## Chapter 22: Variation of Parameters

**22.1 a.** Here, $y_1(x) = x$ and $y_2(x) = x^2$. So the solution is given by

$$y(x) = y_1 u + y_2 v = xu + x^2 v \qquad (\star)$$

where $u = u(x)$ and $v = v(x)$ satisfy the two equations

$$y_1 u' + y_2 v' = 0 \qquad \rightarrowtail \qquad xu' + x^2 v' = 0$$

and

$$y_1{}' u' + y_2{}' v' = \frac{g}{a} \qquad \rightarrowtail \qquad u' + 2xv' = \frac{3\sqrt{x}}{x^2} = 3x^{-3/2} \quad .$$

Solving for $u'$ and $v'$:

$$xu' + x^2 v' = 0 \quad \text{and} \quad u' + 2xv' = 3x^{-3/2}$$

$$\hookrightarrow \qquad u' = -xv' \quad \text{and} \quad -xv' + 2xv' = 3x^{-3/2}$$

$$\hookrightarrow \qquad u' = -xv' \quad \text{and} \quad v' = \frac{3x^{-3/2}}{x} = 3x^{-5/2}$$

$$\hookrightarrow \qquad u' = -x\left[3x^{-5/2}\right] = -3x^{-3/2} \quad \text{and} \quad v' = 3x^{-5/2} \quad .$$

Integrating, we get

$$u = \int u'\,dx = -\int 3x^{-3/2}\,dx = 6x^{-1/2} + c_1 \quad ,$$

and

$$v = \int v'\,dx = \int 3x^{-5/2}\,dx = -2x^{-3/2} + c_2 \quad .$$

Plugging back into formula $(\star)$ for $y$ then yields

$$\begin{aligned} y(x) &= xu + x^2 v \\ &= x\left[6x^{-1/2} + c_1\right] + x^2\left[-2x^{-3/2} + c_2\right] \\ &= 6x^{1/2} + c_1 x - 2x^{1/2} + c_2 x^2 = 4\sqrt{x} + c_1 x + c_2 x^2 \quad . \end{aligned}$$

**22.1 c.** Here, $y_1(x) = \cos(2x)$ and $y_2(x) = \sin(2x)$. So the solution is given by

$$y(x) = y_1 u + y_2 v = \cos(2x)u + \sin(2x)v \qquad (\star)$$

where $u = u(x)$ and $v = v(x)$ satisfy the system

$$\begin{aligned} y_1 u' + y_2 v' &= 0 \\ y_1{}' u' + y_2{}' v' &= \frac{g}{a} \end{aligned} \quad ,$$

which, in this case, is

$$\begin{aligned} \cos(2x)u' + \sin(2x)v' &= 0 \\ -2\sin(2x)u' + 2\cos(2x)v' &= \frac{\csc(2x)}{1} = \frac{1}{\sin(2x)} \end{aligned} \quad .$$

From the first equation in this system, we get

$$v' = -\frac{\cos(2x)}{\sin(2x)}u' \quad .$$

Plugging this into the second equation and continuing:

$$-2\sin(2x)u' + 2\cos(2x)\left[-\frac{\cos(2x)}{\sin(2x)}u'\right] = \frac{1}{\sin(2x)}$$

$$\hookrightarrow \quad -2\left[\sin(2x) + \frac{\cos^2(2x)}{\sin(2x)}\right]u' = \frac{1}{\sin(2x)}$$

$$\hookrightarrow \quad -2\frac{\sin^2(2x) + \cos^2(2x)}{\sin(2x)}u' = \frac{1}{\sin(2x)}$$

$$\hookrightarrow \quad -2\frac{1}{\sin(2x)}u' = \frac{1}{\sin(2x)}$$

$$\hookrightarrow \quad u' = -\frac{1}{2} \quad .$$

Hence, also,

$$v' = -\frac{\cos(2x)}{\sin(2x)}u' = \frac{\cos(2x)}{2\sin(2x)} \quad .$$

Integrating, we get

$$u = \int u'\,dx = -\int \frac{1}{2}\,dx = -\frac{1}{2}x + c_1 \quad ,$$

and

$$v = \int v'\,dx = \int \frac{\cos(2x)}{2\sin(2x)}\,dx = \frac{1}{4}\ln|\sin(2x)| + c_2 \quad .$$

Plugging back into formula (⋆) for $y$ then yields

$$\begin{aligned} y(x) &= \cos(2x)u + \sin(2x)v \\ &= \cos(2x)\left[-\tfrac{1}{2}x + c_1\right] + \sin(2x)\left[\tfrac{1}{4}\ln|\sin(2x)| + c_2\right] \\ &= -\tfrac{1}{2}x\cos(2x) + \tfrac{1}{4}\sin(2x)\ln|\sin(2x)| + c_1\cos(2x) + c_2\sin(2x) \quad . \end{aligned}$$

**22.1 e.** Here, $y_1(x) = e^{2x}$ and $y_2(x) = xe^{2x}$ . So the solution is given by

$$y(x) = y_1 u + y_2 v = e^{2x}u + xe^{2x}v \qquad (\star)$$

where $u = u(x)$ and $v = v(x)$ satisfy the system

$$\begin{aligned} y_1 u' + y_2 v' &= 0 \\ {y_1}'u' + {y_2}'v' &= \frac{g}{a} \end{aligned} \quad ,$$

which, in this case, is

$$\begin{aligned} e^{2x}u' + xe^{2x}v' &= 0 \\ 2e^{2x}u' + [1+2x]e^{2x}v' &= \frac{\left[24x^2+2\right]e^{2x}}{1} = \left[24x^2+2\right]e^{2x} \end{aligned} \quad ,$$

and which, after dividing out the $e^{2x}$, further simplifies to

$$\begin{aligned} u' + xv' &= 0 \\ 2u' + [1+2x]v' &= 24x^2 + 2 \end{aligned} .$$

Solving for $u'$ and $v'$:

$$u' + xv' = 0 \quad \text{and} \quad 2u' + [1+2x]v' = 24x^2 + 2$$

$$\hookrightarrow \quad u' = -xv' \quad \text{and} \quad 2[-xv'] + [1+2x]v' = 24x^2 + 2$$

$$\hookrightarrow \quad u' = -xv' \quad \text{and} \quad v' = 24x^2 + 2$$

$$\hookrightarrow \quad u' = -x[24x^2+2] = -24x^3 - 2x \quad \text{and} \quad v' = 24x^2 + 2 \quad .$$

Integrating, we get

$$u = \int u'\,dx = -\int \left[24x^3 + 2x\right] dx = -6x^4 - x^2 + c_1 \quad ,$$

and

$$v = \int v'\,dx = \int \left[24x^2 + 2\right] dx = 8x^3 + 2x + c_2 \quad .$$

Plugging back into formula ($\star$) for $y$ then yields

$$\begin{aligned} y(x) &= e^{2x}\left[-6x^4 - x^2 + c_1\right] + xe^{2x}\left[8x^3 + 2x + c_2\right] \\ &= \left[2x^4 + x^2 + c_1x + c_2x\right]e^{2x} \quad . \end{aligned}$$

**22.1 g.** The solution is given by

$$y(x) = y_1u + y_2v = xu + x^{-1}v \qquad (\star)$$

where $u = u(x)$ and $v = v(x)$ satisfy

$$y_1u' + y_2v' = 0 \quad \rightarrowtail \quad xu' + x^{-1}v' = 0$$

and

$$y_1{}'u' + y_2{}'v' = \frac{g}{a} \quad \rightarrowtail \quad u' - x^{-2}v' = \frac{\sqrt{x}}{x^2} = x^{-3/2} \quad .$$

Solving for $u'$ and $v'$:

$$xu' + x^{-1}v' = 0 \quad \text{and} \quad u' - x^{-2}v' = x^{-3/2}$$

$$\hookrightarrow \quad u' = -x^{-2}v' \quad \text{and} \quad -x^{-2}v' - x^{-2}v' = x^{-3/2}$$

$$\hookrightarrow \quad u' = -x^{-2}v' \quad \text{and} \quad v' = -\tfrac{1}{2}x^2 \cdot x^{-3/2} = -\tfrac{1}{2}x^{1/2}$$

$$\hookrightarrow \quad u' = -x^{-2}\left[-\tfrac{1}{2}x^{1/2}\right] = \tfrac{1}{2}x^{-3/2} \quad \text{and} \quad v' = = -\tfrac{1}{2}x^{1/2} \quad .$$

Integrating, we get

$$u = \int u' \, dx = \int \frac{1}{2} x^{-3/2} \, dx = -x^{-1/2} + c_1 \quad ,$$

and

$$v = \int v' \, dx = -\int \frac{1}{2} x^{1/2} \, dx = -\frac{1}{3} x^{3/2} + c_2 \quad .$$

Plugging back into formula (⋆) for $y$ then yields

$$\begin{aligned} y(x) &= x \left[ -x^{-1/2} + c_1 \right] + x^{-1} \left[ -\frac{1}{3} x^{3/2} + c_2 \right] \\ &= -\frac{4}{3} x^{1/2} + c_1 x + c_2 x^2 = c_1 x + c_2 x^2 - \frac{4}{3} \sqrt{x} \quad . \end{aligned}$$

**22.1 i.** The solution is given by

$$y(x) = y_1 u + y_2 v = x^2 u + x^2 \ln|x| \, v \qquad (\star)$$

where $u = u(x)$ and $v = v(x)$ satisfy the system

$$\begin{aligned} y_1 u' + y_2 v' &= 0 \\ {y_1}' u' + {y_2}' v' &= \frac{g}{a} \end{aligned} \quad ,$$

which, in this case, is

$$\begin{aligned} x^2 u' + x^2 \ln|x| \, v' &= 0 \\ 2x u' + [2x \ln|x| + x] \, v' &= \frac{x^2}{x^2} = 1 \end{aligned} \quad .$$

From the first equation in this system, we get

$$u' = -\ln|x| \, v' \quad .$$

Plugging this into the second equation and continuing:

$$\begin{aligned} 2x[-\ln|x|] + [2x \ln|x| + x] \, v' &= 1 \\ \hookrightarrow \qquad x v' &= 1 \\ \hookrightarrow \qquad v' &= \frac{1}{x} \quad . \end{aligned}$$

Hence, also,

$$u' = -\ln|x| \, v' = -x^{-1} \ln|x| \quad .$$

Integrating (using integration by parts to compute the integral for $u$ ), we get

$$u = \int u' \, dx = -\int x^{-1} \ln|x| \, dx = -\frac{1}{2} (\ln|x|)^2 + c_1 \quad ,$$

and

$$v = \int v' \, dx = \int \frac{1}{x} \, dx = \ln|x| + c_2 \quad .$$

Plugging back into formula (⋆) for $y$ then yields

$$\begin{aligned} y(x) &= x^2 \left[ -\frac{1}{2} (\ln|x|)^2 + c_1 \right] + x^2 \ln|x| \, [\ln|x| + c_2] \\ &= \frac{1}{2} x^2 (\ln|x|)^2 + c_1 x^2 + c_2 x^2 \ln|x| \quad . \end{aligned}$$

**22.1 k.** The solution is given by

$$y(x) = y_1 u + y_2 v = x^{-1} u + x^{-1} e^{-2x} v \qquad (\star)$$

where $u = u(x)$ and $v = v(x)$ satisfy the system

$$\begin{aligned} y_1 u' + y_2 v' &= 0 \\ y_1{}' u' + y_2{}' v' &= \frac{g}{a} \end{aligned} \quad ,$$

which, in this case, is

$$\begin{aligned} x^{-1} u' + x^{-1} e^{-2x} v' &= 0 \\ -x^{-2} u' - \left[x^{-2} + 2x^{-1}\right] e^{-2x} v' &= \frac{8e^{-2x}}{x} = 8x^{-1} e^{2x} \end{aligned} \quad ,$$

and which further simplifies to

$$\begin{aligned} u' + e^{-2x} v' &= 0 \\ u' + [1 + 2x] e^{-2x} v' &= -8x e^{2x} \end{aligned} \quad .$$

From the first equation in this system, we get

$$u' = -e^{-2x} v' \quad .$$

Plugging this into the second equation and continuing:

$$\left[-e^{-2x} v'\right] + [1 + 2x] e^{-2x} v' = -8x e^{2x}$$

$$\hookrightarrow \qquad 2x e^{-2x} v' = -8x e^{2x}$$

$$\hookrightarrow \qquad v' = -4e^{4x} \quad .$$

Hence, also,

$$u' = -e^{-2x} v' = -e^{-2x} \left[-4e^{4x}\right] = 4e^{2x} \quad .$$

Integrating, we get

$$u = \int u' \, dx = \int 4e^{2x} \, dx = 2e^{2x} + c_1 \quad ,$$

and

$$v = \int v' \, dx = -\int 4e^{4x} \, dx = -e^{4x} + c_2 \quad .$$

Plugging back into formula $(\star)$ for $y$ then yields

$$\begin{aligned} y(x) &= x^{-1} \left[2e^{2x} + c_1\right] + x^{-1} e^{-2x} \left[-e^{4x} + c_2\right] \\ &= x^{-1} e^{2x} + c_1 x^{-1} + c_2 x^{-1} e^{-2x} \quad . \end{aligned}$$

**22.2 a.** First, we must find $y_h$, the general solution to the corresponding homogeneous equation,

$$x^2 y'' - 2xy' - 4y = 0 \quad .$$

Since this is an Euler equation, we try a solution of the form $y(x) = x^r$ :

$$\begin{aligned} 0 &= x^2 y'' - 2xy' - 4y \\ &= x^2 \left[x^r\right]'' - 2x \left[x^r\right]' - 4\left[x^r\right] \\ &= x^2 \left[r(r-1)x^{r-2}\right] - 2x\left[rx^{r-1}\right] - 4\left[x^r\right] \\ &= x^r\left[r(r-1) - 2r - 4\right] = x^r\left[r^2 - 3r - 4\right] \quad . \end{aligned}$$

So,

$$0 = r^2 - 3r - 4 = (r+1)(r-4)$$

$$\hookrightarrow \qquad r = -1 \qquad \text{and} \qquad r = 4$$

$$\hookrightarrow \qquad y_h(x) = c_1 x^{-1} + c_2 x^4 \quad .$$

Thus, to solve the given nonhomogeneous differential equation using variation of parameters, we set

$$y(x) = x^{-1}u + x^4 v \tag{$\star$}$$

where $u$ and $v$ satisfy

$$x^{-1}u' + x^4 v' = 0$$

and

$$-x^{-2}u' + 4x^3 v' = \frac{10/x}{x^2} = 10x^{-3} \quad ,$$

which we can write more simply as the system

$$\begin{aligned} u' + x^5 v' &= 0 \\ -u' + 4x^5 v' &= 10x^{-1} \end{aligned} \quad .$$

Adding these two equations together and solving:

$$-5x^5 v' = -10x^{-1} \quad \rightarrowtail \quad v' = 2x^{-6} \quad .$$

This with the first equation in the system then yields

$$u' = -x^5 v' = -x^5\left[2x^{-6}\right] = -2x^{-1} \quad .$$

Integrating:

$$u(x) = \int u'\,dx = -\int 2x^{-1}\,dx = -2\ln|x| + c_1 \quad ,$$

and

$$v(x) = \int v'\,dx = \int 2x^{-6}\,dx = -\frac{2}{5}x^{-5} + c_2 \quad .$$

Plugging back into formula ($\star$) for $y$ :

$$\begin{aligned} y(x) &= x^{-1}\left[-2\ln|x| + c_1\right] + x^4\left[-\frac{2}{5}x^{-5} + c_2\right] \\ &= -2x^{-1}\ln|x| + \left[c_1 - \frac{2}{5}\right]x^{-1} + c_2 x^4 \\ &= -2x^{-1}\ln|x| + Ax^{-1} + Bx^4 \quad . \end{aligned} \tag{$\star\star$}$$

This is the general solution to the differential equation. Computing it's derivative, we get

$$y'(x) = 2x^{-2}\ln|x| - 2x^{-2} - Ax^{-2} + 4Bx^3 \quad .$$

Applying the initial conditions:

$$3 = y(1) = -2x^{-1}\ln|1| + A\cdot 1^{-1} + B\cdot 1^4 = A + B$$

and

$$\begin{aligned} -15 = y'(1) &= 2\cdot 1^{-2}\ln|x| - 2\cdot 1^{-2} - A\cdot 1^{-2} + 4B\cdot 1^3 \\ &= -2 - A + 4B \quad . \end{aligned}$$

That is,

$$A + B = 3 \qquad \text{and} \qquad -A + 4B = -13 \quad .$$

Solving this simple system yields $A = 5$ and $B = -2$, which, plugged back into formula ($\star\star$) for $y$ gives our final answer:

$$y(x) = -2x^{-1}\ln|x| + 5x^{-1} - 2x^4 \quad .$$

**22.3 a.** Here, $y_1(x) = 1$, $y_2(x) = e^{2x}$ and $y_3(x) = e^{-2x}$. So the solution is given by

$$y(x) = y_1u + y_2v + y_3w = 1\cdot u + e^{2x}v + e^{-2x}w \tag{$\star$}$$

where $u = u(x)$, $v = v(x)$ and $w = w(x)$ satisfy the system

$$\begin{aligned} y_1u' + y_2v' + y_3w' &= 0 \\ {y_1}'u' + {y_2}'v' + {y_3}'w' &= 0 \\ {y_1}''u' + {y_2}''v' + {y_3}''w' &= \frac{g}{a} \end{aligned} \quad ,$$

which, in this case, is

$$\begin{aligned} 1u' + e^{2x}v' + e^{-2x}w' &= 0 \\ 0u' + 2e^{2x}v' - 2e^{-2x}w' &= 0 \\ 0u' + 4e^{2x}v' + 4e^{-2x}w' &= \frac{30e^{3x}}{1} = 30e^{3x} \end{aligned} \quad ,$$

and which, after dividing out common factors in each equation, reduces to

$$u' + e^{2x}v' + e^{-2x}w' = 0 \tag{S1}$$

$$e^{2x}v' - e^{-2x}w' = 0 \quad . \tag{S2}$$

$$e^{2x}v' + e^{-2x}w' = \frac{15}{2}e^{3x} \tag{S3}$$

This is easily solved by adding or subtracting the equations. In particular, subtracting (S3) from (S1) yields

$$u' = -\frac{15}{2}e^{3x} \quad ,$$

adding (S2) and (S3) together gives

$$2e^{2x}v' = \frac{15}{2}e^{3x} \quad \rightarrowtail \quad v' = \frac{15}{4}e^{x} \quad ,$$

and subtracting (S2) from (S3) gives

$$2e^{-2x}w' = \frac{15}{2}e^{3x} \quad \rightarrowtail \quad w' = \frac{15}{4}e^{5x} \quad .$$

Integrating, we obtain

$$u(x) = \int u'\,dx = -\int \frac{15}{2}e^{3x}\,dx = -\frac{5}{2}e^{3x} + c_1 \quad ,$$

$$v(x) = \int v'\,dx = \int \frac{15}{4}e^{x}\,dx = \frac{15}{4}e^{x} + c_2$$

and

$$w(x) = \int w'\,dx = -\int \frac{15}{4}e^{5x}\,dx = \frac{3}{4}e^{5x} + c_3 \quad .$$

The final answer is then given by plugging these back into formula (⋆) for $y$ :

$$\begin{aligned} y(x) &= u + e^{2x}v + e^{-2x}w \\ &= \left[-\frac{5}{2}e^{3x} + c_1\right] + e^{2x}\left[\frac{15}{4}e^{x} + c_2\right] + e^{-2x}\left[\frac{3}{4}e^{5x} + c_3\right] \\ &= \left[-\frac{5}{2} + \frac{15}{4} + \frac{3}{4}\right]e^{3x} + c_1 + c_2e^{2x} + c_3e^{-2x} \\ &= 2e^{3x} + c_1 + c_2e^{2x} + c_3e^{-2x} \quad . \end{aligned}$$

**22.4 a.** Here, $y_1(x) = x$ , $y_2(x) = x^2$ and $y_3(x) = x^3$ . So the solution is given by

$$y(x) = y_1u + y_2v + y_3w = xu + x^2v + x^3w$$

where $u = u(x)$ , $v = v(x)$ and $w = w(x)$ satisfy the system

$$\begin{aligned} y_1u' + y_2v' + y_3w' &= 0 \\ {y_1}'u' + {y_2}'v' + {y_3}'w' &= 0 \\ {y_1}''u' + {y_2}''v' + {y_3}''w' &= \frac{g}{a} \end{aligned} \quad ,$$

which, in this case, is

$$\begin{aligned} xu' + x^2v' + x^3w' &= 0 \\ 1u' + 2xv' + 3x^2w' &= 0 \\ 0u' + 2v' + 6xw' &= \frac{e^{-x}}{x^3} = x^{-3}e^{-x} \end{aligned} \quad .$$

**22.4 c.** Here, $y_1(x) = e^{3x}$ , $y_2(x) = e^{-3x}$ , $y_3(x) = \cos(3x)$ and $y_4(x) = \sin(3x)$ . So

$$y(x) = e^{3x}u_1 + e^{-3x}u_2 + \cos(3x)u_3 + \sin(3x)u_4$$

where

$$\begin{aligned} y_1{u_1}' + y_2{u_2}' + y_3{u_3}' + y_4{u_4}' &= 0 \\ {y_1}'{u_1}' + {y_2}'{u_2}' + {y_3}'{u_3}' + {y_4}'{u_4}' &= 0 \\ {y_1}''{u_1}' + {y_2}''{u_2}' + {y_3}''{u_3}' + {y_4}''{u_4}' &= 0 \\ {y_1}'''{u_1}' + {y_2}'''{u_2}' + {y_3}'''{u_3}' + {y_4}'''{u_4}' &= \frac{g}{a} \end{aligned} \quad ,$$

which, in this case, is

$$\begin{aligned} e^{3x}{u_1}' \;+\; e^{-3x}{u_2}' \;+\; \cos(3x){u_3}' \;+\; \sin(3x){u_4}' \;&=\; 0 \\ 3e^{3x}{u_1}' \;-\; 3e^{-3x}{u_2}' \;-\; 3\sin(3x){u_3}' \;+\; 3\cos(3x){u_4}' \;&=\; 0 \\ 9e^{3x}{u_1}' \;+\; 9e^{-3x}{u_2}' \;-\; 9\cos(3x){u_3}' \;-\; 9\sin(3x){u_4}' \;&=\; 0 \\ 27e^{3x}{u_1}' \;-\; 27e^{-3x}{u_2}' \;+\; 27\sin(3x){u_3}' \;-\; 27\cos(3x){u_4}' \;&=\; \frac{\sinh(x)}{1} \end{aligned} \quad .$$

Dividing out common factors in each equation, this reduces to

$$\begin{aligned} e^{3x}{u_1}' \;+\; e^{-3x}{u_2}' \;+\; \cos(3x){u_3}' \;+\; \sin(3x){u_4}' \;&=\; 0 \\ e^{3x}{u_1}' \;-\; e^{-3x}{u_2}' \;-\; \sin(3x){u_3}' \;+\; \cos(3x){u_4}' \;&=\; 0 \\ e^{3x}{u_1}' \;+\; e^{-3x}{u_2}' \;-\; \cos(3x){u_3}' \;-\; \sin(3x){u_4}' \;&=\; 0 \\ e^{3x}{u_1}' \;-\; e^{-3x}{u_2}' \;+\; \sin(3x){u_3}' \;-\; \cos(3x){u_4}' \;&=\; \frac{1}{27}\sinh(x) \end{aligned} \quad .$$

**22.6.** Recall that, for any sufficiently continuous function $g$,

$$\int_{x_0}^{x_0} g(s)\,ds \;=\; 0 \qquad \text{and} \qquad \frac{d}{dx}\int_{x_0}^{x} g(s)\,ds \;=\; g(x) \quad .$$

Letting $x = x_0$ in formula (22.15) yields

$$\begin{aligned} y_p(x_0) \;&=\; -y_1(x_0)\int_{x_0}^{x_0} \frac{y_2(s)f(s)}{W(s)}\,ds \;+\; y_2(x_0)\int_{x_0}^{x_0} \frac{y_1(s)f(s)}{W(s)}\,ds \\ &=\; -y_1(x_0)\cdot 0 \;+\; y_2(x_0)\cdot 0 \;=\; 0 \quad , \end{aligned}$$

verifying the claim that $y_p(x_0) = 0$.

Differentiating formula (22.15) yields

$$\begin{aligned} {y_p}'(x) \;&=\; \frac{d}{dx}\left[-y_1(x)\int_{x_0}^{x} \frac{y_2(s)f(s)}{W(s)}\,ds \;+\; y_2(x)\int_{x_0}^{x} \frac{y_1(s)f(s)}{W(s)}\,ds\right] \\ &=\; -{y_1}'(x)\int_{x_0}^{x} \frac{y_2(s)f(s)}{W(s)}\,ds \;-\; y_1(x)\frac{d}{dx}\int_{x_0}^{x} \frac{y_2(s)f(s)}{W(s)}\,ds \\ &\qquad +\; {y_2}'(x)\int_{x_0}^{x} \frac{y_1(s)f(s)}{W(s)}\,ds \;+\; y_2(x)\frac{d}{dx}\int_{x_0}^{x} \frac{y_1(s)f(s)}{W(s)}\,ds \\ &=\; -{y_1}'(x)\int_{x_0}^{x} \frac{y_2(s)f(s)}{W(s)}\,ds \;-\; y_1(x)\frac{y_2(x)f(x)}{W(x)} \\ &\qquad +\; {y_2}'(x)\int_{x_0}^{x} \frac{y_1(s)f(s)}{W(s)}\,ds \;+\; y_2(x)\frac{y_1(x)f(x)}{W(x)} \\ &=\; -{y_1}'(x)\int_{x_0}^{x} \frac{y_2(s)f(s)}{W(s)}\,ds \;+\; {y_2}'(x)\int_{x_0}^{x} \frac{y_1(s)f(s)}{W(s)}\,ds \\ &\qquad -\; [\underbrace{y_1(x)y_2(x) - y_2(x)y_1(x)}_{0}]\frac{f(x)}{W(x)} \\ &=\; -{y_1}'(x)\int_{x_0}^{x} \frac{y_2(s)f(s)}{W(s)}\,ds \;+\; {y_2}'(x)\int_{x_0}^{x} \frac{y_1(s)f(s)}{W(s)}\,ds \quad . \end{aligned}$$

Thus,

$$\begin{aligned} {y_p}'(x_0) &= -{y_1}'(x_0) \int_{x_0}^{x_0} \frac{y_2(s) f(s)}{W(s)} \, ds \;+\; {y_2}'(x_0) \int_{x_0}^{x_0} \frac{y_1(s) f(s)}{W(s)} \, ds \\ &= -{y_1}'(x_0) \cdot 0 \;+\; {y_2}'(x_0) \cdot 0 \;=\; 0 \quad , \end{aligned}$$

confirming that ${y_p}'(x_0) = 0$ .