{step}_1(t) \quad .$$

27.3 e. Applying the second translation identity, followed by the first translation identity, we have

$$\mathcal{L}^{-1}\left[\frac{e^{-4s}}{(s-5)^3}\right]\Big|_t = \mathcal{L}^{-1}\left[e^{-4s}\frac{1}{(s-5)^3}\right]\Big|_t = \mathcal{L}^{-1}\left[e^{-4s}F(s)\right]\Big|_t = f(t-4)\,\text{step}_4(t)$$

where

$$F(s) = \frac{1}{(s-5)^3} = G(s-5)$$

and

$$f(t) = \mathcal{L}^{-1}[F(s)]\,t = \mathcal{L}^{-1}[G(s-5)]|_t = e^{5t}g(t) \quad .$$

In this case,

$$G(s-5) = \frac{1}{(s-5)^3} \quad \rightarrowtail \quad G(X) = \frac{1}{X^3}$$

$$\hookrightarrow \quad g(t) = \mathcal{L}^{-1}[G(s)]|_t = \mathcal{L}^{-1}\left[\frac{1}{s^3}\right]\Big|_t = \frac{1}{2}t^2$$

$$\hookrightarrow \quad f(t) = e^{5t}g(t) = \frac{1}{2}t^2e^{5t}$$

$$\hookrightarrow \quad f(X) = \frac{1}{2}X^2e^{5X}$$

$$\hookrightarrow \quad f(t-4) = \frac{1}{2}(t-4)^2e^{5(t-4)}$$

$$\hookrightarrow \quad \mathcal{L}^{-1}\left[\frac{e^{-4s}}{(s-5)^3}\right]\Big|_t = f(t-4)\,\text{step}_4(t) = \frac{1}{2}(t-4)^2e^{5(t-4)}\,\text{step}_4(t) \quad .$$

27.5 a. Computing the inverse transform:

$$\mathcal{L}^{-1}\left[\frac{1-e^{-s}}{s^2}\right]\Big|_t = \mathcal{L}^{-1}\left[\frac{1}{s^2} - e^{-s}\frac{1}{s^2}\right]\Big|_t$$

$$= \mathcal{L}^{-1}\left[\frac{1}{s^2}\right]\Big|_t - \mathcal{L}^{-1}\left[e^{-1s}\underbrace{\frac{1}{s^2}}_{F(s)}\right]\Big|_t = t + f(t-1)\,\text{step}_1(t)$$

where

$$f(t) = \mathcal{L}^{-1}[F(s)]|_t = \mathcal{L}^{-1}\left[\frac{1}{s^2}\right]\Big|_t = t \quad .$$

So,

$$f(X) = X \quad ,$$

$$f(t-1) = t - 1$$

and

$$\mathcal{L}^{-1}\left[\frac{1-e^{-s}}{s^2}\right]\Big|_t = t - f(t-1)\,\text{step}_1(t) = t - (t-1)\,\text{step}_1(t) \quad .$$

Converting to a set of conditional formulas:
If $t < 1$,

$$\mathcal{L}^{-1}\left[\frac{1-e^{-s}}{s^2}\right]\Big|_t = t - (t-1)\,\text{step}_1(t) = t - (t-1)\cdot 0 = t \quad .$$

If $1 < t$,

$$\mathcal{L}^{-1}\left[\frac{1-e^{-s}}{s^2}\right]\Big|_t = t - (t-1)\,\text{step}_1(t)$$

$$= t - (t-1)\cdot 1 = t - (t-1) = 1 \quad .$$

In summary:

$$\mathcal{L}^{-1}\left[\frac{1-e^{-s}}{s^2}\right]\Big|_t = \begin{cases} t & \text{if} \quad t < 1 \\ 1 & \text{if} \quad 1 < t \end{cases} \quad .$$

27.5 c. Computing the inverse transform:

$$\mathcal{L}^{-1}\left[\frac{2}{s^3} - \frac{2+4s}{s^3}e^{-2s}\right]\Big|_t = \mathcal{L}^{-1}\left[\frac{2}{s^3}\right]\Big|_t - \mathcal{L}^{-1}\Big[e^{-2s}\underbrace{\frac{2+4s}{s^3}}_{F(s)}\Big]\Big|_t$$

$$= t^2 - f(t-2)\,\text{step}_2(t)$$

where

$$f(t) = \mathcal{L}^{-1}[F(s)]|_t = \mathcal{L}^{-1}\left[\frac{2+4s}{s^3}\right]\Big|_t$$

$$= \mathcal{L}^{-1}\left[\frac{2}{s^3} + \frac{4}{s^2}\right]\Big|_t$$

$$= \mathcal{L}^{-1}\left[\frac{2}{s^3}\right]\Big|_t + 4\mathcal{L}^{-1}\left[\frac{1}{s^2}\right]\Big|_t = t^2 + 4t \quad .$$

So,

$$f(X) = X^2 + 4X \quad ,$$

$$f(t-2) = (t-2)^2 + 4(t-2) = t^2 - 4$$

and

$$\mathcal{L}^{-1}\left[\frac{2}{s^3} - \frac{2+4s}{s^3}e^{-2s}\right]\Big|_t = t^2 - f(t-2)\,\text{step}_2(t) = t^2 - \left[t^2 - 4\right]\text{step}_2(t) \quad .$$

Converting to a set of conditional formulas:
If $t < 2$,

$$\mathcal{L}^{-1}\left[\frac{2}{s^3} - \frac{2+4s}{s^3}e^{-2s}\right]\Big|_t = t^2 - \left[t^2 - 4\right]\text{step}_2(t)$$

$$= t^2 - \left[t^2 - 4\right]\cdot 0 = t^2 \quad .$$

If $2 \le t$,

$$\mathcal{L}^{-1}\left[\frac{2}{s^3} - \frac{2+4s}{s^3}e^{-2s}\right]\Big|_t = t^2 - \left[t^2 - 4\right]$$

$$= t^2 - \left[t^2 - 4\right]\cdot 1 = 4 \quad .$$

In summary:

$$\mathcal{L}^{-1}\left[\frac{1-e^{-s}}{s^2}\right]\Big|_t = \begin{cases} t^2 & \text{if} \quad t < 2 \\ 4 & \text{if} \quad 2 \le t \end{cases} \quad .$$

27.5 e. Computing the inverse transform:

$$\mathcal{L}^{-1}\left[\frac{(s+4)e^{-12}-8e^{-3s}}{s^2-16}\right]\Bigg|_t = \mathcal{L}^{-1}\left[\frac{(s+4)e^{-12}}{s^2-16}\right]\Bigg|_t - \mathcal{L}^{-1}\Bigg[e^{-3s}\underbrace{\frac{8}{s^2-16}}_{F(s)}\Bigg]\Bigg|_t$$

$$= e^{-12}\mathcal{L}^{-1}\left[\frac{1}{s-4}\right]\Bigg|_t - f(t-3)\,\text{step}_3(t)$$

$$= e^{-12}e^{4t} - f(t-3)\,\text{step}_3(t)$$

where

$$f(t) = \mathcal{L}^{-1}[F(s)]|_t = \mathcal{L}^{-1}\left[\frac{8}{s^2-16}\right]\Bigg|_t$$

$$= \mathcal{L}^{-1}\left[\frac{8}{(s-4)(s+4)}\right]\Bigg|_t$$

$$= \cdots \qquad \text{(Use partial fractions or convolution.)}$$

$$= \left[e^{4t}-e^{-4t}\right] \quad .$$

So,

$$f(X) = \left[e^{4X}-e^{-4X}\right] \quad ,$$

$$f(t-3) = \left[e^{4(t-3)}-e^{-4(t-3)}\right]$$

and

$$\mathcal{L}^{-1}\left[\frac{(s+4)e^{-12}-8e^{-3s}}{s^2-16}\right]\Bigg|_t = e^{-12}e^{4t} - f(t-3)\,\text{step}_3(t)$$

$$= e^{4(t-3)} - \left[e^{4(t-3)}-e^{-4(t-3)}\right]\text{step}_3(t) \quad .$$

Converting to a set of conditional formulas:
If $t < 3$,

$$\mathcal{L}^{-1}\left[\frac{(s+4)e^{-12}-8e^{-3s}}{s^2-16}\right]\Bigg|_t = e^{4(t-3)} - \left[e^{4(t-3)}-e^{-4(t-3)}\right]\text{step}_3(t)$$

$$= e^{4(t-3)} - \left[e^{4(t-3)}-e^{-4(t-3)}\right]\cdot 0 = e^{4(t-3)} \quad .$$

If $3 \le t$,

$$\mathcal{L}^{-1}\left[\frac{(s+4)e^{-12}-8e^{-3s}}{s^2-16}\right]\Bigg|_t = e^{4(t-3)} - \left[e^{4(t-3)}-e^{-4(t-3)}\right]\text{step}_3(t)$$

$$= e^{4(t-3)} - \left[e^{4(t-3)}-e^{-4(t-3)}\right]\cdot 1 = e^{-4(t-3)} \quad .$$

In summary:

$$\mathcal{L}^{-1}\left[\frac{(s+4)e^{-12}-8e^{-3s}}{s^2-16}\right]\Bigg|_t = \begin{cases} e^{4(t-3)} & \text{if} \quad t < 3 \\ e^{-4(t-3)} & \text{if} \quad 3 \le t \end{cases} \quad .$$

27.6 a.

$$\mathcal{L}[y']\big|_s = \mathcal{L}[\text{step}_3(t)]\big|_s$$

$$\hookrightarrow \quad sY(s) - \underbrace{y(0)}_{0} = \frac{e^{-3s}}{s}$$

$$\hookrightarrow \quad Y(s) = e^{-3s}\frac{1}{s^2}$$

$$\hookrightarrow \quad y(t) = \mathcal{L}^{-1}\Big[e^{-3s}\underbrace{\frac{1}{s^2}}_{F(s)}\Big]\Big|_t = f(t-3)\,\text{step}_3(t)$$

where

$$f(t) = \mathcal{L}^{-1}[F(s)]|_t = \mathcal{L}^{-1}\left[\frac{1}{s^2}\right]\Big|_t = t \quad .$$

So,

$$f(X) = X \quad , \quad f(t-3) = t-3$$

and

$$y(t) = f(t-3)\,\text{step}_3(t) = (t-3)\,\text{step}_3(t) \quad .$$

27.6 c.

$$\mathcal{L}[y'']\big|_s = \mathcal{L}[\text{step}_2(t)]\big|_s$$

$$\hookrightarrow \quad s^2Y(s) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0} = \frac{e^{-2s}}{s}$$

$$\hookrightarrow \quad Y(s) = e^{-2s}\frac{1}{s^3}$$

$$\hookrightarrow \quad y(t) = \mathcal{L}^{-1}\Big[e^{-2s}\underbrace{\frac{1}{s^3}}_{F(s)}\Big]\Big|_t = f(t-2)\,\text{step}_2(t)$$

where

$$f(t) = \mathcal{L}^{-1}[F(s)]|_t = \mathcal{L}^{-1}\left[\frac{1}{s^3}\right]\Big|_t = \frac{1}{2}t^2 \quad .$$

So,

$$f(X) = \frac{1}{2}X^2 \quad , \quad f(t-2) = \frac{1}{2}(t-2)^2$$

and

$$y(t) = f(t-2)\,\text{step}_2(t) = \frac{1}{2}(t-2)^2\,\text{step}_2(t) \quad .$$

27.6 e.

$$\mathcal{L}[y''+9y]\big|_s = \mathcal{L}[\text{step}_{10}(t)]\big|_s$$

$$\hookrightarrow \quad \mathcal{L}[y'']\big|_s + 9\mathcal{L}[y]|_s = \frac{e^{-10s}}{s}$$

$$\hookrightarrow \quad \left[s^2Y(s) - s\cdot 0 - 0\right] + 9Y(s) = \frac{e^{-10s}}{s}$$

$$\hookrightarrow \qquad Y(s) = e^{-10s}\frac{1}{s\left(s^2+9\right)}$$

$$\hookrightarrow \qquad y(t) = \mathcal{L}^{-1}\Big[e^{-10s}\underbrace{\frac{1}{s\left(s^2+9\right)}}_{F(s)}\Big]\Big|_t = f(t-10)\,\text{step}_{10}(t)$$

where

$$\begin{aligned} f(t) = \mathcal{L}^{-1}[F(s)]|_t &= \mathcal{L}^{-1}\left[\frac{1}{s\left(s^2+9\right)}\right]\Bigg|_t \\ &= \cdots \qquad \text{(Use partial fractions or convolution.)} \\ &= \frac{1}{9}\big[1-\cos(3t)\big] \quad . \end{aligned}$$

So,

$$f(X) = \frac{1}{9}\big[1-\cos(3X)\big] \quad , \quad f(t-10) = \frac{1}{9}\big[1-\cos(3[t-10])\big]$$

and

$$y(t) = f(t-10)\,\text{step}_{10}(t) = \frac{1}{9}\big[1-\cos(3[t-10])\big]\,\text{step}_{10}(t) \quad .$$

27.7 a. Here,

$$f(t) = \begin{Bmatrix} 0 & \text{if} & t<6 \\ e^{4t} & \text{if} & 6<t \end{Bmatrix} = e^{4t}\begin{Bmatrix} 0 & \text{if} & t<6 \\ 1 & \text{if} & 6<t \end{Bmatrix} = e^{4t}\,\text{step}_6(t) \quad .$$

Applying the translation along the T–axis identity:

$$\mathcal{L}[f(t)]|_s = \mathcal{L}\Big[e^{4t}\,\text{step}_6(t)\Big]\Big|_s = \mathcal{L}\big[g(t-6)\,\text{step}_6(t)\big]\big|_s = e^{-6s}G(s) \quad .$$

In this case,

$$g(t-6) = e^{4t} \quad .$$

Letting $X = t-6$ (hence, $t = X+6$), we get

$$g(X) = e^{4(X+6)} = e^{24}e^{4X} \quad .$$

Thus,

$$g(t) = e^{24}e^{4t} \quad ,$$

$$G(s) = \mathcal{L}[g(t)]|_s = \mathcal{L}\Big[e^{24}e^{4t}\Big]\Big|_s = \frac{e^{24}}{s-4}$$

and

$$\mathcal{L}[f(t)]|_s = \cdots = e^{-6s}G(s) = e^{-6s}\cdot\frac{e^{24}}{s-4} = \frac{1}{s-4}e^{-6(s-4)} \quad .$$

27.7 c. Applying the translation along the T–axis identity:

$$\mathcal{L}\big[t\,\text{step}_6(t)\big]\big|_s = \mathcal{L}\big[f(t-6)\,\text{step}_6(t)\big]\big|_s = e^{-6s}F(s) \quad .$$

In this case,

$$f(t-6) = t \quad .$$

Letting $X = t - 6$ (hence, $t = X + 6$), we get

$$f(X) = X + 6 \quad .$$

Thus,

$$f(t) = t + 6 \quad ,$$

$$F(s) = \mathcal{L}[f(t)]|_s = \mathcal{L}[t+6]|_s = \mathcal{L}[t]|_s + \mathcal{L}[6]|_s = \frac{1}{s^2} + \frac{6}{s}$$

and

$$\mathcal{L}\big[t \operatorname{step}_6(t)\big]\big|_s = \cdots = e^{-6s}F(s) = e^{-6s} \cdot \left[\frac{1}{s^2} + \frac{6}{s}\right] = \frac{1}{s^2}e^{-6s} + \frac{6}{s}e^{-6s} \quad .$$

27.7 e. Applying the translation along the T–axis identity:

$$\mathcal{L}\Big[t^2 \operatorname{step}_6(t)\Big]\Big|_s = \mathcal{L}\big[f(t-6)\operatorname{step}_6(t)\big]\big|_s = e^{-6s}F(s) \quad .$$

In this case,

$$f(t-6) = t^2 \quad .$$

Letting $X = t - 6$ (hence, $t = X + 6$), we get

$$f(X) = (X + 6)^2 = X^2 + 12X + 36 \quad .$$

Thus,

$$f(t) = t^2 + 12t + 36 \quad ,$$

$$F(s) = \mathcal{L}[f(t)]|_s = \mathcal{L}\Big[t^2 + 12t + 36\Big]\Big|_s = \frac{2}{s^3} + \frac{12}{s^2} + \frac{36}{s}$$

and

$$\mathcal{L}\Big[t^2 \operatorname{step}_6(t)\Big]\Big|_s = \cdots = e^{-6s}F(s) = e^{-6s}\left[\frac{2}{s^3} + \frac{12}{s^2} + \frac{36}{s}\right] \quad .$$

27.7 g. Applying the translation along the T–axis identity:

$$\mathcal{L}\big[\sin(2t)\operatorname{step}_{\pi/2}(t)\big]\big|_s = \mathcal{L}\Big[f\Big(t - \frac{\pi}{2}\Big)\operatorname{step}_{\pi/2}(t)\Big]\Big|_s = e^{-\pi s/2}F(s) \quad .$$

In this case,

$$f\Big(t - \frac{\pi}{2}\Big) = \sin(2t) \quad .$$

Letting $X = t - \frac{\pi}{2}$ $\Big($hence, $t = X + \frac{\pi}{2}\Big)$, we get

$$f(X) = \sin\Big(2\Big[X + \frac{\pi}{2}\Big]\Big) = \sin(2X + \pi) = -\sin(2X) \quad .$$

Thus,

$$f(t) = -\sin(2t) \quad ,$$

$$F(s) = \mathcal{L}[f(t)]|_s = -\mathcal{L}[\sin(2t)]|_s = -\frac{2}{s^2+4}$$

and

$$\mathcal{L}\big[\sin(2t)\operatorname{step}_{\pi/2}(t)\big]\big|_s = \cdots = e^{-\pi s/2}F(s)$$

$$= e^{-\pi s/2} \cdot \frac{-2}{s^2+4} = -\frac{2}{s^2+4}e^{-\pi s/2} \quad .$$

27.7 i. Applying the translation along the T–axis identity:

$$\mathcal{L}\left[\sin(2t)\,\mathrm{step}_{\pi/6}(t)\right]\Big|_s \;=\; \mathcal{L}\left[f\left(t-\tfrac{\pi}{6}\right)\mathrm{step}_{\pi/6}(t)\right]\Big|_s \;=\; e^{-\pi s/6}F(s) \quad .$$

In this case,

$$f\left(t-\tfrac{\pi}{6}\right) \;=\; \sin(2t) \quad .$$

Letting $X = t - \pi/6$ (hence, $t = X + \pi/6$), we get

$$\begin{aligned} f(X) \;&=\; \sin\left(2\left[X+\tfrac{\pi}{6}\right]\right) \;=\; \sin\left(2X+\tfrac{\pi}{3}\right) \\ &=\; \sin(2X)\cos\left(\tfrac{\pi}{3}\right) \;+\; \cos(2X)\sin\left(\tfrac{\pi}{3}\right) \\ &=\; \tfrac{1}{2}\sin(2X) \;+\; \tfrac{\sqrt{3}}{2}\cos(2X) \quad . \end{aligned}$$

Thus,

$$f(t) \;=\; \tfrac{1}{2}\sin(2t) \;+\; \tfrac{\sqrt{3}}{2}\cos(2t) \quad ,$$

$$\begin{aligned} F(s) \;=\; \mathcal{L}[f(t)]|_s \;&=\; \mathcal{L}\left[\tfrac{1}{2}\sin(2t) \;+\; \tfrac{\sqrt{3}}{2}\cos(2t)\right]\Big|_s \\ &=\; \tfrac{1}{2}\left[\frac{2}{s^2+4}+\frac{s\sqrt{3}}{s^2+4}\right] \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}\left[\sin(2t)\,\mathrm{step}_{\pi/6}(t)\right]\Big|_s \;&=\; \cdots \;=\; e^{-\pi s/6}F(s) \\ &=\; e^{-\pi s/6}\tfrac{1}{2}\left[\frac{2}{s^2+4}+\frac{s\sqrt{3}}{s^2+4}\right] \\ &=\; \tfrac{1}{2}\left[\frac{2}{s^2+4}+\frac{s\sqrt{3}}{s^2+4}\right]e^{-\pi s/6} \quad . \end{aligned}$$

27.8 a.

$$\begin{aligned} f(t) \;&=\; e^{-4t}\,\mathrm{rect}_{(-\infty,6)}(t) \;+\; 0\cdot\mathrm{rect}_{(6,\infty)}(t) \\ &=\; e^{-4t}\left[1-\mathrm{step}_6(t)\right] \;+\; 0 \;=\; e^{-4t} \;-\; e^{-4t}\,\mathrm{step}_6(t) \quad . \end{aligned}$$

So (using, this time, the first translation identity),

$$\begin{aligned} F(s) \;=\; \mathcal{L}[f(t)]|_s \;&=\; \mathcal{L}\left[e^{-4t}-e^{-4t}\,\mathrm{step}_6(t)\right]\Big|_s \\ &=\; \mathcal{L}\left[e^{-4t}\right]\Big|_s \;-\; \mathcal{L}\left[e^{-4t}\,\mathrm{step}_6(t)\right]\Big|_s \\ &=\; \frac{1}{s+4} \;-\; \frac{e^{-6(s+4)}}{s+4} \quad . \end{aligned}$$

27.8 c.

$$\begin{aligned} f(t) \;&=\; 2\,\mathrm{rect}_{(-\infty,3)}(t) \;+\; 2e^{-4(t-3)}\,\mathrm{rect}_{(3,\infty)}(t) \\ &=\; 2\left[1-\mathrm{step}_3(t)\right] \;+\; 2e^{-4(t-3)}\,\mathrm{step}_3(t) \\ &=\; 2 \;-\; 2\,\mathrm{step}_3(t) \;+\; 2e^{-4(t-3)}\,\mathrm{step}_3(t) \quad . \end{aligned}$$

Hence,

$$F(s) = \mathcal{L}[f(t)]|_s = \mathcal{L}\Big[2 - 2\,\text{step}_3(t) + 2e^{-4(t-3)}\,\text{step}_3(t)\Big]\Big|_s$$

$$= 2\mathcal{L}[1]|_s - 2\mathcal{L}\big[\text{step}_3(t)\big]\big|_s + 2\mathcal{L}\big[\underbrace{e^{-4(t-3)}}_{g(t-3)}\text{step}_3(t)\big]\big|_s$$

$$= \frac{2}{s} - \frac{2e^{-3s}}{s} + 2G(s)e^{-3s} \quad .$$

Clearly,

$$g(t-3) = e^{4(t-3)} \quad \rightarrowtail \quad g(t) = e^{4t}$$

$$\hookrightarrow \qquad G(s) = \mathcal{L}\Big[e^{4t}\Big]\Big|_s = \frac{1}{s-4} \quad .$$

So,

$$F(s) = \frac{2}{s} - \frac{2e^{-3s}}{s} + 2G(s)e^{-3s} = \frac{2}{s}\Big[1 - e^{-3s}\Big] + \frac{2}{s-4}e^{-3s} \quad .$$

27.8 e.
$$f(t) = t^2\,\text{rect}_{(-\infty,3)}(t) + 9\,\text{rect}_{(3,\infty)}(t)$$

$$= t^2\big[1 - \text{step}_3(t)\big] + 9\,\text{step}_3(t) = t^2 - t^2\,\text{step}_3(t) + 9\,\text{step}_3(t) \quad .$$

Hence,

$$F(s) = \mathcal{L}[f(t)]|_s = \mathcal{L}\Big[t^2 - t^2\,\text{step}_3(t) + 9\,\text{step}_3(t)\Big]\Big|_s$$

$$= \mathcal{L}\Big[t^2\Big]\Big|_s - \mathcal{L}\big[\underbrace{t^2}_{g(t-3)}\,\text{step}_3(t)\big]\big|_s + 9\mathcal{L}\big[\text{step}_3(t)\big]\big|_s$$

$$= \frac{2}{s^3} - G(s)e^{-3s} + \frac{9e^{-3s}}{s} \quad .$$

Here,

$$g(\underbrace{t-3}_{X}) = t^2 \quad \rightarrowtail \quad g(X) = (X+3)^2 = X^2 + 6X + 9$$

$$\hookrightarrow \qquad G(s) = \mathcal{L}\Big[t^2 + 6t + 9\Big]\Big|_s = \frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s} \quad .$$

So,

$$F(s) = \frac{2}{s^3} - G(s)e^{-3s} + \frac{9e^{-3s}}{s}$$

$$= \frac{2}{s^3} - \left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right]e^{-3s} + \frac{9e^{-3s}}{s} = \frac{2}{s^3} - \left[\frac{2}{s^3} + \frac{6}{s^2}\right]e^{-3s} \quad .$$

27.8 g.
$$f(t) = 1\,\text{rect}_{(-\infty,2)}(t) + 2\,\text{rect}_{(2,3)}(t) + 4\,\text{rect}_{(3,\infty)}(t)$$

$$= 1\big[1 - \text{step}_2(t)\big] + 2\big[\text{step}_2(t) - \text{step}_3(t)\big] + 4\,\text{step}_3(t)$$

$$= 1 + \text{step}_2(t) + 2\,\text{step}_3(t) \quad .$$

Hence,

$$F(s) = \mathcal{L}[f(t)]|_s = \mathcal{L}\big[1 + \text{step}_2(t) + 2\,\text{step}_3(t)\big]\big|_s$$
$$= \mathcal{L}[1]|_s + \mathcal{L}\big[\text{step}_2(t)\big]\big|_s + 2\mathcal{L}\big[\text{step}_3(t)\big]\big|_s$$
$$= \frac{1}{s} + \frac{e^{-2s}}{s} + 2\frac{e^{-3s}}{s} = \frac{1}{s}\left[1 + e^{-2s} + 2e^{-3s}\right] \quad .$$

27.8 i.

$$f(t) = 0\,\text{rect}_{(-\infty,1)}(t) + (t-1)^2\,\text{rect}_{(1,3)}(t) + 4\,\text{rect}_{(3,\infty)}(t)$$
$$= 0 + (t-1)^2\left[\text{step}_1(t) - \text{step}_3(t)\right] + 4\,\text{step}_3(t)$$
$$= (t-1)^2\,\text{step}_1(t) - (t-1)^2\,\text{step}_3(t) + 4\,\text{step}_3(t) \quad .$$

Hence,

$$F(s) = \mathcal{L}\Big[(t-1)^2\,\text{step}_1(t) - (t-1)^2\,\text{step}_3(t) + 4\,\text{step}_3(t)\Big]\Big|_s$$
$$= \mathcal{L}\big[\underbrace{(t-1)^2}_{g(t-1)}\text{step}_1(t)\big]\big|_s - \mathcal{L}\big[\underbrace{(t-1)^2}_{h(t-3)}\text{step}_3(t)\big]\big|_s + 4\mathcal{L}\big[\text{step}_3(t)\big]\big|_s$$
$$= G(s)e^{-s} - H(s)e^{-3s} + \frac{4e^{-3s}}{s} \quad .$$

Clearly,

$$g(t-1) = (t-1)^2 \quad \rightarrowtail \quad g(X) = X^2$$

$$\hookrightarrow \qquad G(s) = \mathcal{L}[g(t)]|_s = \mathcal{L}\Big[t^2\Big]\Big|_s = \frac{2}{s^3} \quad .$$

Also,

$$h(\underbrace{t-3}_{X}) = (t-1)^2$$

$$\hookrightarrow \qquad h(X) = ([X+3]-1)^2 = X^2 + 4X + 4$$

$$\hookrightarrow \qquad H(s) = \mathcal{L}\Big[t^2 + 4t + 4\Big]\Big|_s = \frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s} \quad .$$

With these formulas for G and H, we can continue computing the formula for F:

$$F(s) = G(s)e^{-s} - H(s)e^{-3s} + \frac{4e^{-3s}}{s}$$
$$= \frac{2}{s^3}e^{-s} - \left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right]e^{-3s} + \frac{4}{s}e^{-3s} = \frac{2}{s^3}e^{-s} - \left[\frac{2}{s^3} + \frac{4}{s^2}\right]e^{-3s} \quad .$$

27.9 a.

$$\text{stair}(t) = \sum_{n=0}^{\infty}(n+1)\,\text{rect}_{(n,n+1)}(t)$$
$$= 1\,\text{rect}_{(0,1)}(t) + 2\,\text{rect}_{(1,2)}(t) + 3\,\text{rect}_{(2,3)}(t) + 4\,\text{rect}_{(3,4)}(t)$$
$$+ \cdots + n\,\text{rect}_{(n-1,n)}(t) + (n+1)\,\text{rect}_{(n,n+1)}(t) + \cdots$$

$$
\begin{aligned}
&= 1\left[\text{step}_0(t) - \text{step}_1(t)\right] \;+\; 2\left[\text{step}_1(t) - \text{step}_2(t)\right] \\
&\quad + \; 3\left[\text{step}_2(t) - \text{step}_3(t)\right] \;+\; 4\left[\text{step}_3(t) - \text{step}_4(t)\right] \;+\; \cdots \\
&\quad + \; n\left[\text{step}_{n-1}(t) - \text{step}_n(t)\right] \;+\; (n+1)\left[\text{step}_n(t) - \text{step}_{n+1}(t)\right] \;+\; \cdots \\
&= \text{step}_0(t) \;+\; \text{step}_1(t) \;+\; \text{step}_2(t) \;+\; \text{step}_3(t) \;+\; \cdots \;+\; \text{step}_n(t) \;+\; \cdots \\
&= \sum_{n=0}^{\infty} \text{step}_n(t) \quad .
\end{aligned}
$$

27.9 b.

$$
\begin{aligned}
\mathcal{L}[\text{stair}(t)]|_s &= \mathcal{L}\left[\sum_{n=0}^{\infty} \text{step}_n(t)\right]\Bigg|_s \\
&= \sum_{n=0}^{\infty} \mathcal{L}\left[\text{step}_n(t)\right]\Big|_s \;=\; \sum_{n=0}^{\infty} \frac{e^{-ns}}{s} \;=\; \frac{1}{s}\sum_{n=0}^{\infty} e^{-ns} \quad .
\end{aligned}
$$

27.9 c.

$$
\mathcal{L}[\text{stair}(t)]|_s \;=\; \frac{1}{s}\sum_{n=0}^{\infty} e^{-ns} \;=\; \frac{1}{s}\sum_{n=0}^{\infty} \left(e^{-s}\right)^n \;=\; \frac{1}{s}\cdot\frac{1}{1-e^{-s}} \;=\; \frac{1}{s\left[1-e^{-s}\right]} \quad .
$$

27.10 a.

$$
\begin{aligned}
&& \mathcal{L}\left[y'\right]\Big|_s &= \mathcal{L}\left[\text{rect}_{(1,3)}(t)\right]\Big|_s \\
&\hookrightarrow & sY(s) \;-\; \underbrace{y(0)}_{0} &= \mathcal{L}\left[\text{step}_1(t) - \text{step}_3(t)\right]\Big|_s \\
&\hookrightarrow & sY(s) &= \frac{e^{-s}}{s} \;-\; \frac{e^{-3s}}{s} \\
&\hookrightarrow & Y(s) &= \frac{e^{-s}}{s^2} \;-\; \frac{e^{-3s}}{s^2} \quad .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
y(t) \;=\; \mathcal{L}\left[\frac{e^{-s}}{s^2} - \frac{e^{-3s}}{s^2}\right]\Bigg|_t &= \mathcal{L}\left[\frac{e^{-s}}{s^2}\right]\Bigg|_t \;-\; \mathcal{L}\left[\frac{e^{-3s}}{s^2}\right]\Bigg|_t \\
&= \mathcal{L}\left[e^{-s}\;\underbrace{\frac{1}{s^2}}_{F(s)}\right]\Bigg|_t \;-\; \mathcal{L}\left[e^{-3s}\;\underbrace{\frac{1}{s^2}}_{F(s)}\right]\Bigg|_t \\
&= f(t-1)\,\text{step}_1(t) \;-\; f(t-3)\,\text{step}_3(t)
\end{aligned}
$$

where

$$
f(t) \;=\; \mathcal{L}^{-1}\left[\frac{1}{s^2}\right]\Bigg|_t \;=\; t \quad .
$$

So,

$$
f(X) \;=\; X
$$

and, letting $X = t-1$ and $X = t-3$, respectively,

$$
\begin{aligned}
y(t) &= f(t-1)\,\text{step}_1(t) \;-\; f(t-3)\,\text{step}_3(t) \\
&= (t-1)\,\text{step}_1(t) \;-\; (t-3)\,\text{step}_3(t)
\end{aligned}
$$

$$= (t-1)\left[\text{step}_1(t) - \text{step}_3(t)\right] + 2\,\text{step}_3(t)$$
$$= (t-1)\,\text{rect}_{(1,3)}(t) + 2\,\text{step}_3(t) \quad .$$

27.10 c.

$$\mathcal{L}\left[y'' + 9y\right]\Big|_s = \mathcal{L}\left[\text{rect}_{(1,3)}(t)\right]\Big|_s$$

$$\hookrightarrow \quad [sY(s) - s\cdot 0 - 0] + 9Y(s) = \mathcal{L}\left[\text{step}_1(t) - \text{step}_3(t)\right]\Big|_s$$

$$\hookrightarrow \quad \left(s^2+9\right)Y(s) = \frac{e^{-s}}{s} - \frac{e^{-3s}}{s}$$

$$\hookrightarrow \quad Y(s) = \frac{e^{-s}}{s\left(s^2+9\right)} - \frac{e^{-3s}}{s\left(s^2+9\right)} \quad .$$

Thus,

$$y(t) = \mathcal{L}\left[\frac{e^{-s}}{s\left(s^2+9\right)} - \frac{e^{-3s}}{s\left(s^2+9\right)}\right]\Bigg|_t$$
$$= \mathcal{L}\Bigg[e^{-s}\underbrace{\frac{1}{s\left(s^2+9\right)}}_{F(s)}\Bigg]\Bigg|_t - \mathcal{L}\Bigg[e^{-3s}\underbrace{\frac{1}{s\left(s^2+9\right)}}_{F(s)}\Bigg]\Bigg|_t$$
$$= f(t-1)\,\text{step}_1(t) - f(t-3)\,\text{step}_3(t)$$

where

$$f(t) = \mathcal{L}^{-1}\left[\frac{1}{s\left(s^2+9\right)}\right]\Bigg|_t$$
$$= \cdots \qquad \text{(Use partial fractions or convolution.)}$$
$$= \frac{1}{9}\left[1 - \cos(3t)\right] \quad .$$

So,

$$f(X) = \frac{1}{9}\left[1 - \cos(3X)\right]$$

and, letting $X = t-1$ and $X = t-3$, respectively,

$$y(t) = f(t-1)\,\text{step}_1(t) - f(t-3)\,\text{step}_3(t)$$
$$= \frac{1}{9}\big[1 - \cos(3[t-1])\,\big]\,\text{step}_1(t) - \frac{1}{9}\big[1 - \cos(3[t-3])\,\big]\,\text{step}_3(t)$$
$$= \frac{1}{9}\Big[\big(\text{step}_1(t) - \text{step}_3(t)\big) + \cos(3[t-3])\,\text{step}_3(t) - \cos(3[t-1])\,\text{step}_1(t)\Big]$$
$$= \frac{1}{9}\Big[\,\text{rect}_{(1,3)}(t) + \cos(3[t-3])\,\text{step}_3(t) - \cos(3[t-1])\,\text{step}_1(t)\Big] \quad .$$

27.11 a. Here we are computing $h * f$ with

$$h(t) = t^2 \qquad \text{and} \qquad f(t) = \text{step}_3(t) \quad .$$

So,

$$t^2 * \text{step}_3(t) = h * f(t) = f * h(t)$$
$$= \int_0^t f(x)h(t-x)\,dx = \int_0^t \text{step}_3(x)(t-x)^2\,dx \quad .$$

If $t < 3$, then $\text{step}_3(t) = 0$, and the above becomes

$$t^2 * \text{step}_3(t) = \int_0^t \text{step}_3(x)(t-x)^2\,dx = \int_0^t 0 \cdot (t-x)^2\,dx = 0 \quad .$$

If $t \geq 3$, then the above becomes

$$\begin{aligned} t^2 * \text{step}_3(t) &= \int_0^t \text{step}_3(x)(t-x)^2\,dx \\ &= \int_0^3 \underbrace{\text{step}_3(x)}_{0}(t-x)^2\,dx + \int_3^t \underbrace{\text{step}_3(x)}_{1}(t-x)^2\,dx \\ &= 0 + \int_3^t (t-x)^2\,dx = -\frac{1}{3}(t-x)^3\Big|_{x=3}^{t} = \frac{1}{3}(t-3)^3 \quad . \end{aligned}$$

In summary,

$$t^2 * \text{step}_3(t) = \begin{cases} 0 & \text{if} \quad t < 3 \\ \frac{1}{3}(t-3)^3 & \text{if} \quad 3 < t \end{cases} \quad .$$

27.11 c. For any $t \geq 0$,

$$\cos(t) * \text{rect}_{(0,\pi)}(t) = \text{rect}_{(0,\pi)}(t) * \cos(t) = \int_0^t \text{rect}_{(0,\pi)}(x)\cos(t-x)\,dx \quad .$$

If $0 \leq t < \pi$, then

$$\begin{aligned} \cos(t) * \text{rect}_{(0,\pi)}(t) &= \int_0^t \underbrace{\text{rect}_{(0,\pi)}(x)}_{1}\cos(t-x)\,dx \\ &= \int_0^t \cos(t-x)\,dx = -\sin(t-x)\Big|_{x=0}^{t} = \sin(t) \quad . \end{aligned}$$

If $t > \pi$, then the above becomes

$$\begin{aligned} \cos(t) * \text{rect}_{(0,\pi)}(t) &= \int_0^t \text{rect}_{(0,\pi)}(x)\cos(t-x)\,dx \\ &= \int_0^\pi \underbrace{\text{rect}_{(0,\pi)}(x)}_{1}\cos(t-x)\,dx + \int_\pi^t \underbrace{\text{rect}_{(0,\pi)}(x)}_{0}\cos(t-x)\,dx \\ &= \int_0^\pi \cos(t-x)\,dx + \int_\pi^t 0\,dx = \sin(\pi) + 0 = 0 \quad . \end{aligned}$$

In summary,

$$\cos(t) * \text{rect}_{(0,\pi)}(t) = \begin{cases} \sin(t) & \text{if} \quad 0 \leq t < \pi \\ 0 & \text{if} \quad \pi < t \end{cases} \quad .$$

27.11 e. For any $t \geq 0$,

$$e^{-2t} * \left[e^{5t}\,\text{rect}_{(1,3)}(t)\right] = \left[e^{5t}\,\text{rect}_{(1,3)}(t)\right] * e^{-2t}$$
$$= \int_0^t \left[e^{5x}\,\text{rect}_{(1,3)}(x)\right] e^{-2(t-x)}\,dx$$
$$= e^{-2t}\int_0^t e^{7x}\,\text{rect}_{(1,3)}(x)\,dx \quad .$$

If $0 \leq t < 1$,

$$e^{-2t} * \left[e^{5t}\,\text{rect}_{(1,3)}(t)\right] = e^{-2t}\int_0^t e^{7x}\underbrace{\text{rect}_{(1,3)}(x)}_{0}\,dx = e^{-2t}\int_0^t 0\,dx = 0 \quad .$$

If $1 < t < 3$,

$$e^{-2t} * \left[e^{5t}\,\text{rect}_{(1,3)}(t)\right] = e^{-2t}\int_0^t e^{7x}\,\text{rect}_{(1,3)}(x)\,dx$$
$$= e^{-2t}\left[\int_0^1 e^{7x}\underbrace{\text{rect}_{(1,3)}(x)}_{0}\,dx + \int_1^t e^{7x}\underbrace{\text{rect}_{(1,3)}(x)}_{1}\,dx\right]$$
$$= e^{-2t}\left[0 + \tfrac{1}{7}\left(e^{7t} - e^{7}\right)\right] = \tfrac{1}{7}\left[e^{5t} - e^{7-2t}\right] \quad .$$

If $3 < t$,

$$e^{-2t} * \left[e^{5t}\,\text{rect}_{(1,3)}(t)\right] = e^{-2t}\int_0^t e^{7x}\,\text{rect}_{(1,3)}(x)\,dx$$
$$= e^{-2t}\left[\int_0^1 e^{7x}\underbrace{\text{rect}_{(1,3)}(x)}_{0}\,dx + \int_1^3 e^{7x}\underbrace{\text{rect}_{(1,3)}(x)}_{1}\,dx\right.$$
$$\left.+ \int_3^t e^{7x}\underbrace{\text{rect}_{(1,3)}(x)}_{0}\,dx\right]$$
$$= e^{-2t}\left[0 + \tfrac{1}{7}\left(e^{21} - e^{7}\right) + 0\right] = \tfrac{1}{7}\left[e^{21-t} - e^{7-2t}\right] \quad .$$

In summary,

$$e^{-2t} * \left[e^{5t}\,\text{rect}_{(1,3)}(t)\right] = \frac{1}{7}\begin{cases} 0 & \text{if} \quad t < 1 \\ e^{5t} - e^{7-2t} & \text{if} \quad 1 < t < 3 \\ e^{21-2t} - e^{7-2t} & \text{if} \quad 3 < t \end{cases} \quad .$$

27.11 g. For any $t \geq 0$,

$$t * f(t) = f(t) * t = \int_0^t f(x)\cdot(t-x)\,dx \quad .$$

If $0 \leq t < 4$,

$$t * f(t) = \int_0^t \underbrace{f(x)}_{\sqrt{x}}\cdot(t-x)\,dx = \int_0^t \left[t x^{1/2} - x^{3/2}\right] dx = \frac{4}{15}t^{5/2} \quad .$$

If $t < 4$,

$$\begin{aligned} t * f(t) &= \int_0^t f(x)\cdot(t-x)\,dx \\ &= \int_0^4 \underbrace{f(x)}_{\sqrt{x}}\cdot(t-x)\,dx + \int_4^t \underbrace{f(x)}_{2}\cdot(t-x)\,dx \\ &= \int_0^4 \left[tx^{1/2} - x^{3/2}\right] dx + \int_4^t 2(t-x)\,dx \\ &= \tfrac{4}{15}4^{5/2} + (t-4)^2 = \tfrac{128}{15} + (t-4)^2 \quad . \end{aligned}$$

In summary,

$$t * f(t) = \frac{1}{15}\begin{cases} 4t^{5/2} & \text{if} \quad t < 4 \\ 128 + 15(t-4)^2 & \text{if} \quad 4 < t \end{cases} \quad .$$

27.12 a. Using theorem 27.2,

$$F(s) = \mathcal{L}[f(t)]|_s = \frac{F_0(s)}{1-e^{-ps}} = \frac{F_0(s)}{1-e^{-3s}} \qquad \text{for} \quad s > 0$$

where

$$\begin{aligned} F_0(s) = \int_0^p f(t)e^{-st}\,dt &= \int_0^3 e^{-2t}e^{-st}\,dt \\ &= \int_0^3 e^{-(s+2)t}e^{-st}\,dt = \frac{1-e^{-3(s+2)}}{s+2} \quad . \end{aligned}$$

So

$$F(s) = \frac{F_0(s)}{1-e^{-3s}} = \frac{1-e^{-3(s+2)}}{(s+2)\left(1-e^{-3s}\right)} \quad .$$

27.12 c. Using theorem 27.2,

$$F(s) = \mathcal{L}[f(t)]|_s = \frac{F_0(s)}{1-e^{-ps}} = \frac{F_0(s)}{1-e^{-2s}} \qquad \text{for} \quad s > 0$$

where

$$\begin{aligned} F_0(s) = \int_0^2 f(t)e^{-st}\,dt &= \int_0^1 \underbrace{f(t)}_{1}e^{-st}\,dt + \int_1^2 \underbrace{f(t)}_{-1}e^{-st}\,dt \\ &= \int_0^1 e^{-st}\,dt - \int_1^2 e^{-st}\,dt \\ &= \frac{1}{s}\left[1-e^{-s}\right] - \frac{1}{s}\left[e^{-s}-e^{-2s}\right] \\ &= \frac{1}{s}\left[1-2e^{-s}+e^{-2s}\right] \quad . \end{aligned}$$

So

$$\begin{aligned} F(s) = \frac{F_0(s)}{1-e^{-2s}} &= \frac{1-2e^{-s}+e^{-2s}}{s\left(1-e^{-2s}\right)} \\ &= \frac{\left(1-e^{-s}\right)^2}{s\left(1-e^{-s}\right)\left(1+e^{-s}\right)} = \frac{1-e^{-s}}{s\left(1+e^{-s}\right)} \quad . \end{aligned}$$

27.12 e. Using theorem 27.2,

$$F(s) = \mathcal{L}[f(t)]|_s = \frac{F_0(s)}{1 - e^{-ps}} = \frac{F_0(s)}{1 - e^{-4s}} \qquad \text{for} \quad s > 0$$

where

$$\begin{aligned} F_0(s) &= \int_0^4 f(t)e^{-st}\,dt \\ &= \int_0^2 \underbrace{f(t)}_{t} e^{-st}\,dt + \int_2^4 \underbrace{f(t)}_{4-t} e^{-st}\,dt \\ &= \int_0^2 te^{-st}\,dt + \int_2^4 (4-t)e^{-st}\,dt \\ &= \frac{1}{s^2}\left[1 - e^{-2s} - 2se^{-2s}\right] + \frac{1}{s^2}\left[2se^{-2s} - e^{-2s} + e^{-4s}\right] \\ &= \frac{1}{s^2}\left[1 - 2e^{-2s} + e^{-4s}\right] \quad . \end{aligned}$$

So

$$\begin{aligned} F(s) &= \frac{F_0(s)}{1 - e^{-4s}} = \frac{1 - 2e^{-2s} + e^{-4s}}{s^2\left(1 - e^{-4s}\right)} \\ &= \frac{\left(1 - e^{-2s}\right)^2}{s^2\left(1 - e^{-2s}\right)\left(1 + e^{-2s}\right)} = \frac{1 - e^{-2s}}{s^2\left(1 + e^{-2s}\right)} \quad . \end{aligned}$$

27.13 a. *For part i:* Theorem 27.5 on page 534 of the text tells us that

$$y(t + p_0) - y(t) = A\cos(\omega_0 t - \phi)$$

where

$$A = \frac{1}{\omega_0 m}\sqrt{(\mathcal{I}_S)^2 + (\mathcal{I}_C)^2} \quad ,$$

$$\mathcal{I}_S = \int_0^{p_0} \cos(\omega_0 x)\, f(x)\,dx \quad , \quad \mathcal{I}_C = -\int_0^{p_0} \sin(\omega_0 x)\, f(x)\,dx$$

and $0 \le \phi < 2\pi$ satisfies both

$$\cos(\phi) = \frac{\mathcal{I}_C}{\sqrt{(\mathcal{I}_S)^2 + (\mathcal{I}_C)^2}} \qquad \text{and} \qquad \sin(\phi) = \frac{\mathcal{I}_S}{\sqrt{(\mathcal{I}_S)^2 + (\mathcal{I}_C)^2}} \quad .$$

In this case, $\omega_0 = \frac{2\pi}{p} = \frac{2\pi}{2} = \pi$,

$$\begin{aligned} \mathcal{I}_S &= \int_0^2 \cos(\pi x)\, f(x)\,dx \\ &= \int_0^1 \cos(\pi x) \underbrace{f(x)}_{1}\,dx + \int_1^2 \cos(\pi x) \underbrace{f(x)}_{0}\,dx = 0 \quad , \end{aligned}$$

$$\begin{aligned}\mathcal{I}_C &= -\int_0^2 \sin(\pi x)\, f(x)\, dx \\ &= -\int_0^1 \sin(\pi x) \underbrace{f(x)}_{1}\, dx \;-\; \int_1^2 \sin(\pi x) \underbrace{f(x)}_{0}\, dx \;=\; \frac{2}{\pi} \quad ,\end{aligned}$$

$$\sqrt{(\mathcal{I}_S)^2 + (\mathcal{I}_C)^2} \;=\; \sqrt{0^2 + (\mathcal{I}_C)^2} \;=\; |\mathcal{I}_C| \;=\; \frac{2}{\pi} \quad ,$$

$$A \;=\; \frac{1}{\omega_0 m}\sqrt{(\mathcal{I}_S)^2 + (\mathcal{I}_C)^2} \;=\; \frac{1}{\pi m}\cdot\frac{2}{\pi} \;=\; \frac{2}{\pi^2 m} \quad ,$$

$$\cos(\phi) \;=\; \frac{\mathcal{I}_C}{\sqrt{(\mathcal{I}_S)^2 + (\mathcal{I}_C)^2}} \;=\; -1$$

and

$$\sin(\phi) \;=\; \frac{\mathcal{I}_S}{\sqrt{(\mathcal{I}_S)^2 + (\mathcal{I}_C)^2}} \;=\; 0 \quad .$$

Clearly, then, $\phi = \pi$ and

$$\begin{aligned} y(t + p_0) \;-\; y(t) &= A\cos(\omega_0 t - \phi) \\ &= \frac{2}{\pi^2 m}\cos(\pi t - \pi) \;=\; \frac{-2}{\pi^2 m}\cos(\pi t) \quad .\end{aligned}$$

For part ii: From formulas (27.22) and (27.18) (on, respectively, pages 535 and 531 of the text), along with the above computations, it follows that the formula for the solution at time $t = \tau + np_0$ is given by

$$\begin{aligned} y(t) &= y(\tau) \;+\; nA\cos(\omega_0\tau - \phi) \\ &= y(\tau) \;+\; n\frac{2}{\pi^2 m}\cos(\pi\tau - \pi) \;=\; y(\tau) \;-\; \frac{2n}{\pi^2 m}\cos(\pi\tau) \quad .\end{aligned}$$

where

$$y(\tau) \;=\; \frac{1}{\omega_0 m}\int_0^\tau \sin(\omega_0[\tau - x])\, f(x)\, dx \;=\; \frac{1}{\pi m}\int_0^\tau \sin(\pi[\tau - x])\, f(x)\, dx \quad .$$

If $0 \le \tau < 1$, then, for the given f,

$$\begin{aligned} y(\tau) &= \frac{1}{\pi m}\int_0^\tau \sin(\pi[\tau - x]) \underbrace{f(x)}_{1}\, dx \\ &= \frac{1}{\pi^2 m}\cos(\pi[\tau - x])\Big|_{x=0}^{\tau} \;=\; \frac{1}{\pi^2 m}\left[1 - \cos(\pi\tau)\right] \quad .\end{aligned}$$

And for $1 \le \tau < 2$,

$$\begin{aligned} y(\tau) &= \frac{1}{\pi m}\int_0^\tau \sin(\pi[\tau - x])\, f(x)\, dx \\ &= \frac{1}{\pi m}\left[\int_0^1 \sin(\pi[\tau - x]) \underbrace{f(x)}_{1}\, dx \;+\; \int_1^\tau \sin(\pi[\tau - x]) \underbrace{f(x)}_{0}\, dx\right]\end{aligned}$$

$$= \frac{1}{\pi^2 m} \cos(\pi[\tau - x])\Big|_{x=0}^{1} + 0$$

$$= \frac{1}{\pi^2 m} [\cos(\pi[\tau - 1]) - \cos(\pi\tau)] = \frac{1}{\pi^2 m} [-2\cos(\pi\tau)] \quad .$$

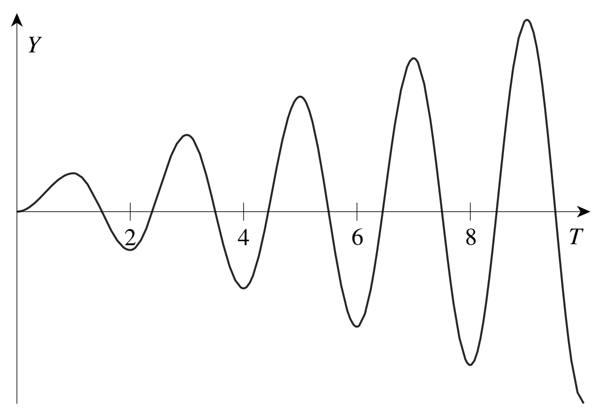
Plugging all the above into the last formula for $y(t)$, we get

$$y(t) = \frac{1}{\pi^2 m} \begin{Bmatrix} 1 - \cos(\pi\tau) & \text{if} \quad 0 \le \tau < 1 \\ -2\cos(\pi\tau) & \text{if} \quad 1 \le \tau < 2 \end{Bmatrix} - \frac{2n}{\pi^2 m} \cos(\pi\tau) \quad .$$

That is,

$$y(t) = \frac{1}{\pi^2 m} \begin{cases} 1 - (2n+1)\cos(\pi\tau) & \text{if} \quad 0 \le \tau < 1 \\ -2(n+1)\cos(\pi\tau) & \text{if} \quad 1 \le \tau < 2 \end{cases} \quad . \qquad (\star)$$

For part iii: Just how you program your computer math package to graph $y(t)$ using formula ($\star$), above, depends on what computer math package you have, and, possibly, which version you have. Just remember that you are graphing a piecewise-defined function that, because of the n in formula ($\star$), changes at every integer value of t. Here is what the author obtained:



27.13 c. *For part i:* Theorem 27.5 on page 534 of the text tells us that

$$y(t + p_0) - y(t) = A\cos(\omega_0 t - \phi)$$

where

$$A = \frac{1}{\omega_0 m}\sqrt{(\mathcal{I}_S)^2 + (\mathcal{I}_C)^2} \quad .$$

In this case, $\omega_0 = \frac{2\pi}{p_0} = \frac{2\pi}{1} = 2\pi$,

$$\mathcal{I}_S = \int_0^{p_0} \cos(\omega_0 x) f(x)\,dx = \int_0^1 \cos(2\pi x)\sin(4\pi x)\,dx$$

and

$$\mathcal{I}_C = -\int_0^{p_0} \sin(\omega_0 x) f(x)\,dx = -\int_0^1 \sin(2\pi x)\sin(4\pi x)\,dx \quad .$$

Using trigonometric identities, these integrals are easily computed. You get $\mathcal{I}_S = 0$ and $\mathcal{I}_C = 0$. Consequently, we also have $A = 0$ and

$$y(t + p_0) - y(t) = A\cos(\omega_0 t - \phi) = 0 \quad .$$

In other words, resonance is not an issue here.

For part ii: We can simply apply convolution formula (27.18) on page 531 of the text:

$$\begin{aligned} y(t) &= \frac{1}{\omega_0 m}\int_0^t \sin(\omega_0[t-x])\, f(x)\,dx \\ &= \frac{1}{2\pi m}\int_0^t \sin(2\pi[t-x])\sin(4\pi x)\ dx \\ &= \cdots \qquad \text{(Use trigonometric identities or integration by parts.)} \\ &= \frac{1}{12\pi^2 m}[2\sin(2\pi t)-\sin(4\pi t)] \quad . \end{aligned} \tag{$\star$}$$

For part iii: Having a computer math package plot formula ($\star$) for y yields