Chapter 26: Convolution

26.2 a. Here $f(x) = e^{3x}$ and $g(t-x) = e^{5(t-x)}$.

So,

$$f*g(t) = \int_0^t f(x)g(t-x)\,dx = \int_0^t e^{3x}e^{5(t-x)}\,dx$$
$$= \int_0^t e^{3x}e^{5t}e^{-5x}\,dx$$
$$= e^{5t}\int_0^t e^{-2x}\,dx$$
$$= e^{5t}\left[-\frac{1}{2}e^{-2x}\right]\Big|_{x=0}^{t}$$
$$= -\frac{1}{2}e^{5t}\left[e^{-2t}-e^{0}\right] = \frac{1}{2}\left[e^{5t}-e^{3t}\right] .$$

26.2 c. Here $f(x) = \sqrt{x} = x^{1/2}$ and $g(t-x) = 6$.

So,

$$f*g(t) = \int_0^t f(x)g(t-x)\,dx = \int_0^t x^{1/2}\cdot 6\,dx = 6\left[\frac{2}{3}x^{3/2}\right]\Big|_{x=0}^{t} = 4t^{3/2} .$$

26.2 e. Here $f(x) = x^2$ and $g(t-x) = (t-x)^2 = t^2 - 2tx + x^2$.

So,

$$f*g(t) = \int_0^t f(x)g(t-x)\,dx = \int_0^t x^2\left[t^2-2tx+x^2\right]dx$$
$$= \int_0^t \left[t^2x^2-2tx^3+x^4\right]dx$$
$$= \left[\frac{1}{3}t^2x^3-\frac{1}{2}tx^4+\frac{1}{5}x^5\right]\Big|_{x=0}^{t}$$
$$= \frac{1}{3}t^5 - \frac{1}{2}t^5 + \frac{1}{5}t^5 - 0 = \frac{1}{30}t^5 .$$

26.2 g. Here $f(x) = \sin(x)$ and, using a well-known trigonometric identity,

$$g(t-x) = \sin(t-x) = \sin(t)\cos(x) - \cos(t)\sin(x) .$$

So,

$$f*g(t) = \int_0^t f(x)g(t-x)\,dx$$
$$= \int_0^t \sin(x)\left[\sin(t)\cos(x) - \cos(t)\sin(x)\right]dx$$
$$= \sin(t)\int_0^t \sin(x)\cos(x)\,dx - \cos(t)\int_0^t \sin^2(x)\,dx$$
$$= \sin(t)\int_0^t \frac{d}{dx}\left[\frac{1}{2}\sin^2(x)\right]dx - \cos(t)\int_0^t \frac{1}{2}\left[1-\cos(2x)\right]dx$$

$$= \frac{1}{2}\sin(x)\sin^2(x)\Big|_{x=0}^{t} \;-\; \frac{1}{2}\cos(t)\left[x - \frac{1}{2}\sin(2x)\right]\Big|_{x=0}^{t}$$

$$= \frac{1}{2}\sin(t)\sin^2(t) \;-\; \frac{1}{2}\cos(t)\left[t - \frac{1}{2}\sin(2t)\right]$$

$$= \frac{1}{2}\sin(t)\sin^2(t) \;-\; \frac{1}{2}\cos(t)\left[t - \frac{1}{2}[2\sin(t)\cos(t)]\right]$$

$$= \frac{1}{2}\left[\sin(t)[\sin^2(t) + \cos^2(t)] - t\cos(t)\right] \;=\; \frac{1}{2}\,[\sin(t) - t\cos(t)] \quad .$$

26.3. We first need to 'compute' all the convolutions in that equation, being careful to use different symbols for the variables of integration in each integral to prevent confusion.

For $f * [g * h]$, we first have

$$g * h(T) \;=\; \int_0^T g(x)h(t-x)\,dx$$

and, thus,

$$f * [g * h](t) \;=\; \int_0^t f(y) \cdot g * h(\underbrace{t-y}_{T})\,dy$$

$$= \int_{y=0}^{t} f(y)\left[\int_{x=0}^{t-y} g(x)h([t-y]-x)\,dx\right] dy \quad .$$

Hence,

$$f * [g * h](t) = \int_{y=0}^{t}\int_{x=0}^{t-y} f(y)g(x)h(t-[x+y])\,dx\,dy \quad . \qquad (\star)$$

For $[f * g] * h$, we first have

$$f * g(s) \;=\; \int_0^s f(z)g(s-z)\,dz$$

and, thus,

$$[f * g] * h(t) \;=\; \int_0^t f * g(s) \cdot h(t-s)\,ds$$

$$= \int_{s=0}^{t}\left[\int_{z=0}^{s} f(z)g(s-z)\,dz\right] h(t-s)\,ds$$

$$= \int_{s=0}^{t}\int_{z=0}^{s} f(z)g(s-z)h(t-s)\,dz\,ds \quad .$$

In the last line, the variable in f (i.e., z) is the variable of integration of the inner integral, while, in ($\star$), the variable in f (i.e., y) is the variable of integration for the outer integral. This suggests changing the order of integration in one of these double integrals. To do this, observe that the result of the last set of computations can be written as

$$[f * g] * h(t) \;=\; \iint_{\mathcal{R}} f(z)g(s-z)h(t-s)\,dA$$

where $\mathcal{R}$ is the region in the SZ–plane containing every point (s, z) such that

$$0 \;\le s \;\le\; t$$

and, for each s,

$$0 \;\le z \;\le\; s \quad .$$

This is the same region as that containing every point (s, z) such that

$$0 \leq z \leq t$$

and, for each z,

$$z \leq s \leq t \quad .$$

Thus,

$$\begin{aligned}[f * g] * h(t) &= \iint_{\mathcal{R}} f(z)g(s-z)h(t-s)\,dA \\ &= \int_{z=0}^{t} \int_{s=z}^{t} f(z)g(s-z)h(t-s)\,ds\,dz \quad .\end{aligned}$$

Applying the simple change of variables $y = z$, followed by the change of variables $x = s - y$ for each y in the inner integral, and then followed by a comparison with formula ($\star$) for $f * [g * h]$ yields

$$\begin{aligned}[f * g] * h(t) &= \int_{z=0}^{t} \int_{s=z}^{t} f(z)g(s-z)h(t-s)\,ds\,dz \\ &= \int_{y=0}^{t} \int_{s=y}^{t} f(y)g(s-y)h(t-s)\,ds\,dy \\ &= \int_{y=0}^{t} \int_{x=0}^{t-y} f(y)g(x)h(t-[x+y])\,dx\,dy \\ &= f * [g * h](t) \quad .\end{aligned}$$

∎

26.4 a.

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{1}{(s-4)(s-3)}\right]\bigg|_t &= \mathcal{L}^{-1}\left[\frac{1}{s-4} \cdot \frac{1}{s-3}\right]\bigg|_t \\ &= \mathcal{L}^{-1}\left[\frac{1}{s-4}\right]\bigg|_t * \mathcal{L}^{-1}\left[\frac{1}{s-3}\right]\bigg|_t \\ &= e^{4t} * e^{3t} \\ &= \int_0^t e^{4x} e^{3(t-x)}\,dx \\ &= e^{3t} \int_0^t e^x\,dx = e^{3t}\left[e^t - e^0\right] = e^{4t} - e^{3t} \quad .\end{aligned}$$

26.4 c.

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{1}{s(s^2+4)}\right]\bigg|_t &= \mathcal{L}^{-1}\left[\frac{1}{s} \cdot \frac{1}{s^2+4}\right]\bigg|_t \\ &= \mathcal{L}^{-1}\left[\frac{1}{s}\right]\bigg|_t * \mathcal{L}^{-1}\left[\frac{1}{s^2+4}\right]\bigg|_t \\ &= 1 * \left[\frac{1}{2}\sin(2t)\right] \\ &= \frac{1}{2}\sin(2t) * 1 \\ &= \frac{1}{2}\int_0^t \sin(2x) \cdot 1\,dx = -\frac{1}{4}\cos(2s)\bigg|_{x=0}^{t} = \frac{1}{4}\left[1 - \cos(2t)\right] \quad .\end{aligned}$$

26.4 e.

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{1}{(s^2+9)^2}\right]\bigg|_t &= \mathcal{L}^{-1}\left[\frac{1}{s^2+9}\cdot\frac{1}{s^2+9}\right]\bigg|_t \\
&= \mathcal{L}^{-1}\left[\frac{1}{s^2+9}\right]\bigg|_t * \mathcal{L}^{-1}\left[\frac{1}{s^2+9}\right]\bigg|_t \\
&= \frac{1}{3}\sin(3t) * \left[\frac{1}{3}\sin(3t)\right] \\
&= \frac{1}{9}\sin(3t) * \sin(3t) \\
&= \frac{1}{9}\int_0^t \sin(3x)\sin(3[t-x])\,dx \\
&= \cdots = \frac{1}{54}\left[\sin(3t) - 3t\cos(3t)\right] \quad .
\end{aligned}$$

(The omitted details of computation are very similar those done in the solution to exercise 26.2 g, above.)

26.4 g.

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{1}{\sqrt{s}(s-3)}\right]\bigg|_t &= \mathcal{L}^{-1}\left[\frac{1}{\sqrt{s}}\cdot\frac{1}{s-3}\right]\bigg|_t \\
&= \mathcal{L}^{-1}\left[\frac{1}{\sqrt{s}}\right]\bigg|_t * \mathcal{L}^{-1}\left[\frac{1}{s-3}\right]\bigg|_t \\
&= \frac{1}{\sqrt{\pi}\sqrt{t}} * e^{3t} \\
&= \frac{1}{\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{x}}e^{3[t-x]}\,dx = \frac{1}{\sqrt{\pi}}e^{3t}\int_0^t \frac{1}{\sqrt{x}}e^{-3x}\,dx \quad .
\end{aligned}$$

26.5 a.

$$\mathcal{L}\big[y'' + 4y\big]\big|_s = \mathcal{L}[f(t)]|_s$$

$$\hookrightarrow \quad \mathcal{L}\big[y''\big]\big|_s + 4\mathcal{L}[y]|_s = F(s)$$

$$\hookrightarrow \quad \big[s^2Y(s) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0}\big] + 4Y(s) = F(s)$$

$$\hookrightarrow \quad \left(s^2+4\right)Y(s) = F(s)$$

$$\hookrightarrow \quad Y(s) = \underbrace{\frac{1}{s^2+4}}_{H(s)}F(s) \quad .$$

Thus, the transfer function is

$$H(s) = \frac{1}{s^2+4} \quad ,$$

the impulse response function is

$$h(t) = \mathcal{L}^{-1}[H(s)]|_t = \mathcal{L}^{-1}\left[\frac{1}{s^2+4}\right]\bigg|_t = \frac{1}{2}\sin(2t) \quad ,$$

and the solution to the differential equation is given by

$$\begin{aligned} y(t) = \mathcal{L}^{-1}[Y(s)]|_t &= \mathcal{L}^{-1}[H(s)F(s)]|_t \\ &= h * f(t) \\ &= \int_0^t h(x)f(t-x)\,dx = \frac{1}{2}\int_0^t \sin(2x) f(t-x)\,dx \quad . \end{aligned}$$

26.5 c.

$$\mathcal{L}\big[y'' - 6y' + 9y\big]\big|_s = \mathcal{L}[f(t)]|_s$$

$$\hookrightarrow \quad \mathcal{L}\big[y''\big]\big|_s - 6\mathcal{L}\big[y'\big]\big|_s + 9\mathcal{L}[y]|_s = F(s)$$

$$\hookrightarrow \quad \big[s^2Y(s) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0}\big] - 6[sY(s) - \underbrace{y(0)}_{0}] + 9Y(s) = F(s)$$

$$\hookrightarrow \quad \left(s^2 - 6s + 9\right)Y(s) = F(s)$$

$$\hookrightarrow \quad Y(s) = \underbrace{\frac{1}{s^2 - 6s + 9}}_{H(s)} F(s) \quad .$$

Thus, the transfer function is

$$H(s) = \frac{1}{s^2 - 6s + 9} = \frac{1}{(s-3)^2}$$

and the impulse response function is

$$h(t) = \mathcal{L}^{-1}[H(s)]|_t = \mathcal{L}^{-1}\left[\frac{1}{(s-3)^2}\right]\Bigg|_t = e^{3t}t$$

(which can be found via the first translation identity or convolution). Hence, finally, the solution to the differential equation is given by

$$\begin{aligned} y(t) = \mathcal{L}^{-1}[Y(s)]|_t &= \mathcal{L}^{-1}[H(s)F(s)]|_t \\ &= h * f(t) \\ &= \int_0^t h(x)f(t-x)\,dx = \frac{1}{2}\int_0^t xe^{3x} f(t-x)\,dx \quad . \end{aligned}$$

26.5 e.

$$\mathcal{L}\big[y''' + 16y'\big]\big|_s = \mathcal{L}[f(t)]|_s$$

$$\hookrightarrow \quad \big[s^3Y(s) - s^2\cdot 0 - s\cdot 0 - 0\big] + 16[sY(s) - 0] = F(s)$$

$$\hookrightarrow \qquad \left(s^3 + 16s\right) Y(s) \; = \; F(s)$$

$$\hookrightarrow \qquad Y(s) \; = \; \underbrace{\frac{1}{s^3 + 16s}}_{H(s)} F(s) \quad .$$

Thus, the transfer function is

$$H(s) \; = \; \frac{1}{s^2 + 16s} \; = \; \frac{1}{s(s^2 + 16)}$$

and the impulse response function is

$$h(t) \; = \; \mathcal{L}^{-1}[H(s)]|_t \; = \; \mathcal{L}^{-1}\left[\frac{1}{s(s^2+16)}\right]\Bigg|_t \; = \; \frac{1}{16}\,[1 - \cos(4t)]$$

(which can be found via partial fractions or convolution [see the solution to exercise 26.4 c]). Hence, finally, the solution to the differential equation is given by

$$\begin{aligned} y(t) \; = \; \mathcal{L}^{-1}[Y(s)]|_t \; &= \; \mathcal{L}^{-1}[H(s)F(s)]|_t \\ &= \; h * f(t) \\ &= \; \int_0^t h(x) f(t-x)\,dx \\ &= \; \frac{1}{16}\int_0^t [1 - \cos(4x)]\, f(t-x)\,dx \quad . \end{aligned}$$

26.6 a.

$$y(t) \; = \; \frac{1}{2}\int_0^t \sin(2x)\,\underbrace{f(t-x)}_{1}\,dx \; = \; \frac{1}{2}\int_0^t \sin(2x)\,dx \; = \; \frac{1}{4}[1 - \cos(2t)] \quad .$$

26.6 c.

$$\begin{aligned} y(t) \; &= \; \frac{1}{2}\int_0^t \sin(2x) f(t-x)\,dx \\ &= \; \frac{1}{2}\int_0^t \sin(2x) e^{3(t-x)}\,dx \; = \; \frac{1}{2}e^{3t}\int_0^t \sin(2x)e^{-3x}\,dx \quad , \end{aligned}$$

which we rewrite as

$$y(t) \; = \; \frac{1}{2}e^{3t}\mathcal{I}$$

where (repeatedly using integration by parts)

$$\begin{aligned} \mathcal{I} \; &= \; \int_0^t \sin(2x)e^{-3x}\,dx \\ &= \; -\frac{1}{3}\sin(2x)e^{-3x}\Big|_{x=0}^{t} \; - \; \int_0^t 2\cos(2x)\left(-\frac{1}{3}e^{3x}\right)dx \\ &= \; -\frac{1}{3}\sin(2t)e^{-3t} \; + \; \frac{2}{3}\int_0^t \cos(2x)e^{-3x}\,dx \\ &= \; -\frac{1}{3}\sin(2t)e^{-3t} \\ &\qquad + \; \frac{2}{3}\left[-\frac{1}{3}\cos(2x)e^{-3x}\Big|_{x=0}^{t} \; - \; \int_0^t (-2\sin(2x))\left(-\frac{1}{3}e^{-3x}\right)dx\right] \\ &= \; -\frac{1}{3}\sin(2t)e^{-3t} \; + \; \frac{2}{9}\left[1 - \cos(2t)e^{-3t}\right] \; - \; \frac{4}{9}\int_0^t \sin(2x)e^{-3x}\,dx \quad . \end{aligned}$$

After cutting out the middle and recognizing that the last integral is $\mathcal{I}$, the above reduces to

$$\mathcal{I} = -\frac{1}{3}\sin(2t)e^{-3t} + \frac{2}{9}\left[1-\cos(2t)e^{-3t}\right] - \frac{4}{9}\mathcal{I} \quad .$$

Solving this for $\mathcal{I}$ yields

$$\mathcal{I} = \frac{1}{13}\left[2 - 2\cos(2t)e^{-3t} - 3\sin(2t)e^{-3t}\right] \quad ,$$

and this, along with our last formula for y, gives us

$$y(t) = \frac{1}{2}e^{3t}\mathcal{I} = \frac{1}{13}\left[e^{3t} - \cos(2t) - \frac{3}{2}\sin(2t)\right]$$

26.6 e.

$$\begin{aligned} y(t) &= \frac{1}{2}\int_0^t \sin(2x) f(t-x)\,dx \\ &= \frac{1}{2}\int_0^t \sin(2x)\sin(\alpha[t-x])\,dx \\ &= \cdots \qquad \text{(Use integration by parts repeatedly, as in the solution to exercise 26.6 c.)} \\ &= \frac{\alpha\sin(2t) - 2\sin(\alpha t)}{2\alpha^2 - 8} \quad . \end{aligned}$$

26.7 a.

$$y(t) = \int_0^t xe^{3x} \underbrace{f(t-x)}_{1}\,dx = \int_0^t xe^{3x}\,dx = \frac{1}{9}\left[3te^{3t} - e^{3t} + 1\right] \quad .$$

26.7 c.

$$\begin{aligned} y(t) = \int_0^t xe^{3x} f(t-x)\,dx &= \int_0^t xe^{3x}e^{3[t-x]}\,dx \\ &= e^{3t}\int_0^t x\,dx = \frac{1}{2}t^2e^{3t} \quad . \end{aligned}$$

26.7 e.

$$\begin{aligned} y(t) = \int_0^t xe^{3x} f(t-x)\,dx &= \int_0^t xe^{3x}e^{\alpha[t-x]}\,dx \\ &= e^{\alpha t}\int_0^t xe^{(3-\alpha)x}\,dx \\ &= e^{\alpha t}\left[\left(\frac{x}{3-\alpha} - \frac{1}{(3-\alpha)^2}\right)e^{(3-\alpha)x}\right]\Bigg|_{x=0}^{t} \\ &= \left[\frac{t}{3-\alpha} - \frac{1}{((3-\alpha)^2}\right]e^{3t} + \frac{1}{(3-\alpha)^2}e^{\alpha t} \quad . \end{aligned}$$

26.8 a.

$$\begin{aligned} y(t) &= \frac{1}{16}\int_0^t [1-\cos(4x)]\underbrace{f(t-x)}_{1} \\ &= \frac{1}{16}\int_0^t [1-\cos(4x)]\,dx = \frac{1}{16}\left[t - \frac{1}{4}\sin(4t)\right] = \frac{1}{64}\left[4t - \sin(4t)\right] \quad . \end{aligned}$$

26.8 c.

$$\begin{aligned} y(t) &= \frac{1}{16}\int_0^t [1-\cos(4x)]\, f(t-x) \\ &= \frac{1}{16}\int_0^t [1-\cos(4x)]\, e^{3[t-x]}\, dx \\ &= \frac{1}{16}\int_0^t \left[e^{3[t-x]} - \cos(4x)e^{3[t-x]}\right] dx \\ &= \frac{1}{16}\left[\frac{1}{3}e^{3t} - \frac{1}{3} - \int_0^t \cos(4x)e^{3[t-x]}\, dx\right] \\ &= \cdots \qquad \text{(Use integration by parts repeatedly, as in the solution to exercise 26.6 c.)} \\ &= \frac{1}{1200}\left[16e^{3t} - 25 + 9\cos(4t) - 12\sin(4t)\right] \quad . \end{aligned}$$

26.8 e.

$$\begin{aligned} y(t) &= \frac{1}{16}\int_0^t [1-\cos(4x)]\, f(t-x)\, dx \\ &= \frac{1}{16}\int_0^t [1-\cos(4x)]\sin(\alpha[t-x])\, dx \\ &= \frac{1}{16}\int_0^t \left[\sin(\alpha[t-x]) - \cos(4x)\sin(\alpha[t-x])\right] dx \\ &= \frac{1}{16}\left[\frac{1}{\alpha}[1-\cos(\alpha t) - \int_0^t \cos(4x)\sin(\alpha[t-x])\, dx\right] \\ &= \cdots \qquad \text{(Use integration by parts repeatedly, as in the solution to exercise 26.6 c.)} \\ &= \frac{1}{16}\left[\frac{1}{\alpha}[1-\cos(\alpha t) + \frac{\alpha}{\alpha^2-16}[\cos(\alpha t) - \cos(4t)]\right] \\ &= \frac{1}{16}\left[\frac{1}{\alpha} + \frac{16}{\alpha\left(\alpha^2-16\right)}\cos(\alpha t) - \frac{\alpha}{\alpha^2-16}\cos(4t)\right] \\ &= \frac{1}{16\alpha\left(\alpha^2-16\right)}\left[\alpha^2 - 16 + 16\cos(\alpha t) - \alpha^2\cos(4t)\right] \quad . \end{aligned}$$

26.9 a. Since $t > 0$,

$$\begin{aligned} |f * g(t)| &= \left|\int_0^t f(x)g(t-x)\, dx\right| \\ &\le \int_0^t |f(x)g(t-x)|\, dx \\ &= \int_0^t |f(x)|\,|g(t-x)|\, dx \\ &< \int_0^t M_f e^{s_0 x} \cdot M_g e^{s_0[t-x]}\, dx \\ &= M_f M_g e^{s_0 t}\int_0^t 1\, dx \;=\; M_f M_g e^{s_0 t}\, t \quad . \end{aligned}$$

26.9 b. It follows immediately from lemma 24.8 on page 478 in the text.