Chapter 21: Springs: Part II (Forced Vibrations)

21.2 a. $$\kappa = \left|\frac{F_0}{y_0}\right| = \frac{mg}{|\text{equilibrium length} - \text{natural length}|} = \frac{0.01 \times 9.8}{|0.12 - 0.1|} = 4.9 \quad \left(\frac{\text{kg}}{\text{sec}^2}\right) \quad .$$

21.2 b. $$\omega_0 = \sqrt{\frac{\kappa}{m}} = \sqrt{\frac{4.9}{0.01}} = 7\sqrt{10} \quad (\text{/sec}) \quad ,$$

and $$\nu_0 = \frac{\omega_0}{2\pi} = \frac{7\sqrt{10}}{2\pi} \quad (\text{hertz}) \quad .$$

21.3 a i. In this case,

$$\kappa = \left|\frac{F_0}{y_0}\right| = \frac{mg}{|\text{equilib. length} - \text{natural length}|} = \frac{25 \times 9.8}{|0.9 - 1|} = 2450 \quad \left(\frac{\text{kg}}{\text{sec}^2}\right) \quad .$$

21.3 a ii. $$\omega_0 = \sqrt{\frac{\kappa}{m}} = \sqrt{\frac{2450}{25}} = \sqrt{98} = 7\sqrt{2} \quad (\text{/sec}) \quad .$$

21.3 a iii. $$\nu_0 = \frac{\omega_0}{2\pi} = \frac{7\sqrt{2}}{2\pi} \approx 1.576 \quad \text{times per second (i.e., hertz)} \quad .$$

21.3 c. We have

$$\kappa = \left|\frac{F_0}{y_0}\right| = \frac{mg}{|\text{equilib. length} - \text{natural length}|} \quad .$$

So,

$$\text{mass} = m = \frac{\kappa \times |\text{equilib. length} - \text{natural length}|}{g}$$
$$= \frac{2450 \times |0.85 - 1.0|}{9.8} = 37.5 \quad (\text{kg}) \quad .$$

21.4 a i. Since the chicken's flapping agrees with the natural frequency of the system, formula (21.6) applies, which, with the given data, becomes

$$y(t) = \frac{F_0}{2m\omega_0} t \sin(\omega_0 t) = \frac{3}{2 \cdot 2(2\pi \cdot 6)} t \sin((2\pi \cdot 6)t) = \frac{t}{16\pi} \sin(12\pi t) \quad .$$

21.4 a ii. According to the given data, the spring breaks when the amplitude $A(t) = \frac{1}{16\pi} t$ (found in the above problem) reaches half the spring's natural length of 1 meter. So we solve $A(t) = \frac{1}{2}$ for t:

$$\frac{1}{16\pi} t = \frac{1}{2} \quad \rightarrowtail \quad t = \frac{16\pi}{2} = 8\pi \approx 25.1 \quad (\text{seconds}) \quad .$$

21.4 b i. Applying formula (21.8):

$$\begin{aligned} y_p(t) &= \frac{F_0}{m\left[(\omega_0)^2 - \eta^2\right]}\cos(\eta t) \\ &= \frac{3}{2\left[(2\pi\cdot 6)^2 - (2\pi\cdot 3)^2\right]}\cos((2\pi\cdot 3)t) \;=\; \frac{1}{72\pi^2}\cos(6\pi t) \quad . \end{aligned}$$

21.4 b ii. The amplitude of the oscillations induced by the chicken flapping is only

$$\frac{1}{72\pi^2} \approx 0.001407 \text{ meters} \quad ,$$

which is much below the half meter required to break the spring. So the spring does not break.

21.5 a. In this case, we already know that the general solution is given by formula (21.9). Thus,

$$y_\eta(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{F_0}{m\left[(\omega_0)^2 - \eta^2\right]}\cos(\eta t)$$

and

$$y_\eta{}'(t) = -c_1\omega_0\sin(\omega_0 t) + c_2\omega_0\cos(\omega_0 t) + \frac{F_0\eta}{m\left[(\omega_0)^2 - \eta^2\right]}\sin(\eta t) \quad .$$

Applying the initial conditions, we get

$$\begin{aligned} 0 = y_\eta(0) &= c_1\cos(\omega_0 0) + c_2\sin(\omega_0 0) + \frac{F_0}{m\left[(\omega_0)^2 - \eta^2\right]}\cos(\eta 0) \\ &= c_1 + \frac{F_0}{m\left[(\omega_0)^2 - \eta^2\right]} \end{aligned}$$

and

$$\begin{aligned} 0 = y_\eta{}'(0) &= -c_1\omega_0\sin(\omega_0 0) + c_2\omega_0\cos(\omega_0 0) + \frac{F_0\eta}{m\left[(\omega_0)^2 - \eta^2\right]}\sin(\eta 0) \\ &= c_2\omega_0 \quad . \end{aligned}$$

Hence,

$$c_1 = -\frac{F_0}{m\left[(\omega_0)^2 - \eta^2\right]} \quad , \quad c_2 = 0$$

and

$$\begin{aligned} y_\eta(t) &= -\frac{F_0}{m\left[(\omega_0)^2 - \eta^2\right]}\cos(\omega_0 t) + 0\sin(\omega_0 t) + \frac{F_0}{m\left[(\omega_0)^2 - \eta^2\right]}\cos(\eta t) \\ &= \frac{F_0}{m\left[(\omega_0)^2 - \eta^2\right]}\left[\cos(\eta t) - \cos(\omega_0 t)\right] \quad . \end{aligned}$$

21.5 b. In this case, we already know the general solution is given by formula 21.7. Applying this formula, we get

$$y_\eta(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{F_0}{2m\omega_0}t\sin(\omega_0 t)$$

and

$$y_\eta{}'(t) = -c_1\omega_o\sin(\omega_0 t) + c_2\omega_0\cos(\omega_0 t) + \frac{F_0}{2m\omega_0}\left[\sin(\omega_0 t) + \omega_0 t\cos(\omega_0 t)\right] \quad .$$

Applying the initial conditions, it is clear that the equations reduce to

$$0 = y_\eta(0) = c_1 \qquad \text{and} \qquad 0 = {y_\eta}'(0) = 0c_2\omega_0 \quad .$$

Thus, $c_1 = c_2 = 0$ and the formula for y_η reduces to

$$y_\eta(t) = \frac{F_0}{2m\omega_0} t \sin(\omega_0 t) \quad .$$

21.5 c. Naively attempting to compute the limit by simply replacing η with ω_0 yields

$$\lim_{\eta\to\omega_0} y_\eta(t) = \lim_{\eta\to\omega_0} \frac{F_0}{m\left[(\omega_0)^2 - \eta^2\right]}\left[\cos(\eta t) - \cos(\omega_0 t)\right] = \frac{0}{0} \quad ,$$

which is indeterminant. So we must use L'Hôpital's rule to compute the limit:

$$\begin{aligned}
\lim_{\eta\to\omega_0} y_\eta(t) &= \lim_{\eta\to\omega_0} \frac{F_0}{m\left[(\omega_0)^2 - \eta^2\right]}\left[\cos(\eta t) - \cos(\omega_0 t)\right] \\
&= \lim_{\eta\to\omega_0} \frac{\dfrac{d}{d\eta} F_0\left[\cos(\eta t) - \cos(\omega_0 t)\right]}{\dfrac{d}{d\eta} m\left[(\omega_0)^2 - \eta^2\right]} \\
&= \lim_{\eta\to\omega_0} \frac{F_0\left[-t\sin(\eta t) - 0\right]}{m\left[0 - 2\eta\right]} \\
&= \frac{F_0\left[-t\sin(\omega_0 t) - 0\right]}{m\left[0 - 2\omega_0\right]} = \frac{F_0}{2m\omega_0} t\sin(\omega_0 t) = y_{\omega_0}(t) \quad .
\end{aligned}$$