Chapter 24: Differentiation and the Laplace Transform

24.1 a. Letting $Y(s) = \mathcal{L}[y(t)]|_s$,

$$\mathcal{L}\big[y' + 4y\big]\big|_s \;=\; \mathcal{L}[0]|_s$$

$$\hookrightarrow \quad \mathcal{L}\big[y'\big]\big|_s \;+\; 4\mathcal{L}[y]|_s \;=\; 0$$

$$\hookrightarrow \quad [sY(s) - \underbrace{y(0)}_{3}] \;+\; 4Y(s) \;=\; 0$$

$$\hookrightarrow \quad (s+4)Y(s) \;=\; 3 \quad \rightarrowtail \quad Y(s) \;=\; \frac{3}{s+4} \quad .$$

24.1 c.

$$\mathcal{L}\big[y' + 3y\big]\big|_s \;=\; \mathcal{L}\big[\text{step}_4(t)\big]\big|_s$$

$$\hookrightarrow \quad \mathcal{L}\big[y'\big]\big|_s \;+\; 3\mathcal{L}[y]|_s \;=\; \frac{e^{-4s}}{s}$$

$$\hookrightarrow \quad [sY(s) - \underbrace{y(0)}_{0}] \;+\; 3Y(s) \;=\; \frac{e^{-4s}}{s}$$

$$\hookrightarrow \quad (s+3)Y(s) \;=\; \frac{e^{-4s}}{s} \quad \rightarrowtail \quad Y(s) \;=\; \frac{e^{-4s}}{s(s+3)} \quad .$$

24.1 e.

$$\mathcal{L}\big[y'' + 4y\big]\big|_s \;=\; \mathcal{L}\Big[20e^{4t}\Big]\Big|_s$$

$$\hookrightarrow \quad \mathcal{L}\big[y''\big]\big|_s \;+\; 4\mathcal{L}[y]|_s \;=\; 20\mathcal{L}\Big[e^{4t}\Big]\Big|_s$$

$$\hookrightarrow \quad [s^2Y(s) - s\,\underbrace{y(0)}_{3} - \underbrace{y'(0)}_{12}] \;+\; 4Y(s) \;=\; 20 \cdot \frac{1}{s-4}$$

$$\hookrightarrow \quad (s^2+4)Y(s) \;-\; 3s \;-\; 12 \;=\; \frac{20}{s-4}$$

$$\hookrightarrow \quad Y(s) \;=\; \frac{20}{(s^2+4)(s-4)} \;+\; \frac{3s+12}{s^2+4} \;=\; \frac{3s^2-28}{(s-4)\left(s^2+4\right)} \quad .$$

24.1 g.

$$\mathcal{L}\big[y'' + 4y\big]\big|_s \;=\; \mathcal{L}\big[3\,\text{step}_2(t)\big]\big|_s$$

$$\hookrightarrow \quad \mathcal{L}\big[y''\big]\big|_s \;+\; 4\mathcal{L}[y]|_s \;=\; 3\mathcal{L}\big[\text{step}_2(t)\big]\big|_s$$

$$\hookrightarrow \quad [s^2Y(s) - s\,\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{5}] \;+\; 4Y(s) \;=\; 3\frac{e^{2s}}{s}$$

$$\hookrightarrow \quad \left(s^2+4\right)Y(s) \;-\; 0s \;-\; 5 \;=\; \frac{3e^{2s}}{s}$$

$$\hookrightarrow \quad Y(s) \;=\; \frac{3e^{2s}}{s\left(s^2+4\right)} \;+\; \frac{5}{s^2+4} \quad .$$

24.1 i.
$$\mathcal{L}\big[y'' - 5y' + 6y\big]\big|_s = \mathcal{L}\Big[t^2e^{4t}\Big]\Big|_s$$

$$\hookrightarrow \quad \mathcal{L}\big[y''\big]\big|_s - 5\mathcal{L}\big[y'\big]\big|_s + 6\mathcal{L}[y]|_s = \mathcal{L}\Big[e^{4t}t^2\Big]\Big|_s$$

$$\hookrightarrow \quad [s^2Y(s) - s\underbrace{y(0)}_{0}] - \underbrace{y'(0)}_{2}]$$
$$- 5[sY(s) - \underbrace{y(0)}_{0} + 6Y(s) = \mathcal{L}\big[e^{4t}\underbrace{t^2}_{f(t)}\big]\big|_s$$

$$\hookrightarrow \quad \Big(s^2 - 5s + 6\Big)Y(s) - 2 = F(s-4) \quad .$$

Here,
$$F(s) = \mathcal{L}[f(t)]|_s = \mathcal{L}\Big[t^2\Big]\Big|_s = \frac{2!}{s^{2+1}} = \frac{2}{s^3}$$

$$\hookrightarrow \quad F(X) = \frac{2}{X^3} \quad \rightarrowtail \quad F(s-4) = \frac{2}{(s-4)^3} \quad .$$

Using this, we rewrite the last line in our computations for Y and continue:
$$\Big(s^2 - 5s + 6\Big)Y(s) - 2 = F(s-4) = \frac{2}{(s-4)^3}$$

$$\hookrightarrow \quad Y(s) = \frac{2}{s^2 - 5s + 6} + \frac{2}{(s-4)^3\big(s^2 - 5s + 6\big)} \quad .$$

24.1 k.
$$\mathcal{L}\big[y'' - 4y' + 13y\big]\big|_s = \mathcal{L}\big[e^{2t}\underbrace{\sin(3t)}_{f(t)}\big]\big|_s$$

$$\hookrightarrow \quad \mathcal{L}\big[y''\big]\big|_s - 4\mathcal{L}\big[y'\big]\big|_s + 13\mathcal{L}[y]|_s = F(s-2)$$

$$\hookrightarrow \quad [s^2Y(s) - s\underbrace{y(0)}_{4} - \underbrace{y'(0)}_{3}]$$
$$- 4[sY(s) - \underbrace{y(0)}_{4}] + 13Y(s) = \frac{3}{(s-2)^2 + 3^2}$$

$$\hookrightarrow \quad \Big(s^2 - 4s + 13\Big)Y(s) - 4s - 3 - 4(-4) = \frac{3}{s^2 - 4s + 13}$$

$$\hookrightarrow \quad \Big(s^2 - 4s + 13\Big)Y(s) - 4s + 13 = \frac{3}{s^2 - 4s + 13}$$

$$\hookrightarrow \quad Y(s) = \frac{4s - 13}{s^2 - 4s + 13} + \frac{3}{\big(s^2 - 4s + 13\big)^2} \quad .$$

24.1 m.
$$\mathcal{L}\big[y''' - 27y\big]\big|_s = \mathcal{L}\Big[e^{-3t}\Big]\Big|_s$$

$$\hookrightarrow \quad \mathcal{L}\big[y'''\big]\big|_s - 27\mathcal{L}[y]|_s = \frac{1}{s - (-3)}$$

$$\hookrightarrow \quad [s^3Y(s) - s^2y(0) - sy'(0) - y''(0)] \; - \; 27Y(s) \; = \; \frac{1}{s+3}$$

$$\hookrightarrow \quad \left(s^3 - 27\right)Y(s) \; - \; 2s^2 \; - \; 3s \; - \; 4 \; = \; \frac{1}{s+3}$$

$$\hookrightarrow \quad Y(s) \; = \; \frac{2s^2+3s+4}{s^3-27} \; + \; \frac{1}{(s+3)\left(s^3-27\right)} \quad .$$

24.2 a.

$$\mathcal{L}[t\underbrace{\cos(3t)}_{f(t)}]|_s \; = \; \mathcal{L}[tf(t)]|_s \; = \; -\frac{dF}{ds}$$

where

$$F(s) \; = \; \mathcal{L}[\cos(3t)]|_s \; = \; \frac{s}{s^2+9} \quad .$$

Using this with the first line, we have

$$\mathcal{L}[t\cos(3t)]|_s \; = \; -\frac{d}{ds}\left[\frac{s}{s^2+9}\right] \; = \; -\frac{1\left(s^2+9\right) - s(2s)}{\left(s^2+9\right)^2} \; = \; \frac{s^2-9}{\left(s^2+9\right)^2} \quad .$$

24.2 c.

$$\mathcal{L}[t\underbrace{e^{-7t}}_{f(t)}]|_s \; = \; \mathcal{L}[tf(t)]|_s \; = \; -\frac{dF}{ds}$$

where

$$F(s) \; = \; \mathcal{L}\left[e^{-7t}\right]\Big|_s \; = \; \frac{1}{s+7} \quad .$$

Combining the above two lines, we have

$$\mathcal{L}\left[te^{-7t}\right]\Big|_s \; = \; -\frac{d}{ds}\left[\frac{1}{s+7}\right] \; = \; -\frac{-1}{(s+7)^2} \; = \; \frac{1}{(s+7)^2} \quad .$$

24.2 e.

$$\mathcal{L}\left[t\,\text{step}(t-3)\right]\Big|_s \; = \; -\frac{d}{ds}\left[\mathcal{L}\left[\text{step}(t-3)\right]\Big|_s\right]$$

$$= \; -\frac{d}{ds}\left[\frac{e^{-3s}}{s}\right] \; = \; -\frac{-3e^{-3s}s - e^{-3s}}{s^2} \; = \; \frac{1+3s}{s^2}e^{-3s} \quad .$$

24.3 a.

$$\mathcal{L}[t\underbrace{\sin(\omega t)}_{f(t)}]|_s \; = \; \mathcal{L}[tf(t)]|_s \; = \; -\frac{dF}{ds}$$

where

$$F(s) \; = \; \mathcal{L}[\sin(\omega t)]|_s \; = \; \frac{\omega}{s^2+\omega^2} \quad .$$

Combining these two lines, we have

$$\mathcal{L}[t\sin(\omega t)]|_s \; = \; -\frac{d}{ds}\left[\frac{\omega}{s^2+\omega^2}\right] \; = \; -\frac{-\omega(2s)}{\left(s^2+\omega^2\right)^2} \; = \; \frac{2\omega s}{\left(s^2+\omega^2\right)^2} \quad .$$

24.4 a. First note that, by the differentiation identities and given initial conditions,

$$\mathcal{L}[ty]|_s = -\frac{dY}{ds} \quad ,$$

$$\mathcal{L}\left[\frac{dy}{dt}\right]\Big|_s = sY(s) - y(0) = sY(s) - 1$$

and

$$\begin{aligned}\mathcal{L}\left[t\frac{d^2y}{dt^2}\right]\Big|_s &= -\frac{d}{ds}\left[\mathcal{L}\left[\frac{d^2y}{dt^2}\right]\Big|_s\right] \\ &= -\frac{d}{ds}\left[s^2Y(s) - sy(0) - y'(0)\right] \\ &= -\frac{d}{ds}\left[s^2Y(s) - s\right] \\ &= -\left[2sY(s) + s^2\frac{dY}{ds} - 1\right] = -s^2\frac{dY}{ds} - 2sY(s) + 1 \quad .\end{aligned}$$

So,

$$\mathcal{L}\left[t\frac{d^2y}{dt^2} + \frac{dy}{dt} + ty\right]\Big|_s = \mathcal{L}[0]|_s$$

$$\hookrightarrow \quad \mathcal{L}\left[t\frac{d^2y}{dt^2}\right]\Big|_s + \mathcal{L}\left[\frac{dy}{dt}\right]\Big|_s + \mathcal{L}[ty]|_s = 0$$

$$\hookrightarrow \quad \left[-s^2\frac{dY}{ds} - 2sY(s) + 1\right] + [sY(s) - 1] - \frac{dY}{ds} = 0$$

$$\hookrightarrow \quad -\left(s^2+1\right)\frac{dY}{ds} - 2sY + (1-1) = 0$$

$$\hookrightarrow \quad \left(s^2+1\right)\frac{dY}{ds} + 2sY = 0 \quad .$$

24.4 c. Keeping in mind that $y(t) = J_0(t)$ in this problem, we see that

$$Y(0) = \int_0^\infty y(t)e^{-0t}\,dt = \int_0^\infty J_0(t)\,dt = 1 \quad .$$

24.5 a. Identify and apply the appropriate identities one-by-one, and carefully keeping track of the different functions you end up dealing with. Beginning with the computation of the transform of the given function and identifying the first identity to be used yields

$$\mathcal{L}\left[te^{4t}\sin(3t)\right]\Big|_s = \mathcal{L}\big[t\,\underbrace{e^{4t}\sin(3t)}_{f(t)}\big]\big|_s = \mathcal{L}[tf(t)]|_s = -\frac{dF}{ds} \qquad (\star)$$

where

$$F(s) = \mathcal{L}[f(t)]|_s = \mathcal{L}\left[e^{4t}\sin(3t)\right]\Big|_s = \mathcal{L}\big[e^{4t}\underbrace{\sin(3t)}_{g(t)}\big]\big|_s = G(s-4)$$

where

$$G(s) = \mathcal{L}[g(t)]|_s = \mathcal{L}[\sin(3t)]|_s = \frac{3}{s^2+9} \quad .$$

Thus,

$$G(X) = \frac{3}{X^2+9} \quad .$$

Returning to the formula for F and using $X = s-4$, we get

$$F(s) = G(\underbrace{s-4}_{X}) = \frac{3}{X^2+9} = \frac{3}{(s-4)^2+9} \quad .$$

And, thus, equation ($\star$) becomes

$$\mathcal{L}\Big[te^{4t}\sin(3t)\Big]\Big|_s = -\frac{dF}{ds} = -\frac{d}{ds}\left[\frac{3}{(s-4)^2+9}\right]$$

$$= -\frac{-3\cdot 2(s-4)}{\left[(s-4)^2+9\right]^2} = \frac{6(s-4)}{\left[(s-4)^2+9\right]^2} \quad .$$

24.5 c. $$\mathcal{L}\Big[te^{4t}\operatorname{step}(t-3)\Big]\Big|_s = \mathcal{L}\big[t\,\underbrace{e^{4t}\operatorname{step}(t-3)}_{f(t)}\big]\big|_s = \mathcal{L}[tf(t)]|_s = -\frac{dF}{ds} \qquad (\star)$$

where

$$F(s) = \mathcal{L}[f(t)]|_s = \mathcal{L}\big[e^{4t}\underbrace{\operatorname{step}(t-3)}_{g(t)}\big]\big|_s = G(s-4)$$

where

$$G(s) = \mathcal{L}[g(t)]|_s = \mathcal{L}\big[\operatorname{step}(t-3)\big]\big|_s = \frac{e^{-3s}}{s} \quad .$$

Thus,

$$G(X) = \frac{e^{-3X}}{X} \quad .$$

Returning to the formula for F and using $X = s-4$, we get

$$F(s) = G(\underbrace{s-4}_{X}) = \frac{e^{-3X}}{X} = \frac{e^{-3(s-4)}}{s-4} \quad .$$

And, thus, equation ($\star$) becomes

$$\mathcal{L}\Big[te^{4t}\operatorname{step}(t-3)\Big]\Big|_s = -\frac{dF}{ds} = -\frac{d}{ds}\left[\frac{e^{-3(s-4)}}{s-4}\right]$$

$$= -\frac{-3e^{-3(s-4)}(s-4) - e^{-3(s-4)}\cdot 1}{(s-4)^2}$$

$$= \frac{3s-11}{(s-4)^2}e^{-3(s-4)} \quad .$$

24.7. $$\mathcal{L}[\operatorname{Si}(t)]|_s = \mathcal{L}\left[\int_0^t \underbrace{\frac{\sin(\tau)}{\tau}}_{f(\tau)}\,d\tau\right]\Bigg|_s = \frac{F(s)}{s}$$

where

$$F(s) = \mathcal{L}\left[\frac{\sin(t)}{t}\right]\Bigg|_s = \arctan\left(\frac{1}{s}\right) \quad .$$

So,

$$\mathcal{L}[\operatorname{Si}(t)]|_s = \frac{F(s)}{s} = \frac{\arctan\left(\frac{1}{s}\right)}{s} = \frac{1}{s}\arctan\left(\frac{1}{s}\right) \quad .$$

24.8 a. Since letting $t = 0$ in the given function yields $^0/_0$, we must compute the limit using L'Hôpital's rule:

$$\lim_{t\to 0} \frac{1 - e^{-t}}{t} = \lim_{t\to 0} \frac{\frac{d}{dt}\left[1 - e^{-t}\right]}{\frac{d}{dt}[t]} = \lim_{t\to 0} \frac{e^{-t}}{1} = 1 \quad ,$$

which is finite. So theorem 24.6 on page 475 applies, giving us

$$\mathcal{L}\left[\frac{1 - e^{-t}}{t}\right]\Big|_s = \mathcal{L}\left[\frac{f(t)}{t}\right]\Big|_s = \int_s^\infty F(\sigma)\,d\sigma$$

where

$$F(s) = \mathcal{L}[f(t)]|_s = \mathcal{L}\left[1 - e^{-t}\right]\Big|_s$$
$$= \mathcal{L}[1]|_s - \mathcal{L}\left[e^{-t}\right]\Big|_s = \frac{1}{s} - \frac{1}{s+1} \qquad \text{for} \quad s > 0 \quad .$$

Thus, for $s > 0$,

$$\begin{aligned}
\mathcal{L}\left[\frac{1 - e^{-t}}{t}\right]\Big|_s &= \int_s^\infty F(\sigma)\,d\sigma \\
&= \int_s^\infty \left[\frac{1}{\sigma} - \frac{1}{\sigma+1}\right] d\sigma \\
&= \Big[\ln|\sigma| - \ln|\sigma+1|\Big]\Big|_s^\infty \\
&= \ln\left(\frac{\sigma}{\sigma+1}\right)\Big|_s^\infty \\
&= \lim_{\sigma\to\infty} \ln\left(\frac{\sigma}{\sigma+1}\right) - \ln\left(\frac{s}{s+1}\right) \\
&= \underbrace{\lim_{\sigma\to\infty} \ln\left(\frac{1}{1+1/\sigma}\right)}_{0} + \ln\left(\frac{s+1}{s}\right) = \ln\left(1 + \frac{1}{s}\right) \quad .
\end{aligned}$$

24.8 c. Since letting $t = 0$ in the given function yields $^0/_0$, we must compute the limit using L'Hôpital's rule:

$$\lim_{t\to 0} \frac{e^{-2t} - e^{3t}}{t} = \lim_{t\to 0} \frac{-2e^{-2t} - 3e^{3t}}{1} = -5 \quad ,$$

which is finite. So theorem 24.6 on page 475 applies, giving us

$$\mathcal{L}\left[\frac{e^{-2t} - e^{3t}}{t}\right]\Big|_s = \mathcal{L}\left[\frac{f(t)}{t}\right]\Big|_s = \int_s^\infty F(\sigma)\,d\sigma$$

where

$$F(s) = \mathcal{L}[f(t)]|_s = \mathcal{L}\left[e^{-2t} - e^{3t}\right]\Big|_s = \frac{1}{s+2} - \frac{1}{s-3} \qquad \text{for} \quad s > 3 \quad .$$

Thus, for $s > 3$,

$$\begin{aligned}
\mathcal{L}\left[\frac{e^{-2t} - e^{3t}}{t}\right]\Big|_s &= \int_s^\infty F(\sigma)\,d\sigma \\
&= \int_s^\infty \left[\frac{1}{\sigma+2} - \frac{1}{\sigma-3}\right] d\sigma
\end{aligned}$$

$$= \left[\ln|\sigma+2| - \ln|\sigma-3|\right]\Big|_s^\infty$$
$$= \ln\left(\frac{\sigma+2}{\sigma-3}\right)\Big|_s^\infty$$
$$= \underbrace{\lim_{\sigma\to\infty}\ln\left(\frac{\sigma+2}{\sigma-3}\right)}_{0} - \ln\left(\frac{s+2}{s-3}\right) = \ln\left(\frac{s-3}{s+2}\right) \quad .$$

24.8 e. Since letting $t = 0$ in the given function yields $^0/_0$, we must compute the limit using L'Hôpital's rule:

$$\lim_{t\to 0}\frac{1-\cosh(t)}{t} = \lim_{t\to 0}\frac{-\sinh(t)}{1} = 0 \quad ,$$

which is finite. So theorem 24.6 on page 475 applies, giving us

$$\mathcal{L}\left[\frac{1-\cosh(t)}{t}\right]\Big|_s = \mathcal{L}\left[\frac{f(t)}{t}\right]\Big|_s = \int_s^\infty F(\sigma)\,d\sigma$$

where

$$F(s) = \mathcal{L}[f(t)]|_s = \mathcal{L}[1-\cosh(t)]|_s$$
$$= \mathcal{L}\left[1 - \frac{e^t+e^{-t}}{2}\right]\Big|_s$$
$$= \mathcal{L}[1]|_s - \frac{1}{2}\mathcal{L}\left[e^t\right]\Big|_s - \frac{1}{2}\mathcal{L}\left[e^{-t}\right]\Big|_s$$
$$= \frac{1}{s} - \frac{1}{2}\cdot\frac{1}{s-1} - \frac{1}{2}\cdot\frac{1}{s+1} \qquad \text{for} \quad s > 1 \quad .$$

So, for $s > 1$,

$$\mathcal{L}\left[\frac{1-\cosh(t)}{t}\right]\Big|_s = \int_s^\infty F(\sigma)\,d\sigma$$
$$= \int_s^\infty \left[\frac{1}{\sigma} - \frac{1}{2}\cdot\frac{1}{\sigma-1} - \frac{1}{2}\cdot\frac{1}{\sigma+1}\right] d\sigma$$
$$= \left[\ln|\sigma| - \frac{1}{2}\ln|\sigma+1| - \frac{1}{2}\ln|\sigma-1|\right]\Big|_s^\infty$$
$$= \frac{1}{2}\left[2\ln|\sigma| - \ln\left|\sigma^2-1\right|\right]\Big|_s^\infty$$
$$= \frac{1}{2}\ln\left(\frac{\sigma^2}{\sigma^2-1}\right)\Big|_s^\infty$$
$$= \frac{1}{2}\left[\lim_{\sigma\to\infty}\ln\left(\frac{\sigma^2}{\sigma^2-1}\right) - \ln\left(\frac{s^2}{s^2-1}\right)\right]$$
$$= \frac{1}{2}\left[0 + \ln\left(\frac{s^2-1}{s^2}\right)\right] = \frac{1}{2}\ln\left(1-\frac{1}{s^2}\right) \quad .$$