Chapter 22: Variation of Parameters

22.1 a. Here, \(y_1(x) = x\) and \(y_2(x) = x^2\). So the solution is given by

\[
y(x) = y_1u + y_2v = xu + x^2v
\]

where \(u = u(x)\) and \(v = v(x)\) satisfy the two equations

\[
y_1u' + y_2v' = 0 \quad \implies \quad xu' + x^2v' = 0
\]

and

\[
y_1'u' + y_2'v' = \frac{g}{a} \quad \implies \quad u' + 2xv' = \frac{3\sqrt{x}}{x^2} = 3x^{-\frac{3}{2}}.
\]

Solving for \(u'\) and \(v'\):

\[
xu' + x^2v' = 0 \quad \text{and} \quad u' + 2xv' = 3x^{-\frac{3}{2}}
\]

\[
\implies u' = -xv' \quad \text{and} \quad -xv' + 2xv' = 3x^{-\frac{3}{2}}
\]

\[
\implies u' = -xv' \quad \text{and} \quad v' = \frac{3x^{-\frac{3}{2}}}{x} = 3x^{-\frac{5}{2}}
\]

\[
\implies u' = -x \left[3x^{-\frac{5}{2}}\right] = -3x^{-\frac{3}{2}} \quad \text{and} \quad v' = 3x^{-\frac{5}{2}}.
\]

Integrating, we get

\[
u = \int u'dx = -\int 3x^{-\frac{3}{2}}dx = 6x^{-\frac{1}{2}} + c_1.
\]

and

\[
v = \int v'dx = \int 3x^{-\frac{5}{2}}dx = -2x^{-\frac{3}{2}} + c_2.
\]

Plugging back into formula (\(*\)) for \(y\) then yields

\[
y(x) = xu + x^2v
\]

\[
= x \left[6x^{-\frac{1}{2}} + c_1\right] + x^2 \left[-2x^{-\frac{3}{2}} + c_2\right]
\]

\[
= 6x^{\frac{1}{2}} + c_1x - 2x^{\frac{1}{2}} + c_2x^2 = 4\sqrt{x} + c_1x + c_2x^2.
\]

22.1 c. Here, \(y_1(x) = \cos(2x)\) and \(y_2(x) = \sin(2x)\). So the solution is given by

\[
y(x) = y_1u + y_2v = \cos(2x)u + \sin(2x)v
\]

where \(u = u(x)\) and \(v = v(x)\) satisfy the system

\[
y_1u' + y_2v' = 0
\]

\[
y_1'u' + y_2'v' = \frac{g}{a}
\]

which, in this case, is

\[
\cos(2x)u' + \sin(2x)v' = 0
\]

\[
-2\sin(2x)u' + 2\cos(2x)v' = \frac{\csc(2x)}{1} = \frac{1}{\sin(2x)}.
\]
From the first equation in this system, we get
\[ v' = -\frac{\cos(2x)}{\sin(2x)} u' \, . \]
Plugging this into the second equation and continuing:
\[ -2 \sin(2x)u' + 2 \cos(2x) \left[ -\frac{\cos(2x)}{\sin(2x)} u' \right] = \frac{1}{\sin(2x)} \]
\[ \leftrightarrow \quad -2 \left[ \sin(2x) + \frac{\cos^2(2x)}{\sin(2x)} \right] u' = \frac{1}{\sin(2x)} \]
\[ \leftrightarrow \quad -2 \frac{\sin^2(2x) + \cos^2(2x)}{\sin(2x)} u' = \frac{1}{\sin(2x)} \]
\[ \leftrightarrow \quad -2 \frac{1}{\sin(2x)} u' = \frac{1}{\sin(2x)} \]
\[ \leftrightarrow \quad u' = -\frac{1}{2} \, . \]
Hence, also,
\[ v' = -\frac{\cos(2x)}{\sin(2x)} u' = \frac{\cos(2x)}{2\sin(2x)} \, . \]
Integrating, we get
\[ u = \int u' \, dx = -\int \frac{1}{2} \, dx = -\frac{1}{2} x + c_1 \, , \]
and
\[ v = \int v' \, dx = \int \frac{\cos(2x)}{2\sin(2x)} \, dx = \frac{1}{4} \ln |\sin(2x)| + c_2 \, . \]
Plugging back into formula (⋆) for \( y \) then yields
\[ y(x) = \cos(2x)u + \sin(2x)v \]
\[ = \cos(2x) \left[ -\frac{1}{2} x + c_1 \right] + \sin(2x) \left[ \frac{1}{4} \ln |\sin(2x)| + c_2 \right] \]
\[ = -\frac{1}{2} x \cos(2x) + \frac{1}{4} \sin(2x) \ln |\sin(2x)| + c_1 \cos(2x) + c_2 \sin(2x) \, . \]

22.1 e. Here, \( y_1(x) = e^{2x} \) and \( y_2(x) = xe^{2x} \). So the solution is given by
\[ y(x) = y_1 u + y_2 v = e^{2x}u + xe^{2x}v \quad (⋆) \]
where \( u = u(x) \) and \( v = v(x) \) satisfy the system
\[ y_1 u' + y_2 v' = 0 \]
\[ y_1' u' + y_2' v' = \frac{g}{a} \, , \]
which, in this case, is
\[ e^{2x} u' + xe^{2x} v' = 0 \]
\[ 2e^{2x} u' + [1 + 2x]e^{2x} v' = \left[ \frac{24x^2 + 2}{1} \right] e^{2x} = \left[ 24x^2 + 2 \right] e^{2x} \, . \]
and which, after dividing out the $e^{2x}$, further simplifies to

$$u' + xv' = 0$$

$$2u' + [1 + 2x]v' = 24x^2 + 2$$

Solving for $u'$ and $v'$:

$$u' + xv' = 0 \quad \text{and} \quad 2u' + [1 + 2x]v' = 24x^2 + 2$$

$$\implies u' = -xv' \quad \text{and} \quad 2[-xv'] + [1 + 2x]v' = 24x^2 + 2$$

$$\implies u' = -xv' \quad \text{and} \quad v' = 24x^2 + 2$$

$$\implies u' = -x[24x^2 + 2] = -24x^3 - 2x \quad \text{and} \quad v' = 24x^2 + 2$$

Integrating, we get

$$u = \int u' \, dx = -\int \left[24x^3 + 2x\right] \, dx = -6x^4 - x^2 + c_1$$

and

$$v = \int v' \, dx = \int \left[24x^2 + 2\right] \, dx = 8x^3 + 2x + c_2$$

Plugging back into formula $(\star)$ for $y$ then yields

$$y(x) = e^{2x} \left[-6x^4 - x^2 + c_1\right] + xe^{2x} \left[8x^3 + 2x + c_2\right]$$

$$= \left[2x^4 + x^2 + c_1x + c_2x\right] e^{2x}$$

22.1g. The solution is given by

$$y(x) = y_1u + y_2v = xu + x^{-1}v \quad (\star)$$

where $u = u(x)$ and $v = v(x)$ satisfy

$$y_1u' + y_2v' = 0 \implies xu' + x^{-1}v' = 0$$

and

$$y_1u' + y_2v' = \frac{g}{a} \implies u' - x^{-2}v' = \frac{\sqrt{x}}{x^2} = x^{-3/2}$$

Solving for $u'$ and $v'$:

$$xu' + x^{-1}v' = 0 \quad \text{and} \quad u' - x^{-2}v' = x^{-3/2}$$

$$\implies u' = -x^{-2}v' \quad \text{and} \quad -x^{-2}v' - x^{-2}v' = x^{-3/2}$$

$$\implies u' = -x^{-2}v' \quad \text{and} \quad v' = -\frac{1}{2}x^{1/2} \cdot x^{-3/2} = -\frac{1}{2}x^{1/2}$$

$$\implies u' = -x^{-2} \left[-\frac{1}{2}x^{1/2}\right] = \frac{1}{2}x^{-3/2} \quad \text{and} \quad v' = = -\frac{1}{2}x^{1/2}.$$
Integrating, we get
\[ u = \int u' \, dx = \int \frac{1}{2} x^{-3/2} \, dx = -x^{-1/2} + c_1 , \]
and
\[ v = \int v' \, dx = -\int \frac{1}{2} x^{1/2} \, dx = -\frac{1}{3} x^{3/2} + c_2 . \]

Plugging back into formula (⋆) for \( y \) then yields
\[ y(x) = x \left[ -x^{-1/2} + c_1 \right] + x^{-1} \left[ -\frac{1}{3} x^{3/2} + c_2 \right] = -\frac{4}{3} x^{1/2} + c_1 x + c_2 x^2 = c_1 x + c_2 x^2 - \frac{4}{3} \sqrt{x} . \]

22.1 i. The solution is given by
\[ y(x) = y_1 u + y_2 v = x^2 u + x^2 \ln |x| v \quad (\star) \]
where \( u = u(x) \) and \( v = v(x) \) satisfy the system
\[ y_1 u' + y_2 v' = 0 \]
\[ y_1 u' + y_2 v' = \frac{g}{a} , \]
which, in this case, is
\[ x^2 u' + x^2 \ln |x| v' = 0 \]
\[ 2x u' + [2x \ln |x| + x] v' = \frac{x^2}{x^2} = 1 . \]

From the first equation in this system, we get
\[ u' = -\ln |x| v' . \]

Plugging this into the second equation and continuing:
\[ 2x [-\ln |x|] + [2x \ln |x| + x] v' = 1 \]
\[ \leftrightarrow \]
\[ x v' = 1 \]
\[ \leftrightarrow \]
\[ v' = \frac{1}{x} . \]

Hence, also,
\[ u' = -\ln |x| v' = -x^{-1} \ln |x| . \]

Integrating (using integration by parts to compute the integral for \( u \)), we get
\[ u = \int u' \, dx = -\int x^{-1} \ln |x| \, dx = -\frac{1}{2} (\ln |x|)^2 + c_1 , \]
and
\[ v = \int v' \, dx = \int x^{-1} \, dx = \ln |x| + c_2 . \]

Plugging back into formula (⋆) for \( y \) then yields
\[ y(x) = x^2 \left[ -\frac{1}{2} (\ln |x|)^2 + c_1 \right] + x^2 \ln |x| [\ln |x| + c_2] \]
\[ = \frac{1}{2} x^2 (\ln |x|)^2 + c_1 x^2 + c_2 x^2 \ln |x| . \]
22.1 k. The solution is given by

\[ y(x) = y_1u + y_2v = x^{-1}u + x^{-1}e^{-2x}v \] (*

where \( u = u(x) \) and \( v = v(x) \) satisfy the system

\[
\begin{align*}
    y_1' &+ y_2' = 0 \\
    y_1' &+ y_2' = g \\
\end{align*}
\]

which, in this case, is

\[
\begin{align*}
    x^{-1}u' + x^{-1}e^{-2x}v' &= 0 \\
    -x^{-2}u' - [x^{-2} + 2x^{-1}]e^{-2x}v' &= \frac{8e^{-2x}}{x} = 8x^{-1}e^{2x} \\
\end{align*}
\]

and which further simplifies to

\[
\begin{align*}
    u' + e^{-2x}v' &= 0 \\
    u' + [1 + 2x]e^{-2x}v' &= -8xe^{2x} \\
\end{align*}
\]

From the first equation in this system, we get

\[ u' = -e^{-2x}v' \]

Plugging this into the second equation and continuing:

\[
\begin{align*}
    \left[-e^{-2x}v\right] + [1 + 2x]e^{-2x}v' &= -8xe^{2x} \\
    2xe^{-2x}v' &= -8xe^{2x} \\
    v' &= -4e^{4x} \\
\end{align*}
\]

Hence, also,

\[ u' = -e^{-2x}v' = -e^{-2x}\left[-4e^{4x}\right] = 4e^{2x} \]

Integrating, we get

\[ u = \int u'dx = \int 4e^{2x}dx = 2e^{2x} + c_1 \]

and

\[ v = \int v'dx = -\int 4e^{4x}dx = -e^{4x} + c_2 \]

Plugging back into formula (*) for \( y \) then yields

\[ y(x) = x^{-1}\left[2e^{2x} + c_1\right] + x^{-1}e^{-2x}\left[-e^{4x} + c_2\right] = x^{-1}e^{2x} + c_1x^{-1} + c_2x^{-1}e^{-2x} \]

22.2 a. First, we must find \( y_h \), the general solution to the corresponding homogeneous equation,

\[ x^2y'' - 2xy' - 4y = 0 \]
Since this is an Euler equation, we try a solution of the form \( y(x) = x^r \):

\[
0 = x^2 y'' - 2xy' - 4y
\]

\[
= x^2 \left[ x^r \right]'' - 2x \left[ x^r \right]' - 4 \left[ x^r \right]
\]

\[
= x^2 \left[ r(r - 1)x^{r-2} \right] - 2x \left[ rx^{r-1} \right] - 4 \left[ x^r \right]
\]

\[
= x^r \left[ r(r - 1) - 2r - 4 \right] = x^r \left[ r^2 - 3r - 4 \right].
\]

So,

\[
0 = r^2 - 3r - 4 = (r + 1)(r - 4)
\]

\[\mapsto \quad r = -1 \quad \text{and} \quad r = 4\]

\[\mapsto \quad y_h(x) = c_1 x^{-1} + c_2 x^4.\]

Thus, to solve the given nonhomogeneous differential equation using variation of parameters, we set

\[
y(x) = x^{-1} u + x^4 v \quad (**)
\]

where \( u \) and \( v \) satisfy

\[
x^{-1} u' + x^4 v' = 0
\]

and

\[
- x^{-2} u' + 4x^3 v' = \frac{10/x}{x^2} = 10x^{-3},
\]

which we can write more simply as the system

\[
u' + x^5 v' = 0
\]

\[-u' + 4x^5 v' = 10x^{-1}.
\]

Adding these two equations together and solving:

\[-5x^5 v' = -10x^{-1} \quad \mapsto \quad v' = 2x^{-6}.
\]

This with the first equation in the system then yields

\[
u' = -x^5 v' = -x^5 \left[ 2x^{-6} \right] = -2x^{-1}.
\]

Integrating:

\[
u(x) = \int u' \, dx = - \int 2x^{-1} \, dx = -2 \ln |x| + c_1.
\]

and

\[
v(x) = \int v' \, dx = \int 2x^{-6} \, dx = -\frac{2}{5} x^{-5} + c_2.
\]

Plugging back into formula \((**)\) for \( y \):

\[
y(x) = x^{-1} [-2 \ln |x| + c_1] + x^4 \left[ -\frac{2}{5} x^{-5} + c_2 \right]
\]

\[
= -2x^{-1} \ln |x| + \left[ c_1 - \frac{2}{5} \right] x^{-1} + c_2 x^4
\]

\[
= -2x^{-1} \ln |x| + Ax^{-1} + Bx^4. \quad (***)
\]
This is the general solution to the differential equation. Computing its derivative, we get
\[ y'(x) = 2x^{-2} \ln |x| - 2x^{-2} - Ax^{-2} + 4Bx^3. \]

Applying the initial conditions:
\[ 3 = y(1) = -2x^{-1} \ln |1| + A \cdot 1^{-1} + B \cdot 1^4 = A + B \]
and
\[ -15 = y'(1) = 2 \cdot 1^{-2} \ln |x| - 2 \cdot 1^{-2} - A \cdot 1^{-2} + 4B \cdot 1^3 = -2 - A + 4B. \]

That is,
\[ A + B = 3 \quad \text{and} \quad -A + 4B = -13. \]

Solving this simple system yields \( A = 5 \) and \( B = -2 \), which, plugged back into formula \((\star \star)\) for \( y \) gives our final answer:
\[ y(x) = -2x^{-1} \ln |x| + 5x^{-1} - 2x^4. \]

22.3 a. Here, \( y_1(x) = 1 \), \( y_2(x) = e^{2x} \) and \( y_3(x) = e^{-2x}. \) So the solution is given by
\[ y(x) = y_1u + y_2v + y_3w = 1 \cdot u + e^{2x}v + e^{-2x}w \quad (\star) \]
where \( u = u(x), v = v(x) \) and \( w = w(x) \) satisfy the system
\[ y_1u' + y_2v' + y_3w' = 0 \]
\[ y_1u' + y_2v' + y_3w' = 0 \]
\[ y_1u'' + y_2v'' + y_3w'' = \frac{g}{a} \]
which, in this case, is
\[ 1u' + e^{2x}v' + e^{-2x}w' = 0 \]
\[ 0u' + 2e^{2x}v' - 2e^{-2x}w' = 0 \]
\[ 0u' + 4e^{2x}v' + 4e^{-2x}w' = \frac{30e^{3x}}{1} = 30e^{3x} \]
and which, after dividing out common factors in each equation, reduces to
\[ u' + e^{2x}v' + e^{-2x}w' = 0 \quad (S1) \]
\[ e^{2x}v' - e^{-2x}w' = 0 \quad (S2) \]
\[ e^{2x}v' + e^{-2x}w' = \frac{15}{2} e^{3x} \quad (S3) \]
This is easily solved by adding or subtracting the equations. In particular, subtracting \((S3)\) from \((S1)\) yields
\[ u' = -\frac{15}{2} e^{3x}, \]
adding \((S2)\) and \((S3)\) together gives
\[ 2e^{2x}v' = \frac{15}{2} e^{3x} \implies v' = \frac{15}{4} e^x. \]
and subtracting (S2) from (S3) gives
\[ 2e^{-2x}w' = \frac{15}{2} e^{3x} \implies w' = \frac{15}{4} e^{5x}. \]

Integrating, we obtain
\[
\begin{align*}
u(x) &= \int u' \, dx = -\int \frac{15}{2} e^{3x} \, dx = -\frac{5}{2} e^{3x} + c_1, \\
v(x) &= \int u' \, dx = \int \frac{15}{4} e^{x} \, dx = \frac{15}{4} e^{x} + c_2
\end{align*}
\]
and
\[
\begin{align*}
u(x) &= \int w' \, dx = -\int \frac{15}{4} e^{5x} \, dx = \frac{3}{4} e^{5x} + c_3.
\end{align*}
\]
The final answer is then given by plugging these back into formula (\*) for \( y \):
\[
y(x) = u + e^{2x}v + e^{-2x}w = \left[ -\frac{5}{2} e^{3x} + c_1 \right] + e^{2x} \left[ \frac{15}{4} e^{x} + c_2 \right] + e^{-2x} \left[ \frac{3}{4} e^{5x} + c_3 \right] \\
= \left[ -\frac{5}{2} e^{3x} + \frac{15}{4} e^{x} + \frac{3}{4} e^{5x} + c_3 \right] \\
= 2e^{3x} + c_1 + c_2 e^{2x} + c_3 e^{-2x}.
\]

22.4.a. Here, \( y_1(x) = x \), \( y_2(x) = x^2 \) and \( y_3(x) = x^3 \). So the solution is given by
\[
y(x) = y_1u + y_2v + y_3w = xu + x^2v + x^3w
\]
where \( u = u(x) \), \( v = v(x) \) and \( w = w(x) \) satisfy the system
\[
\begin{align*}
y_1u' + y_2v' + y_3w' &= 0 \\
y_1'u' + y_2'v' + y_3'w' &= 0 \\
y_1''u' + y_2''v' + y_3''w' &= \frac{g}{a}
\end{align*}
\]
which, in this case, is
\[
\begin{align*}
xu' + x^2v' + x^3w' &= 0 \\
u' + 2xv' + 3x^2w' &= 0 \\
0u' + 2v' + 6xw' &= \frac{e^{-x}}{x^3} = x^{-3} e^{-x}
\end{align*}
\]

22.4.c. Here, \( y_1(x) = e^{3x} \), \( y_2(x) = e^{-3x} \), \( y_3(x) = \cos(3x) \) and \( y_4(x) = \sin(3x) \). So
\[
y(x) = e^{3x}u_1 + e^{-3x}u_2 + \cos(3x)u_3 + \sin(3x)u_4
\]
where
\[
\begin{align*}
y_1u_1' + y_2u_2' + y_3u_3' + y_4u_4' &= 0 \\
y_1'u_1' + y_2'u_2' + y_3'u_3' + y_4'u_4' &= 0 \\
y_1''u_1' + y_2''u_2' + y_3''u_3' + y_4''u_4' &= 0 \\
y_1'''u_1' + y_2'''u_2' + y_3'''u_3' + y_4'''u_4' &= \frac{g}{a}
\end{align*}
\]
which, in this case, is

\[

\begin{align*}
    e^{3x}u_1' + e^{-3x}u_2' + \cos(3x)u_3' + \sin(3x)u_4' &= 0 \\
    3e^{3x}u_1' - 3e^{-3x}u_2' - 3\sin(3x)u_3' + 3\cos(3x)u_4' &= 0 \\
    9e^{3x}u_1' + 9e^{-3x}u_2' - 9\cos(3x)u_3' - 9\sin(3x)u_4' &= 0 \\
    27e^{3x}u_1' - 27e^{-3x}u_2' + 27\sin(3x)u_3' - 27\cos(3x)u_4' &= \frac{\sinh(x)}{1}
\end{align*}
\]

Dividing out common factors in each equation, this reduces to

\[
\begin{align*}
    e^{3x}u_1' + e^{-3x}u_2' + \cos(3x)u_3' + \sin(3x)u_4' &= 0 \\
    e^{3x}u_1' - e^{-3x}u_2' - \sin(3x)u_3' + \cos(3x)u_4' &= 0 \\
    e^{3x}u_1' + e^{-3x}u_2' - \cos(3x)u_3' - \sin(3x)u_4' &= 0 \\
    e^{3x}u_1' - e^{-3x}u_2' + \sin(3x)u_3' - \cos(3x)u_4' &= \frac{1}{27}\sinh(x)
\end{align*}
\]

22.6. Recall that, for any sufficiently continuous function \( g \),

\[
\int_{x_0}^x g(s) \, ds = 0 \quad \text{and} \quad \frac{d}{dx} \int_{x_0}^x g(s) \, ds = g(x).
\]

Letting \( x = x_0 \) in formula (22.15) yields

\[
y_p(x_0) = -y_1(x_0) \int_{x_0}^{x_0} y_2(s) f(s) \frac{W(s)}{W(x)} \, ds + y_2(x_0) \int_{x_0}^{x_0} y_1(s) f(s) \frac{W(s)}{W(x)} \, ds \\
= -y_1(x_0) \cdot 0 + y_2(x_0) \cdot 0 = 0,
\]

verifying the claim that \( y_p(x_0) = 0 \).

Differentiating that \( y_p(x_0) = 0 \),

\[
y_p'(x) = \frac{d}{dx} \left[ -y_1(x) \int_{x_0}^x y_2(s) f(s) \frac{W(s)}{W(x)} \, ds + y_2(x) \int_{x_0}^x y_1(s) f(s) \frac{W(s)}{W(x)} \, ds \right]
\]

\[
= -y_1'(x) \int_{x_0}^x y_2(s) f(s) \frac{W(s)}{W(x)} \, ds - y_1(x) \frac{d}{dx} \int_{x_0}^x y_2(s) f(s) \frac{W(s)}{W(x)} \, ds \\
+ y_2'(x) \int_{x_0}^x y_1(s) f(s) \frac{W(s)}{W(x)} \, ds + y_2(x) \frac{d}{dx} \int_{x_0}^x y_1(s) f(s) \frac{W(s)}{W(x)} \, ds
\]

\[
= -y_1'(x) \int_{x_0}^x y_2(s) f(s) \frac{W(s)}{W(x)} \, ds - y_1(x) y_2'(x) \frac{f(x)}{W(x)} \\
+ y_2'(x) \int_{x_0}^x y_1(s) f(s) \frac{W(s)}{W(x)} \, ds + y_2(x) y_1'(x) \frac{f(x)}{W(x)} \\
= -y_1'(x) \int_{x_0}^x y_2(s) f(s) \frac{W(s)}{W(x)} \, ds + y_2'(x) \int_{x_0}^x y_1(s) f(s) \frac{W(s)}{W(x)} \, ds \\
- \left[ y_1(x) y_2'(x) - y_2(x) y_1'(x) \right] \frac{f(x)}{W(x)} \\
= -y_1'(x) \int_{x_0}^x y_2(s) f(s) \frac{W(s)}{W(x)} \, ds + y_2'(x) \int_{x_0}^x y_1(s) f(s) \frac{W(s)}{W(x)} \, ds.
\]
Thus,

\[ y'_p(x_0) = -y'_1(x_0) \int_{x_0}^{x_0} \frac{y_2(s)f(s)}{W(s)} \, ds + y'_2(x_0) \int_{x_0}^{x_0} \frac{y_1(s)f(s)}{W(s)} \, ds \]

\[ = -y'_1(x_0) \cdot 0 + y'_2(x_0) \cdot 0 = 0 , \]

confirming that \( y'_p(x_0) = 0 \).