

Chapter 25: The Inverse Laplace Transform

25.1 a. From the table: $\mathcal{L}^{-1}\left[\frac{1}{s-6}\right]\Big|_t = e^{6t}$.

25.1 c. $\mathcal{L}^{-1}\left[\frac{1}{s^2}\right]\Big|_t = \mathcal{L}^{-1}\left[\frac{1!}{s^{1+1}}\right]\Big|_t = t^1 = t$.

25.1 e. $\mathcal{L}^{-1}\left[\frac{5}{s^2+25}\right]\Big|_t = \mathcal{L}^{-1}\left[\frac{5}{s^2+5^2}\right]\Big|_t = \sin(5t)$.

25.2 a. $\mathcal{L}^{-1}\left[\frac{6}{s+2}\right]\Big|_t = \mathcal{L}^{-1}\left[6 \cdot \frac{1}{s-(-2)}\right]\Big|_t = 6\mathcal{L}^{-1}\left[\frac{1}{s-(-2)}\right]\Big|_t = 6e^{-2t}$.

25.2 c.

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{3}{\sqrt{s}} - \frac{8}{s-4}\right]\Big|_t &= 3\mathcal{L}^{-1}\left[\frac{1}{\sqrt{s}}\right]\Big|_t - 8\mathcal{L}^{-1}\left[\frac{1}{s-4}\right]\Big|_t \\ &= 3\mathcal{L}^{-1}\left[\frac{1}{\sqrt{\pi}} \frac{\sqrt{\pi}}{\sqrt{s}}\right]\Big|_t - 8e^{4t} \\ &= \frac{3}{\sqrt{\pi}}\mathcal{L}^{-1}\left[\frac{\sqrt{\pi}}{\sqrt{s}}\right]\Big|_t - 8e^{4t} \\ &= \frac{3}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{t}} - 8e^{4t} = \frac{3}{\sqrt{\pi t}} - 8e^{4t} \end{aligned}$$

25.2 e.

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{3s+1}{s^2+25}\right]\Big|_t &= \mathcal{L}^{-1}\left[\frac{3s}{s^2+25} + \frac{1}{s^2+25}\right]\Big|_t \\ &= \mathcal{L}^{-1}\left[3\frac{s}{s^2+5^2} + \frac{1}{5}\frac{5}{s^2+5^2}\right]\Big|_t \\ &= 3\mathcal{L}^{-1}\left[\frac{s}{s^2+5^2}\right]\Big|_t + \frac{1}{5}\mathcal{L}^{-1}\left[\frac{5}{s^2+5^2}\right]\Big|_t \\ &= 3\cos(5t) + \frac{1}{5}\sin(5t) \end{aligned}$$

25.3 a. From exercise 24.3, we have $\mathcal{L}[t \sin(\omega t)]_s = \frac{2\omega s}{(s^2 + \omega^2)^2}$.

By the definition of the inverse Laplace transform and linearity, it then follows that

$$t \sin(\omega t) = \mathcal{L}^{-1}\left[\frac{2\omega s}{(s^2 + \omega^2)^2}\right]\Big|_t = 2\omega\mathcal{L}^{-1}\left[\frac{s}{(s^2 + \omega^2)^2}\right]\Big|_t$$

Dividing through by 2ω then yields

$$\mathcal{L}^{-1}\left[\frac{s}{(s^2 + \omega^2)^2}\right]\Big|_t = \frac{t}{2\omega} \sin(\omega t) \text{ .}$$

$$25.4 \text{ a.} \quad \mathcal{L}[y' + 9y]|_s = \mathcal{L}[0]|_s$$

$$\hookrightarrow \mathcal{L}[y']|_s + 9\mathcal{L}[y]|_s = 0$$

$$\hookrightarrow [sY(s) - \underbrace{y(0)}_4] + 9Y(s) = 0$$

$$\hookrightarrow (s + 9)Y(s) - 4 = 0 \quad \rightsquigarrow \quad Y(s) = \frac{4}{s + 9} .$$

Taking the inverse:

$$y(t) = \mathcal{L}^{-1}[Y(s)]|_t = \mathcal{L}^{-1}\left[\frac{4}{s+9}\right]|_t = 4\mathcal{L}^{-1}\left[\frac{1}{s-(-9)}\right]|_t = 4e^{-9t} .$$

25.5 a. First, we must find the partial fraction expansion:

$$\begin{aligned} \frac{7s + 5}{(s + 2)(s - 1)} &= \frac{A}{s + 2} + \frac{B}{s - 1} \\ &= \frac{A(s - 1)}{(s + 2)(s - 1)} + \frac{B(s + 2)}{(s - 1)(s + 2)} = \frac{A(s - 1) + B(s + 2)}{(s + 2)(s - 1)} . \end{aligned}$$

Cutting out the middle and multiplying through by $(s + 2)(s - 1)$ then gives

$$A(s - 1) + B(s + 2) = 7s + 5 .$$

Solving for A and B is easily done by, respectively, plugging $s = -2$ and $s = 1$ into the last equation and “solving”:

$$A(-2 - 1) + B(-2 + 2) = 7(-2) + 5 \quad \rightsquigarrow \quad A = \frac{7(-2) + 5}{-3} = 3$$

and

$$A(1 - 1) + B(1 + 2) = 7 \cdot 1 + 5 \quad \rightsquigarrow \quad B = \frac{7 + 5}{1 + 2} = 4 .$$

Thus, the partial fraction expansion of our function is

$$\frac{7s + 5}{(s + 2)(s - 1)} = \frac{A}{s + 2} + \frac{B}{s - 1} = \frac{3}{s + 2} + \frac{4}{s - 1} .$$

Now we can take the inverse transform:

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{7s + 5}{(s + 2)(s - 1)}\right]|_s &= \mathcal{L}^{-1}\left[\frac{3}{s + 2} + \frac{4}{s - 1}\right]|_s \\ &= 3\mathcal{L}^{-1}\left[\frac{1}{s + 2}\right]|_s + 4\mathcal{L}^{-1}\left[\frac{1}{s - 1}\right]|_s = 3e^{-2t} + 4e^t . \end{aligned}$$

$$\begin{aligned} 25.5 \text{ c.} \quad \frac{1}{s^2 - 4} &= \frac{1}{(s - 2)(s + 2)} \\ &= \frac{A}{s - 2} + \frac{B}{s + 2} \\ &= \frac{A(s + 2)}{(s - 2)(s + 2)} + \frac{B(s - 2)}{(s + 2)(s - 2)} = \frac{A(s + 2) + B(s - 2)}{(s - 2)(s + 2)} . \end{aligned}$$

So,

$$A(s + 2) + B(s - 2) = 1 \quad .$$

Letting $s = 2$ yields

$$A(2 + 2) + B(2 - 2) = 1 \quad \rightsquigarrow \quad A = \frac{1}{4} \quad ,$$

while letting $s = -1$ yields

$$A(-2 + 2) + B(-2 - 2) = 1 \quad \rightsquigarrow \quad B = -\frac{1}{4} \quad .$$

Thus,

$$\frac{1}{s^2 - 4} = \frac{A}{s - 2} + \frac{B}{s + 2} = \frac{1/4}{s - 2} + \frac{-1/4}{s + 2} \quad ,$$

and

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{1}{s^2 - 4}\right]\Big|_t &= \mathcal{L}^{-1}\left[\frac{1/4}{s - 2} + \frac{-1/4}{s + 2}\right]\Big|_t \\ &= \frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s - 2}\right]\Big|_t - \frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s + 2}\right]\Big|_t = \frac{1}{4}e^{2t} - \frac{1}{4}e^{-2t} \quad . \end{aligned}$$

25.5 e.

$$\begin{aligned} \frac{1}{s^3 - 4s^2} &= \frac{1}{s^2(s - 4)} \\ &= \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s - 4} = \frac{A(s - 4) + Bs(s - 4) + Cs^2}{s^2(s - 4)} \quad . \end{aligned}$$

So

$$A(s - 4) + Bs(s - 4) + Cs^2 = 1 \quad .$$

Multiplying this out and gathering like terms yields

$$(B + C)s^2 + (A - 4B)s - 4A = 1 = 0s^2 + 0s + 1 \quad ,$$

giving us the system

$$\begin{aligned} B + C &= 0 \\ A - 4B &= 0 \quad , \\ -4A &= 1 \end{aligned}$$

which, in turn, means that

$$A = -\frac{1}{4} \quad , \quad B = \frac{A}{4} = -\frac{1}{16} \quad \text{and} \quad C = -B = \frac{1}{16} \quad .$$

So

$$\frac{1}{s^3 - 4s^2} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s - 4} = -\frac{1/4}{s^2} - \frac{1/16}{s} + \frac{1/16}{s - 4} \quad ,$$

and

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{1}{s^3 - 4s^2}\right]\Big|_t &= \mathcal{L}^{-1}\left[-\frac{1/4}{s^2} - \frac{1/16}{s} + \frac{1/16}{s - 4}\right]\Big|_t \\ &= -\frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s^2}\right]\Big|_t - \frac{1}{16}\mathcal{L}^{-1}\left[\frac{1}{s}\right]\Big|_t + \frac{1}{16}\mathcal{L}^{-1}\left[\frac{1}{s - 4}\right]\Big|_t \\ &= -\frac{1}{4}t^1 - \frac{1}{16} \cdot 1 + \frac{1}{16}e^{4t} = \frac{1}{16}e^{4t} - \frac{1}{4}t - \frac{1}{16} \quad . \end{aligned}$$

$$25.5 \text{ g.} \quad \frac{5s^2 + 6s - 40}{(s+6)(s^2+16)} = \frac{A}{s+6} + \frac{Bs+C}{s^2+16} = \frac{A(s^2+16) + (Bs+C)(s+6)}{(s+6)(s^2+16)} .$$

So

$$A(s^2+16) + (Bs+C)(s+6) = 5s^2 + 6s - 40 . \quad (\star)$$

Multiplying this out and gathering like terms yields

$$(A+B)s^2 + (6B+C)s + (16A+6C) = 5s^2 + 6s - 40 ,$$

giving us the system

$$\begin{aligned} A + B &= 5 \\ 6B + C &= 6 \\ 16A + 6C &= 40 \end{aligned} . \quad (\star\star)$$

The first coefficient, A , is easily found after first letting $s = -6$ in equation (\star) :

$$A[(-6)^2 + 16] + (B[-6] + C)(-6 + 6) = 5[-6]^2 + 6[-6] - 40$$

$$\hookrightarrow A = \frac{5[-6]^2 + 6[-6] - 40}{[-6]^2 + 16} = 2 .$$

From this and system $(\star\star)$, we then obtain

$$B = 5 - A = 5 - 2 = 3 \quad \text{and} \quad C = 6 - 6B = 6 - 6 \cdot 3 = -12 .$$

So,

$$\begin{aligned} \frac{5s^2 + 6s - 40}{(s+6)(s^2+16)} &= \frac{A}{s+6} + \frac{Bs+C}{s^2+16} \\ &= \frac{2}{s+6} + \frac{3s-12}{s^2+16} = \frac{2}{s+6} + \frac{3s}{s^2+16} - \frac{12}{s^2+16} , \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}^{-1} \left[\frac{5s^2 + 6s - 40}{(s+6)(s^2+16)} \right] \Big|_t &= \mathcal{L}^{-1} \left[\frac{2}{s+6} + \frac{3s}{s^2+16} - \frac{12}{s^2+16} \right] \Big|_t \\ &= 2\mathcal{L}^{-1} \left[\frac{1}{s+6} \right] \Big|_t + 3\mathcal{L}^{-1} \left[\frac{s}{s^2+16} \right] \Big|_t - 12\mathcal{L}^{-1} \left[\frac{1}{s^2+16} \right] \Big|_t \\ &= 2e^{-6t} + 3\mathcal{L}^{-1} \left[\frac{s}{s^2+4^2} \right] \Big|_t - \frac{12}{4}\mathcal{L}^{-1} \left[\frac{4}{s^2+4^2} \right] \Big|_t \\ &= 2e^{-6t} + 3\cos(4t) - 3\sin(4t) . \end{aligned}$$

$$\begin{aligned} 25.5 \text{ i.} \quad \frac{6s^2 + 62s + 92}{(s+1)(s^2+10s+21)} &= \frac{6s^2 + 62s + 92}{(s+1)(s+3)(s+7)} \\ &= \frac{A}{s+1} + \frac{B}{s+3} + \frac{C}{s+7} \\ &= \frac{A(s+3)(s+7) + B(s+1)(s+7) + C(s+1)(s+3)}{(s+1)(s+3)(s+7)} . \end{aligned}$$

So

$$A(s+3)(s+7) + B(s+1)(s+7) + C(s+1)(s+3) = 6s^2 + 62s + 92 \quad . \quad (\star)$$

Letting $s = -1$ in equation (\star) :

$$\begin{aligned} A(-1+3)(-1+7) + B \cdot 0 \cdot (-1+7) + C \cdot 0 \cdot (-1+3) \\ = 6[-1]^2 + 62[-1] + 92 \end{aligned}$$

$$\hookrightarrow A = \frac{6[-1]^2 + 62[-1] + 92}{(-1+3)(-1+7)} = 3 \quad .$$

Letting $s = -3$ in equation (\star) :

$$\begin{aligned} A \cdot 0 \cdot (-3+7) + B(-3+1)(-3+7) + C(-3+1) \cdot 0 \\ = 6[-3]^2 + 62[-3] + 92 \end{aligned}$$

$$\hookrightarrow B = \frac{6[-3]^2 + 62[-3] + 92}{(-3+1)(-3+7)} = 5 \quad .$$

Letting $s = -7$ in equation (\star) :

$$\begin{aligned} A(-7+3) \cdot 0 + B(-7+1) \cdot 0 + C(-7+1)(-7+3) \\ = 6[-7]^2 + 62[-7] + 92 \end{aligned}$$

$$\hookrightarrow C = \frac{6[-7]^2 + 62[-7] + 92}{(-7+1)(-7+3)} = -2 \quad .$$

Thus,

$$\begin{aligned} \frac{6s^2 + 62s + 92}{(s+1)(s^2 + 10s + 21)} &= \frac{A}{s+1} + \frac{B}{s+3} + \frac{C}{s+7} \\ &= \frac{3}{s+1} + \frac{5}{s+3} - \frac{2}{s+7} \quad , \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}^{-1} \left[\frac{6s^2 + 62s + 92}{(s+1)(s^2 + 10s + 21)} \right] \Big|_t \\ &= \mathcal{L}^{-1} \left[\frac{3}{s+1} + \frac{5}{s+3} - \frac{2}{s+7} \right] \Big|_t \\ &= 3\mathcal{L}^{-1} \left[\frac{1}{s+1} \right] \Big|_t + 5\mathcal{L}^{-1} \left[\frac{1}{s+3} \right] \Big|_t - 2\mathcal{L}^{-1} \left[\frac{1}{s+7} \right] \Big|_t \\ &= 3e^{-t} + 5e^{-3t} - 2e^{-7t} \quad . \end{aligned}$$

25.6 a. Taking the transform:

$$\mathcal{L}[y'' - 9y] \Big|_s = \mathcal{L}[0] \Big|_s$$

$$\hookrightarrow \mathcal{L}[y''] \Big|_s - 9\mathcal{L}[y] \Big|_s = 0$$

$$\Leftrightarrow [s^2 Y(s) - s \underbrace{y(0)}_4 - \underbrace{y'(0)}_9] - 9Y(s) = 0$$

$$\Leftrightarrow (s^2 - 9)Y(s) - 4s - 9 = 0 \quad \rightsquigarrow \quad Y(s) = \frac{4s + 9}{s^2 - 9} .$$

Finding the partial fraction expansion for Y and then taking the inverse transform:

$$\frac{4s + 9}{s^2 - 9} = \frac{4s + 9}{(s - 3)(s + 3)} = \frac{A}{s - 3} + \frac{B}{s + 3} = \frac{A(s + 3) + B(s - 3)}{(s - 3)(s + 3)} .$$

So,

$$A(s + 3) + B(s - 3) = 4s + 9 .$$

Letting $s = 3$ and $s = -3$, respectively, in this last equation:

$$A(3 + 3) + B \cdot 0 = 4 \cdot 3 + 9 \quad \rightsquigarrow \quad A = \frac{4 \cdot 3 + 9}{3 + 3} = \frac{7}{2} ,$$

and

$$A \cdot 0 + B(-3 - 3) = 4[-3] + 9 \quad \rightsquigarrow \quad B = \frac{4[-3] + 9}{-3 - 3} = \frac{1}{2} .$$

Thus,

$$Y(s) = \frac{A}{s - 3} + \frac{B}{s + 3} = \frac{7/2}{s - 3} + \frac{1/2}{s + 3} ,$$

and

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}[Y(s)]|_t = \mathcal{L}^{-1}\left[\frac{7/2}{s - 3} + \frac{1/2}{s + 3}\right]|_t \\ &= \frac{7}{2}\mathcal{L}^{-1}\left[\frac{1}{s - 3}\right]|_t + \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s + 3}\right]|_t = \frac{7}{2}e^{3t} + \frac{1}{2}e^{-3t} . \end{aligned}$$

25.6 c. Taking the transform:

$$\mathcal{L}[y'' + 8y' + 7y]|_s = \mathcal{L}[165e^{4t}]|_s$$

$$\Leftrightarrow \mathcal{L}[y'']|_s + 8\mathcal{L}[y']|_s + 7\mathcal{L}[y]|_s = 165\mathcal{L}[e^{4t}]|_s$$

$$\begin{aligned} \Leftrightarrow [s^2 Y(s) - s \underbrace{y(0)}_8 - \underbrace{y'(0)}_1] \\ + 8[sY(s) - \underbrace{y(0)}_8] + 7Y(s) &= \frac{165}{s - 4} \end{aligned}$$

$$\Leftrightarrow (s^2 + 8s + 7)Y(s) - 8s - 65 = \frac{165}{s - 4}$$

Solving for Y :

$$\begin{aligned} Y(s) &= \frac{165}{(s - 4)(s^2 + 8s + 7)} + \frac{8s + 65}{s^2 + 8s + 7} \\ &= \frac{165 + (8s + 65)(s - 4)}{(s - 4)(s^2 + 8s + 7)} = \frac{8s^2 + 33s - 95}{(s - 4)(s + 7)(s + 1)} . \end{aligned}$$

Finding the partial fraction expansion of Y :

$$\begin{aligned} Y(s) &= \frac{8s^2 + 33s - 95}{(s-4)(s+7)(s+1)} \\ &= \frac{A}{s-4} + \frac{B}{s+7} + \frac{C}{s+1} \\ &= \frac{A(s+7)(s+1) + B(s-4)(s+1) + C(s-4)(s+7)}{(s-4)(s+7)(s+1)}, \end{aligned}$$

which requires that

$$A(s+7)(s+1) + B(s-4)(s+1) + C(s-4)(s+7) = 8s^2 + 33s - 95 \quad (\star)$$

Letting $s = 4$ in equation (\star) :

$$\begin{aligned} A(4+7)(4+1) + B \cdot 0 \cdot (4+1) + C \cdot 0 \cdot (4+7) \\ = 8 \cdot 4^2 + 33 \cdot 4 - 95 \end{aligned}$$

$$\Leftrightarrow A = \frac{8 \cdot 4^2 + 33 \cdot 4 - 95}{(4+7)(4+1)} = 3 \quad .$$

Letting $s = -7$ in equation (\star) :

$$\begin{aligned} A \cdot 0 \cdot (-7+1) + B(-7-4)(-7+1) + C(-7-4) \cdot 0 \\ = 8[-7]^2 + 33[-7] - 95 \end{aligned}$$

$$\Leftrightarrow B = \frac{8[-7]^2 + 33[-7] - 95}{(-7-4)(-7+1)} = 1 \quad .$$

Letting $s = -1$ in equation (\star) :

$$\begin{aligned} A(-1+7) \cdot 0 + B(-1-4) \cdot 0 + C(-1-4)(-1+7) \\ = 8[-1]^2 + 33[-1] - 95 \end{aligned}$$

$$\Leftrightarrow C = \frac{8[-1]^2 + 33[-1] - 95}{(-1-4)(-1+7)} = 4 \quad .$$

So,

$$Y(s) = \frac{A}{s-4} + \frac{B}{s+7} + \frac{C}{s+1} = \frac{3}{s-4} + \frac{1}{s+7} + \frac{4}{s+1} \quad ,$$

and

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}[Y(s)]|_t = \mathcal{L}^{-1}\left[\frac{3}{s-4} + \frac{1}{s+7} + \frac{4}{s+1}\right]|_t \\ &= 3\mathcal{L}^{-1}\left[\frac{1}{s-4}\right]|_t + \mathcal{L}^{-1}\left[\frac{1}{s-(-7)}\right]|_t + 4\mathcal{L}^{-1}\left[\frac{1}{s-(-1)}\right]|_t \\ &= 3e^{4t} + e^{-7t} + 4e^{-t} \quad . \end{aligned}$$

25.7 a. Using the translation identity, we have

$$\mathcal{L}^{-1}\left[\frac{1}{(s-7)^5}\right]|_t = \mathcal{L}^{-1}[F(s-7)]|_t = e^{7t}f(t) \quad (\star)$$

where

$$F(s - 7) = \frac{1}{(s - 7)^5} .$$

Replacing $s - 7$ with X , this means

$$F(X) = \frac{1}{X^5} .$$

Hence,

$$F(s) = \frac{1}{s^5}$$

and

$$f(t) = \mathcal{L}^{-1}[F(s)]|_t = \mathcal{L}^{-1}\left[\frac{1}{s^5}\right]|_t = \frac{1}{4!}\mathcal{L}^{-1}\left[\frac{4!}{s^{4+1}}\right]|_t = \frac{1}{24}t^4 .$$

Thus, equation (★) becomes

$$\mathcal{L}^{-1}\left[\frac{1}{(s - 7)^5}\right]|_t = \dots = e^{7t}f(t) = e^{7t}\frac{1}{24}t^4 .$$

25.7 c. Since the denominator does not factor, we “complete the square” of the denominator:

$$s^2 - 6s + 45 = \underbrace{s^2 - 2 \cdot 3s + 3^2}_{(s-3)^2} - 3^2 + 45 = (s - 3)^2 + 36 .$$

So,

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{s}{s^2 - 6s + 45}\right]|_t &= \mathcal{L}^{-1}\left[\frac{s}{(s - 3)^2 + 36}\right]|_t \\ &= \mathcal{L}^{-1}[F(s - 3)]|_t = e^{3t}f(t) \end{aligned} \quad (\star)$$

with

$$F(s - 3) = \frac{s}{(s - 3)^2 + 36} .$$

To get $F(s)$, first let $X = s - 3$ (equivalently, $s = X + 3$) in this equation:

$$F(X) = \frac{X + 3}{X^2 + 36} ,$$

which means

$$F(s) = \frac{s + 3}{s^2 + 36} = \frac{s}{s^2 + 6^2} + \frac{3}{s^2 + 6^2} .$$

Thus,

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}[F(s)]|_t \\ &= \mathcal{L}^{-1}\left[\frac{s}{s^2 + 6^2} + \frac{3}{s^2 + 6^2}\right]|_t \\ &= \mathcal{L}^{-1}\left[\frac{s}{s^2 + 6^2}\right]|_t + \frac{3}{6}\mathcal{L}^{-1}\left[\frac{6}{s^2 + 6^2}\right]|_t = \cos(6t) + \frac{1}{2}\sin(6t) , \end{aligned}$$

and equation (★) continues as

$$\mathcal{L}^{-1}\left[\frac{s}{s^2 - 6s + 45}\right]|_t = \dots = e^{3t}f(t) = e^{3t}\left[\cos(6t) + \frac{1}{2}\sin(6t)\right] .$$

25.7 e. Since the denominator factors as

$$s^2 + 8s + 16 = (s + 4)^2 = (s - [-4])^2 ,$$

we can apply the translation identity as follows:

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{1}{s^2 + 8s + 16}\right]\Big|_t &= \mathcal{L}^{-1}\left[\frac{1}{(s - [-4])^2}\right]\Big|_t \\ &= F(s - [-4]) = e^{-4t} f(t) . \end{aligned} \quad (\star)$$

with

$$F(s - [-4]) = \frac{1}{(s - [-4])^2} .$$

Letting $X = s - [-4]$ this becomes

$$F(X) = \frac{1}{X^2} .$$

So,

$$F(s) = \frac{1}{s^2} , \quad f(t) = \mathcal{L}^{-1}[F(s)]\Big|_t = \mathcal{L}^{-1}\left[\frac{1}{s^2}\right]\Big|_t = t ,$$

and equation (\star) becomes

$$\mathcal{L}^{-1}\left[\frac{1}{s^2 + 8s + 16}\right]\Big|_t = \dots = e^{-4t} f(t) = e^{-4t} t = te^{-4t} .$$

25.7 g. Since

$$s^2 + 12s + 40 = s^2 + 12s + 36 + 4 = (s + 6)^2 + 4 ,$$

we have

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{1}{s^2 + 12s + 40}\right]\Big|_t &= \mathcal{L}^{-1}\left[\frac{1}{(s + 6)^2 + 4}\right]\Big|_t \\ &= F(s + 6) = F(s - [-6]) = e^{-6t} f(t) \end{aligned} \quad (\star)$$

with

$$F(s + 6) = \frac{1}{(s + 6)^2 + 4} .$$

Letting $X = s + 6$, this becomes

$$F(X) = \frac{1}{X^2 + 4} .$$

Hence,

$$F(s) = \frac{1}{s^2 + 4} = \frac{1}{s^2 + 2^2} ,$$

$$f(t) = \mathcal{L}^{-1}[F(s)]\Big|_t = \mathcal{L}^{-1}\left[\frac{1}{s^2 + 2^2}\right]\Big|_t = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{s^2 + 2^2}\right]\Big|_t = \frac{1}{2}\sin(2t) ,$$

and equation (\star) becomes

$$\mathcal{L}^{-1}\left[\frac{1}{s^2 + 12s + 40}\right]\Big|_t = \dots = e^{-6t} f(t) = e^{-6t} \left[\frac{1}{2}\sin(2t)\right] = \frac{1}{2}\sin(2t)e^{-6t} .$$

25.8 a.

$$\mathcal{L}[y'' - 8y' + 17y]|_t = \mathcal{L}[0]|_t$$

$$\Leftrightarrow \mathcal{L}[y'']|_t - 8\mathcal{L}[y']|_t + 17\mathcal{L}[y]|_t = 0$$

$$\Leftrightarrow [s^2Y(s) - s\underbrace{y(0)}_3 - \underbrace{y'(0)}_{12}] - 8[sY(s) - \underbrace{y(0)}_3] + 17Y(s) = 0$$

$$\Leftrightarrow (s^2 - 8s + 17)Y(s) - 3s + 12 = 0 \quad .$$

So,

$$Y(s) = \frac{3s - 12}{s^2 - 8s + 17} = \frac{3s - 12}{s^2 - 8s + 16 + 1} = \frac{3s - 12}{(s - 4)^2 + 1} \quad ,$$

and

$$y(t) = \mathcal{L}^{-1}\left[\frac{3s - 12}{(s - 4)^2 + 1}\right]|_t = \mathcal{L}^{-1}[F(s - 4)]|_t = e^{4t}f(t) \quad (\star)$$

with

$$F(s - 4) = \frac{3s - 12}{(s - 4)^2 + 1} \quad .$$

Letting $X = s - 4$ (equivalently, $s = X + 4$) we have

$$F(X) = \frac{3[X + 4] - 12}{X^2 + 1} = \frac{3X}{X^2 + 1} \quad .$$

Thus,

$$F(s) = \frac{3s}{s^2 + 1} \quad ,$$

$$f(t) = \mathcal{L}^{-1}\left[\frac{3s}{s^2 + 1}\right]|_t = 3\mathcal{L}^{-1}\left[\frac{s}{s^2 + 1}\right]|_t = 3\cos(t) \quad ,$$

and

$$y(t) = \dots = e^{4t}f(t) = 3\cos(t)e^{4t} \quad .$$

25.8 c.

$$\mathcal{L}[y'' + 6y' + 13y]|_t = \mathcal{L}[0]|_t$$

$$\Leftrightarrow \mathcal{L}[y'']|_t + 6\mathcal{L}[y']|_t + 13\mathcal{L}[y]|_t = 0$$

$$\Leftrightarrow [s^2Y(s) - s\underbrace{y(0)}_2 - \underbrace{y'(0)}_8] + 6[sY(s) - \underbrace{y(0)}_2] + 13Y(s) = 0$$

$$\Leftrightarrow (s^2 + 6s + 13)Y(s) - 2s - 20 = 0 \quad .$$

So,

$$Y(s) = \frac{2s + 20}{s^2 + 6s + 13} = \frac{2s + 20}{s^2 + 6s + 9 + 4} \\ = \frac{2s + 20}{(s + 3)^2 + 4} = \frac{2s + 20}{(s - [-3])^2 + 4} \quad ,$$

and

$$y(t) = \mathcal{L}^{-1} \left[\frac{2s + 20}{(s - [-3])^2 + 4} \right]_t = \mathcal{L}^{-1}[F(s - [-3])]_t = e^{-3t} f(t) \quad (\star)$$

with

$$F(s - [-3]) = \frac{2s + 20}{(s - [-3])^2 + 4} \quad .$$

Letting $X = s - [-3] = s + 3$ (equivalently, $s = X - 3$), we have

$$F(X) = \frac{2[X - 3] + 20}{X^2 + 4} = \frac{2X + 14}{X^2 + 4} \quad .$$

Thus,

$$F(s) = \frac{2s + 14}{s^2 + 4} = \frac{2s}{s^2 + 2^2} + \frac{7 \cdot 2}{s^2 + 2^2} \quad ,$$

$$f(t) = \mathcal{L}^{-1} \left[\frac{2s}{s^2 + 2^2} + \frac{7 \cdot 2}{s^2 + 2^2} \right]_t \\ = 2\mathcal{L}^{-1} \left[\frac{s}{s^2 + 2^2} \right]_t + 7\mathcal{L}^{-1} \left[\frac{2}{s^2 + 2^2} \right]_t = 2 \cos(2t) + 7 \sin(2t) \quad ,$$

and

$$y(t) = \dots = e^{-3t} f(t) = e^{-3t} [2 \cos(2t) + 7 \sin(2t)] \quad .$$

25.9 a.

$$\mathcal{L}[y'']_t = \mathcal{L}[e^t \sin(t)]_s$$

$$\Leftrightarrow s^2 Y(s) - \underbrace{s y(0)}_0 - \underbrace{y'(0)}_0 = \mathcal{L}[e^{1t} \sin(t)]_s \\ = \mathcal{L}[e^{1t} \underbrace{\sin(t)}_{f(t)}]_s$$

$$\Leftrightarrow s^2 Y(s) = F(s - 1)$$

where

$$F(s) = \mathcal{L}[f(t)]_s = \mathcal{L}[\sin(t)]_s = \frac{1}{s^2 + 1} \quad .$$

So, the last line containing Y becomes

$$s^2 Y(s) = F(\underbrace{s - 1}_X) = F(X) = \frac{1}{X^2 + 1} = \frac{1}{(s - 1)^2 + 1} \quad .$$

Solving for Y and starting to find the corresponding partial fraction expansion leads to

$$Y(s) = \frac{1}{s^2((s - 1)^2 + 1)} \\ = \frac{A}{s^2} + \frac{B}{s} + \frac{Cs + D}{(s - 1)^2 + 1} \\ = \frac{A((s - 1)^2 + 1) + Bs((s - 1)^2 + 1) + (Cs + D)s^2}{s^2((s - 1)^2 + 1)} \quad ,$$

which requires that

$$A((s-1)^2 + 1) + Bs((s-1)^2 + 1) + (Cs + D)s^2 = 1 \quad .$$

Multiplying this out and gathering like terms then yields

$$\begin{aligned} s^3 + [A - 2B + D]s^2 + [-2A + 2B]s + 2A &= 1 \\ &= 0s^3 + 0s^2 + 0s + 1 \quad , \end{aligned}$$

which, in turn, requires that

$$\begin{aligned} B + C &= 0 \quad , \\ A - 2B + D &= 0 \quad , \\ -2A + 2B &= 0 \quad , \end{aligned}$$

and

$$2A = 1 \quad .$$

From this it follows that

$$A = \frac{1}{2} \quad , \quad B = A = \frac{1}{2} \quad , \quad D = 2B - A = \frac{1}{2} \quad , \quad C = -B = -\frac{1}{2} \quad ,$$

and, thus, the partial fraction expansion of Y is

$$\begin{aligned} Y(s) &= \frac{A}{s^2} + \frac{B}{s} + \frac{Cs + D}{(s-1)^2 + 1} \\ &= \frac{1}{2} \left(\frac{1}{s^2} + \frac{1}{s} + \frac{-s+1}{(s-1)^2 + 1} \right) = \frac{1}{2} \left(\frac{1}{s^2} + \frac{1}{s} - \frac{s-1}{(s-1)^2 + 1} \right) \quad . \end{aligned}$$

Consequently,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left[\frac{1}{2} \left(\frac{1}{s^2} + \frac{1}{s} - \frac{s-1}{(s-1)^2 + 1} \right) \right]_t \\ &= \frac{1}{2} \left(\mathcal{L}^{-1} \left[\frac{1}{s^2} \right]_t + \mathcal{L}^{-1} \left[\frac{1}{s} \right]_t - \mathcal{L}^{-1} \left[\frac{s-1}{(s-1)^2 + 1} \right]_t \right) \\ &= \frac{1}{2} \left(t + 1 - \mathcal{L}^{-1} [G(s-1)]_t \right) = \frac{1}{2} \left(t + 1 - e^{1t} g(t) \right) \end{aligned}$$

where

$$G(s-1) = \frac{s-1}{(s-1)^2 + 1} \quad .$$

Letting $X = s - 1$, this becomes

$$G(X) = \frac{X}{X^2 + 1} \quad .$$

Hence,

$$g(t) = \mathcal{L}^{-1} [G(s)]_t = \mathcal{L}^{-1} \left[\frac{s}{s^2 + 1} \right]_t = \cos(t) \quad ,$$

and the last formula above for y becomes

$$y(t) = \frac{1}{2} \left(t + 1 - e^{1t} g(t) \right) = \frac{1}{2} \left(t + 1 - e^t \cos(t) \right) \quad .$$

$$\begin{aligned}
 \text{25.9 c.} \quad & \mathcal{L}[y'' - 9y]|_s = \mathcal{L}[24e^{-3t}]|_s \\
 \Leftrightarrow & \mathcal{L}[y'']|_s - \mathcal{L}[9y]|_s = 24\mathcal{L}[e^{-3t}]|_s \\
 \Leftrightarrow & [s^2Y(s) - s\underbrace{y(0)}_6 - \underbrace{y'(0)}_2] - 9Y(s) = \frac{24}{s - [-3]t} \\
 \Leftrightarrow & (s^2 - 9)Y(s) - 6s - 2 = \frac{24}{s + 3}
 \end{aligned}$$

So,

$$\begin{aligned}
 Y(s) &= \frac{24}{(s+3)(s^2-9)} + \frac{6s+2}{s^2-9} \\
 &= \frac{24 + (6s+2)(s+3)}{(s+3)(s^2-9)} \\
 &= \frac{6s^2 + 20s + 30}{(s+3)^2(s-3)} \\
 &= \frac{A}{(s+3)^2} + \frac{B}{s+3} + \frac{C}{s-3} = \frac{A(s-3) + B(s+3)(s-3) + C(s+3)^2}{(s+3)^2(s-3)} .
 \end{aligned}$$

This requires that

$$A(s-3) + B(s+3)(s-3) + C(s+3)^2 = 6s^2 + 20s + 30 \quad . \quad (\star)$$

Multiplying this out and gathering like terms yields

$$[B+C]s^2 + [A+6C]s + [-3A-9B+9C] = 6s^2 + 20s + 30 \quad ,$$

which, in turn, yields the system

$$\begin{aligned}
 B + C &= 6 \\
 A + 6C &= 20 \quad . \\
 -3A - 9B + 9C &= 30
 \end{aligned} \quad (\star\star)$$

Letting $s = -3$ in (\star) :

$$\begin{aligned}
 A(-3-3) + B \cdot 0 \cdot (-3-3) + C \cdot 0^2 &= 6[-3]^2 + 20[-3] + 30 \\
 \Leftrightarrow A &= \frac{6[-3]^2 + 20[-3] + 30}{-3-3} = -4 \quad .
 \end{aligned}$$

Letting $s = 3$ in (\star) :

$$\begin{aligned}
 A \cdot 0 + B(3+3) \cdot 0 + C(3+3)^2 &= 6[3]^2 + 20[3] + 30 \\
 \Leftrightarrow C &= \frac{2[3]^2 + 20[3] + 30}{(3+3)^2} = 4 \quad .
 \end{aligned}$$

Using these values with the first equation in system $(\star\star)$:

$$B = 6 - C = 6 - 4 = 2 \quad .$$

Thus,

$$Y(s) = \frac{A}{(s+3)^2} + \frac{B}{s+3} + \frac{C}{s-3} = -\frac{4}{(s+3)^2} + \frac{2}{s+3} + \frac{4}{s-3} ,$$

and

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[-\frac{4}{(s+3)^2} + \frac{2}{s+3} + \frac{4}{s-3}\right]\Big|_t \\ &= -4\mathcal{L}^{-1}\left[\frac{1}{(s+3)^2}\right]\Big|_t + 2\mathcal{L}^{-1}\left[\frac{1}{s+3}\right]\Big|_t + 4\mathcal{L}^{-1}\left[\frac{1}{s-3}\right]\Big|_t \\ &= -4\mathcal{L}^{-1}[F(s-[-3])]\Big|_t + 2e^{-3t} + 4e^{3t} \\ &= -4e^{-3t}f(t) + 2e^{-3t} + 4e^{3t} \end{aligned}$$

with

$$F(s-[-3]) = F(s+3) = \frac{1}{(s+3)^2} \rightsquigarrow F(X) = \frac{1}{X^2}$$

$$\hookrightarrow f(t) = \mathcal{L}^{-1}[F(s)]\Big|_t = \mathcal{L}^{-1}\left[\frac{1}{s^2}\right]\Big|_y = t .$$

So,

$$\begin{aligned} y(t) &= -4e^{-3t}f(t) + 2e^{-3t} + 4e^{3t} \\ &= -4e^{-3t}t + 2e^{-3t} + 4e^{3t} = 2e^{-3t} - 4te^{-3t} + 4e^{3t} . \end{aligned}$$

25.10 a. From theorem 24.5 on page 473 and the definition of the inverse transform, we have

$$\mathcal{L}^{-1}\left[\frac{F(s)}{s}\right]\Big|_t = \int_0^t f(\tau) d\tau \quad \text{with } f(\tau) = \mathcal{L}^{-1}[F(s)]\Big|_\tau .$$

In this problem,

$$\frac{1}{s(s^2+9)} = \frac{F(s)}{s} \quad \text{with } F(s) = \frac{1}{s^2+9} .$$

So,

$$f(\tau) = \mathcal{L}\left[\frac{1}{s^2+9}\right]\Big|_\tau = \frac{1}{3}\mathcal{L}\left[\frac{3}{s^2+3^2}\right]\Big|_\tau = \frac{1}{3}\sin(3\tau) ,$$

and

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{1}{s(s^2+9)}\right]\Big|_t &= \mathcal{L}^{-1}\left[\frac{F(s)}{s}\right]\Big|_t = \int_0^t f(\tau) d\tau \\ &= \int_0^t \frac{1}{3}\sin(3\tau) d\tau = \frac{1}{9}[1 - \cos(3t)] . \end{aligned}$$

25.10 c. In this problem,

$$\frac{1}{s(s-3)^2} = \frac{F(s)}{s} \quad \text{with } F(s) = \frac{1}{(s-3)^2} .$$

So,

$$f(\tau) = \mathcal{L}\left[\frac{1}{(s-3)^2}\right]\Big|_\tau = \mathcal{L}[F(s-3)]\Big|_\tau = e^{3t}f(t) = \dots = \tau e^{3\tau} ,$$

and

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{1}{s(s-3)^2}\right]\Big|_t &= \mathcal{L}^{-1}\left[\frac{F(s)}{s}\right]\Big|_t \\ &= \int_0^t f(\tau) d\tau \\ &= \int_0^t \tau e^{3\tau} d\tau \\ &= \frac{1}{3}\tau e^{3\tau}\Big|_0^t - \frac{1}{3}\int_0^t e^{3\tau} d\tau \\ &= \frac{1}{3}te^{3t} - \frac{1}{9}[e^{3t} - 1] = \frac{1}{9}[1 + (3t - 1)e^{3t}] \quad .\end{aligned}$$