Chapter 23: The Laplace Transform

23.6 a.

$$\begin{aligned} F(s) = \mathcal{L}[4]|_s &= \int_0^\infty 4e^{-st}\,dt \\ &= 4\int_0^\infty e^{-st}\,dt \\ &= \frac{4}{-s}\,e^{-st}\Big|_0^\infty \\ &= -\frac{4}{s}\Big[\underbrace{\lim_{t\to\infty} e^{-st}}_{0 \text{ if } s>0} - e^{-s\cdot 0}\Big] = \frac{4}{s} \qquad \text{for} \quad s > 0 \quad . \end{aligned}$$

23.6 c.

$$\begin{aligned} F(s) = \mathcal{L}[f(t)]|_s &= \int_0^\infty f(t)e^{-st}\,dt \\ &= \int_0^3 \underbrace{f(t)}_{2}\,e^{-st}\,dt + \int_3^\infty \underbrace{f(t)}_{0}\,e^{-st}\,dt \\ &= \int_0^3 2e^{-st}\,dt + \int_3^\infty 0\,e^{-st}\,dt \\ &= \frac{2}{-s}\,e^{-st}\Big|_0^3 + 0 = -\frac{2}{s}\Big[e^{-s\cdot 3} - 1\Big] = \frac{2}{s}\Big[1 - e^{-3s}\Big] \quad . \end{aligned}$$

23.6 e.

$$\begin{aligned} F(s) = \mathcal{L}[f(t)]|_s &= \int_0^\infty f(t)e^{-st}\,dt \\ &= \int_0^4 \underbrace{f(t)}_{e^{2t}}\,e^{-st}\,dt + \int_4^\infty \underbrace{f(t)}_{0}\,e^{-st}\,dt \\ &= \int_0^4 e^{2t}e^{-st}\,dt + \int_4^\infty 0\,e^{-st}\,dt \\ &= \int_0^4 e^{(2-s)t}\,dt + 0 \\ &= \frac{1}{2-s}\,e^{(2-s)t}\Big|_0^4 + 0 \\ &= \frac{1}{2-s}\Big[e^{(2-s)4} - 1\Big] = \frac{1}{s-2}\Big[1 - e^{4(2-s)}\Big] \quad . \end{aligned}$$

23.6 g. Using integration by parts,

$$\begin{aligned} F(s) = \mathcal{L}[f(t)]|_s &= \int_0^\infty f(t)e^{-st}\,dt \\ &= \int_0^1 te^{-st}\,dt + \int_1^\infty 0\,e^{-st}\,dt \\ &= -\frac{t}{s}e^{-st}\Big|_0^1 - \int_0^1 \left(-\frac{1}{s}\right)e^{-st}\,dt + 0 \end{aligned}$$

$$= -\frac{1}{s}e^{-s\cdot 1} + \frac{0}{s}e^{-s\cdot 0} - \frac{1}{s^2}\left[e^{-s\cdot 1} - 1\right]$$
$$= \frac{1}{s^2}\left[1 - e^{-s}\right] - \frac{1}{s}e^{-s} \quad .$$

23.7 a. Using formula (23.7), $\mathcal{L}\left[t^4\right]\Big|_s = \frac{4!}{s^{4+1}} = \frac{4\cdot 3\cdot 2\cdot 1}{s^5} = \frac{24}{s^5} \quad \text{for} \quad s > 0 \quad .$

23.7 c. Using formula (23.8), $\mathcal{L}\left[e^{7t}\right]\Big|_s = \frac{1}{s-7} \quad \text{for} \quad s > 7 \quad .$

23.7 e. Using formula (23.8), $\mathcal{L}\left[e^{-7t}\right]\Big|_s = \frac{1}{s-(-7)} = \frac{1}{s+7} \quad \text{for} \quad s > -7 \quad .$

23.8 a. $\mathcal{L}[\sin(3t)]|_s = \frac{3}{s^2+3^2} = \frac{3}{s^2+9} \quad \text{for} \quad s > 0 \quad .$

23.8 c. $\mathcal{L}[7]|_s = \mathcal{L}[7\cdot 1]|_s = 7\mathcal{L}[1]|_s = 7\cdot\frac{1}{s} = \frac{7}{s} \quad \text{for} \quad s > 0 \quad .$

23.8 e.

$$\mathcal{L}[\sinh(4t)]|_s = \mathcal{L}\left[\frac{e^{4t}-e^{-4t}}{2}\right]\Bigg|_s = \frac{1}{2}\mathcal{L}\left[e^{4t}-e^{-4t}\right]\Big|_s$$
$$= \frac{1}{2}\left[\mathcal{L}\left[e^{4t}\right]\Big|_s - \mathcal{L}\left[e^{-4t}\right]\Big|_s\right]$$
$$= \frac{1}{2}\left[\frac{1}{s-4} - \frac{1}{s+4}\right]$$
$$= \frac{1}{2}\left[\frac{(s+4)-(s-4)}{(s-4)(s+4)}\right] = \frac{4}{s^2-16} \quad .$$

For the above to hold, we need both $s > 4$ and $s > -4$. But since $4 > -4$, it suffices to have $s > 4$.

23.8 g.

$$\mathcal{L}\left[6e^{2t} + 8e^{-3t}\right]\Big|_s = 6\mathcal{L}\left[e^{2t}\right]\Big|_s + 8\mathcal{L}\left[e^{-3t}\right]\Big|_s$$
$$= 6\cdot\frac{1}{s-2} + 8\cdot\frac{1}{s+3} = \frac{6}{s-2} + \frac{8}{s+3} \quad .$$

For the above to hold, we need both $s > 2$ and $s > -3$. But since $2 > -3$, it suffices to have $s > 2$.

23.8 i. For $s > 0$,

$$\mathcal{L}[3\cos(2t) - 4\sin(2t)]|_s = 3\mathcal{L}[\cos(2t)]|_s - 4\mathcal{L}[\sin(2t)]|_s$$
$$= 3\cdot\frac{s}{s^2+2^2} - 4\cdot\frac{2}{s^2+2^2} = \frac{3s-8}{s^2+4} \quad .$$

23.9 a.

$$\mathcal{L}\left[t^{3/2}\right]\Big|_s = \frac{\Gamma\left(\frac{3}{2}+1\right)}{t^{3/2+1}} = \frac{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)}{s^{5/2}}$$

$$= \frac{3}{2}\Gamma\left(\frac{1}{2}+1\right)s^{-5/2}$$

$$= \frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)s^{-5/2} = \frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}s^{-5/2} = \frac{3}{4}\sqrt{\pi}s^{-5/2} \quad .$$

23.9 c.

$$\mathcal{L}\left[t^{-1/3}\right]\Big|_s = \frac{\Gamma\left(-\frac{1}{3}+1\right)}{s^{-1/3+1}} = \frac{\Gamma\left(\frac{2}{3}\right)}{s^{2/3}} = \Gamma\left(\frac{2}{3}\right)s^{-2/3} \quad .$$

23.10 a.

$$1 - \text{step}_2(t) = 1 - \begin{Bmatrix} 1 & \text{if} & t < 2 \\ 0 & \text{if} & 2 \le t \end{Bmatrix} = \begin{Bmatrix} 1-1 & \text{if} & t < 2 \\ 1-0 & \text{if} & 2 \le t \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 & \text{if} & t < 2 \\ 1 & \text{if} & 2 \le t \end{Bmatrix} = f(t) \quad .$$

23.10 b.

$$\mathcal{L}[f(t)]|_s = \mathcal{L}\big[1 - \text{step}_2(t)\big]\big|_s = \mathcal{L}[1]|_s - \mathcal{L}\big[\text{step}_2(t)\big]\big|_s$$

$$= \frac{1}{s} - \frac{e^{-2s}}{s} = \frac{1}{s}\left[1 - e^{-2s}\right] \quad .$$

23.11 a.

$$\mathcal{L}\left[te^{4t}\right]\Big|_s = \mathcal{L}\big[e^{4t}\underbrace{t}_{f(t)}\big]\big|_s = \mathcal{L}\big[e^{4t}f(t)\big]\big|_s = F(s-4) \quad .$$

Here,

$$F(s) = \mathcal{L}[f(t)]|_s = \mathcal{L}[t]|_s = \frac{1}{s^2} \qquad \text{for} \quad s > 0 \quad .$$

So, for any $X > 0$,

$$F(X) = \frac{1}{X^2} \quad ,$$

and we complete the computations we started above with

$$\mathcal{L}\left[te^{4t}\right]\Big|_s = \cdots = F(\underbrace{s-4}_{X}) = F(X) = \frac{1}{X^2} = \frac{1}{(s-4)^2} \quad ,$$

keeping in mind that we must have $s - 4 = X > 0$; that is, $s > 4$.

23.11 c.

$$\mathcal{L}\left[e^{2t}\sin(3t)\right]\Big|_s = \mathcal{L}\big[e^{2t}\underbrace{\sin(3t)}_{f(t)}\big]\big|_s = \mathcal{L}\big[e^{2t}f(t)\big]\big|_s = F(s-2)$$

with

$$F(s) = \mathcal{L}[f(t)]|_s = \mathcal{L}[\sin(3t)]|_s = \frac{3}{s^2+9} \qquad \text{for} \quad s > 0$$

$$\hookrightarrow \qquad F(X) = \frac{3}{X^2+9} \qquad \text{for} \quad X > 0$$

$$\hookrightarrow \qquad F(s-2) = \frac{3}{(s-2)^2+9} \qquad \text{for} \quad s-2 > 0 \quad .$$

So the first line in our computations continues as

$$\mathcal{L}\left[e^{2t}\sin(3t)\right]\Big|_s = \cdots = F(s-2) = \frac{3}{(s-2)^2+9} \qquad \text{for} \quad s > 2 \quad .$$

23.11 e. $$\mathcal{L}\left[e^{3t}\sqrt{t}\right]\Big|_s = \mathcal{L}\big[e^{3t}\underbrace{t^{1/2}}_{f(t)}\big]\big|_s = \mathcal{L}\big[e^{3t}f(t)\big]\big|_s = F(s-3)$$

with

$$F(s) = \mathcal{L}[f(t)]|_s = \mathcal{L}\left[t^{1/2}\right]\Big|_s = \frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{1/2+1}} \qquad \text{for} \quad s > 0$$

$$\hookrightarrow \qquad F(s) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)s^{-3/2} = \frac{\sqrt{\pi}}{2}s^{-3/2} \qquad \text{for} \quad s > 0$$

$$\hookrightarrow \qquad F(X) = \frac{\sqrt{\pi}}{2}X^{-3/2} \qquad \text{for} \quad X > 0$$

$$\hookrightarrow \qquad F(s-3) = \frac{\sqrt{\pi}}{2}(s-3)^{-3/2} \qquad \text{for} \quad s-3 > 0 \quad .$$

So the first line in our computations continues as

$$\mathcal{L}\left[e^{3t}\sqrt{t}\right]\Big|_s = \cdots = F(s-3) = \frac{\sqrt{\pi}}{2}(s-3)^{-3/2} \qquad \text{for} \quad s > 3 \quad .$$

23.12 a. $$\mathcal{L}\big[t^n e^{\alpha t}\big]\big|_s = \mathcal{L}\big[e^{\alpha t}\underbrace{t^n}_{f(t)}\big]\big|_s = \mathcal{L}\big[e^{\alpha t}f(t)\big]\big|_s = F(s-\alpha) \quad .$$

Here,

$$F(s) = \mathcal{L}[f(t)]|_s = \mathcal{L}\big[t^n\big]\big|_s = \frac{n!}{s^{n+1}} \qquad \text{for} \quad s > 0 \quad .$$

So,

$$F(X) = \frac{n!}{X^{n+1}} \qquad \text{for} \quad X > 0 \quad .$$

In particular,

$$F(s-\alpha) = \frac{n!}{(s-\alpha)^{n+1}} \qquad \text{for} \quad s-\alpha > 0 \quad ,$$

and the computations started above are completed with

$$\mathcal{L}\big[t^n e^{\alpha t}\big]\big|_s = \cdots = F(s-\alpha) = \frac{n!}{(s-\alpha)^{n+1}} \qquad \text{for} \quad s > \alpha \quad .$$

23.12 c. $$\mathcal{L}\big[e^{\alpha t}\cos(\omega t)\big]\big|_s = \mathcal{L}\big[e^{\alpha t}\underbrace{\cos(\omega t)}_{f(t)}\big]\big|_s = \mathcal{L}\big[e^{\alpha t}f(t)\big]\big|_s = F(s-\alpha)$$

with

$$F(s) = \mathcal{L}[f(t)]|_s = \mathcal{L}[\cos(\omega t)]|_s = \frac{s}{s^2+\omega^2} \qquad \text{for} \quad s > 0$$

$$\hookrightarrow \qquad F(X) = \frac{X}{X^2+\omega^2} \qquad \text{for} \quad X > 0$$

$$\hookrightarrow \qquad F(s-\alpha) = \frac{s-\alpha}{(s-\alpha)^2+\omega^2} \qquad \text{for} \quad s-\alpha > 0 \quad .$$

So the first line in our computations continues as

$$\mathcal{L}\big[e^{\alpha t}\cos(\omega t)\big]\big|_s \;=\; \cdots \;=\; F(s-\alpha) \;=\; \frac{s-\alpha}{(s-\alpha)^2+\omega^2} \qquad \text{for} \quad s > \alpha \quad .$$

23.13 a. Using the definition

$$\Gamma(x) \;=\; \int_0^\infty u^{x-1}e^{-u}\,du \qquad \text{for} \quad x > 0$$

along with integration by parts, we have

$$\begin{aligned}
\Gamma(\sigma+1) \;&=\; \int_0^\infty u^{(\sigma+1)-1}e^{-u}\,du \\
&=\; \int_0^\infty u^{\sigma}e^{-u}\,du \\
&=\; u^{\sigma}\left(-e^{-u}\right)\Big|_{u=0}^{\infty} \;-\; \int_0^\infty \left(\sigma u^{\sigma-1}\right)\left(-e^{-u}\right)du \\
&=\; -\lim_{u\to\infty} u^{\sigma}e^{-u} \;+\; 0^{\sigma}e^{0} \;+\; \sigma\int_0^\infty u^{\sigma-1}e^{-u}\,du \\
&=\; -\lim_{u\to\infty} u^{\sigma}e^{-u} \;+\; \sigma\Gamma(\sigma) \qquad \text{for} \quad \sigma > 0 \quad .
\end{aligned}$$

In addition, using L'Hôpital's rule, it is easily seen that, for any real value σ ,

$$\lim_{u\to\infty} u^{\sigma}e^{-u} \;=\; 0 \quad .$$

So, the first term in the last formula derived above for $\Gamma(\sigma+1)$ vanishes, leaving us with

$$\Gamma(\sigma+1) \;=\; \sigma\Gamma(\sigma) \qquad \text{for} \quad \sigma > 0 \quad .$$

23.14 a. Clearly $\lim_{t\to 0^+} f(t)$ is finite, and the only discontinuity on $(0,\infty)$ is a jump discontinuity at $t = 3$. So the function is piecewise continuous on $(0,\infty)$.

23.14 c. Since $\sin(t)$ is continuous at every point on $(-\infty,\infty)$, it is automatically is continuous at every point on $(0,\infty)$ and has a finite limit at $t = 0$. So the function is piecewise continuous on $(0,\infty)$.

23.14 e. Since $\lim_{t\to\pi/2}|\tan(t)| = \infty$, the function is not piecewise continuous on $(0,\infty)$.

23.14 g. This function "blows up" (becomes infinite) as $t \to 0^+$. So it is not piecewise continuous on $(0,\infty)$.

23.14 i. Since $\lim_{t\to 1}\left|\frac{1}{t^2-1}\right| = \infty$, the function is not piecewise continuous on $(0,\infty)$.

23.14 k. On any finite subinterval of $(0, \infty)$, $\text{stair}(t)$ is continuous at every point except at the finite number of integral values of t in the interval. But those discontinuities are all finite jumps. Also, $\lim_{t\to 0^+} \text{stair}(t) = 1$ (which is finite). So $\text{stair}(t)$ is piecewise continuous on $(0, \infty)$.

23.15. First of all, since $g(t)$ is continuous at $t = t_0$,

$$\lim_{t\to t_0^+} g(t) = g(t_0) = \lim_{t\to t_0^-} g(t) \quad .$$

Thus,

$$\begin{aligned}\lim_{t\to t_0^+} fg(t) &= \lim_{t\to t_0^+} \big(f(t)\cdot g(t)\big) \\ &= \left(\lim_{t\to t_0^+} f(t)\right)\left(\lim_{t\to t_0^+} g(t)\right) = \left(\lim_{t\to t_0^+} f(t)\right) g(t_0)\end{aligned}$$

and

$$\begin{aligned}\lim_{t\to t_0^-} fg(t) &= \lim_{t\to t_0^-} \big(f(t)\cdot g(t)\big) \\ &= \left(\lim_{t\to t_0^-} f(t)\right)\left(\lim_{t\to t_0^-} g(t)\right) = \left(\lim_{t\to t_0^-} f(t)\right) g(t_0) \quad .\end{aligned}$$

Hence,

$$\begin{aligned}\text{the jump in } fg \text{ at } t_0 &= \lim_{t\to t_0^+} fg(t) - \lim_{t\to t_0^-} fg(t) \\ &= \left(\lim_{t\to t_0^+} f(t)\right) g(t_0) - \left(\lim_{t\to t_0^-} f(t)\right) g(t_0) \\ &= \left(\lim_{t\to t_0^+} f(t) - \lim_{t\to t_0^-} f(t)\right) g(t_0) \\ &= \big(\text{the jump in } f \text{ at } t_0\big) g(t_0) \quad ,\end{aligned}$$

23.16 a. Since $e^a \le e^b$ whenever $a \le b$, we clearly have

$$\left|e^{3t}\right| \le 1\cdot e^{s_0 t} \qquad \text{whenever} \quad 0 \le t \quad \text{and} \quad 3 \le s_0 \quad .$$

So e^{3t} is of exponential order s_0 for any $s_0 \ge 3$.

23.16 c. Applying the test, it is clear that

$$\lim_{t\to\infty} \left[te^{3t}\right] e^{-s_0 t} = \lim_{t\to\infty} te^{(3-s_0)t} = \infty \qquad \text{if} \quad s_0 \le 3 \quad .$$

However, if $3 < s_0$ then $e^{-(s_0-3)t} \to 0$ as $t \to \infty$, and a naive computation of the limit yields

$$\lim_{t\to\infty} \left[te^{3t}\right] e^{-s_0 t} = \lim_{t\to\infty} te^{-(s_0-3)t} = \text{“}\infty \times 0\text{”} \quad ,$$

telling us that we must compute this limit via L'Hôpital's rule. Doing so,

$$\begin{aligned}\lim_{t\to\infty}\left[te^{3t}\right]e^{-s_0t} &= \lim_{t\to\infty} te^{-(s_0-3)t}\\ &= \lim_{t\to\infty}\frac{t}{e^{(s_0-3)t}}\\ &= \lim_{t\to\infty}\frac{1}{(s_0-3)e^{(s_0-3)t}} = \lim_{t\to\infty}\frac{1}{(s_0-3)}e^{-(s_0-3)t} = 0 \quad .\end{aligned}$$

Hence, the test applies and tells us that te^{3t} is of exponential order s_0 for any $s_0 > 3$.

23.16 e. By basic algebra, we clearly have

$$|\sin(t)| \leq 1 \leq 1\cdot e^{s_0t} \qquad \text{whenever} \quad 0 \leq t \quad \text{and} \quad 0 \leq s_0 \quad .$$

So e^{3t} is of exponential order s_0 for any $s_0 \geq 0$.

23.17 a. For convenience, let $f(t) = t^{\alpha}e^{-\sigma t}$. Observe that f is a continuous, differentiable function on $(0,\infty)$ and that, since σ and α are positive,

$$f(0) = 0 \qquad , \qquad \lim_{t\to\infty} f(t) = 0$$

and

$$f(t) > 0 \qquad \text{for} \quad 0 < t < \infty \quad .$$

By basic calculus, we know the maximum value of f on $[0,\infty)$ must then occur where the derivative is zero,

$$0 = f'(t) = \alpha t^{\alpha-1}e^{-\sigma t} - \sigma t^{\alpha}e^{-\sigma t} = [\alpha - \sigma t]t^{\alpha-1}e^{-\sigma t}$$

$$\hookrightarrow \qquad 0 = \alpha - \sigma t \quad \rightarrowtail \quad t = \frac{\alpha}{\sigma} \quad .$$

So $t = \frac{\alpha}{\sigma}$ is where $f(t)$ is maximum on $[0,\infty)$, and this maximum is

$$M_{\alpha,\sigma} = f\left(\frac{\alpha}{\sigma}\right) = \left(\frac{\alpha}{\sigma}\right)^{\alpha}e^{-\sigma\cdot\alpha/\sigma} = \left(\frac{\alpha}{\sigma}\right)^{\alpha}e^{-\alpha} \quad .$$