Chapter 20: Method of Undetermined Coefficients

20.1 a. For \( y_p \): Let \( y_p(x) = Ae^{2x} \). Then

\[
52e^{2x} = y'' + 9y = \left[Ae^{2x}\right]'' + 9Ae^{2x} = 4Ae^{2x} + 9Ae^{2x} = 13Ae^{2x}.
\]

Solving for \( A \), we get \( A = \frac{52}{13} = 4 \). Thus,

\[
y_p(x) = 4e^{2x}.
\]

For \( y_h \):

\[
y'' + 9y = 0
\]

\[
\iff r^2 + 9 = 0 \iff r = \pm\sqrt{-9} = \pm3i
\]

\[
y_h(x) = c_1 \cos(3x) + c_2 \sin(3x).
\]

The general solution to the nonhomogeneous equation is then

\[
y(x) = y_p(x) + y_h(x) = 4e^{2x} + c_1 \cos(3x) + c_2 \sin(3x).
\]

20.1 c. For \( y_p \): Let \( y(x) = Ae^{-4x} \). Then

\[
30e^{-4x} = y'' + 4y' - 5y
\]

\[
= \left[Ae^{-4x}\right]'' + 4\left[Ae^{-4x}\right]' - 5Ae^{-4x}
\]

\[
= 16Ae^{-4x} - 16Ae^{-4x} - 5Ae^{-4x} = -5Ae^{-4x}.
\]

Solving for \( A \), we get \( A = \frac{30}{-5} = -6 \). Thus,

\[
y_p(x) = -6e^{-4x}.
\]

For \( y_h \):

\[
y'' + 4y' - 5y = 0
\]

\[
\iff 0 = r^2 + 4r - 5 = (r - 1)(r + 5)
\]

\[
r = 1 \quad \text{and} \quad r = -5
\]

\[
y_h(x) = c_1 e^x + c_2 e^{-5x}.
\]

The general solution to the nonhomogeneous equation is then

\[
y(x) = y_p(x) + y_h(x) = -6e^{-4x} + c_1 e^x + c_2 e^{-5x}.
\]
20.2. For \( y_p \): Let \( y(x) = y_p(x) = Ae^{3x} \). Then

\[
-5e^{3x} = y'' - 3y' - 10y \\
= \left[Ae^{3x}\right]'' - 3\left[Ae^{3x}\right]' - 10Ae^{3x} \\
= 9Ae^{3x} - 9Ae^{3x} - 10Ae^{3x} = -10Ae^{3x}.
\]

Solving for \( A \), we get \( A = \frac{-5}{-10} = \frac{1}{2} \). Thus,

\[
y_p(x) = \frac{1}{2}e^{3x}.
\]

For \( y_h \):

\[
y'' - 3y' - 10y = 0
\]

\[
\Leftrightarrow \quad 0 = r^2 - 3r - 10 = (r + 2)(r - 5)
\]

\[
\Leftrightarrow \quad r = -2 \quad \text{and} \quad r = 5
\]

\[
y_h(x) = c_1e^{-2x} + c_2e^{5x}.
\]

The general solution to the nonhomogeneous equation is then

\[
y(x) = y_p(x) + y_h(x) = \frac{1}{2}e^{3x} + c_1e^{-2x} + c_2e^{5x}, \quad (\star)
\]

and the derivative is

\[
y'(x) = \frac{3}{2}e^{3x} - 2c_1e^{-2x} + 5c_2e^{5x}.
\]

Applying the initial conditions, we have

\[
y(0) = \frac{1}{2}e^{0} + c_1e^{-2\cdot0} + c_2e^{5\cdot0} = \frac{1}{2} + c_1 + c_2
\]

and

\[
y'(0) = \frac{3}{2}e^{3\cdot0} - 2c_1e^{-2\cdot0} + 5c_2e^{5\cdot0} = \frac{3}{2} - 2c_1 + 5c_2.
\]

Solving for the constants and plugging them back into formula (\( \star \)) for \( y \):

\[
5 = \frac{1}{2} + c_1 + c_2 \quad \text{and} \quad 3 = \frac{3}{2} - 2c_1 + 5c_2
\]

\[
\Leftrightarrow \quad c_1 = \frac{9}{2} - c_2 \quad \text{and} \quad \frac{3}{2} = -2\left[\frac{9}{2} - c_2\right] + 5c_2
\]

\[
\Leftrightarrow \quad c_1 = \frac{9}{2} - c_2 \quad \text{and} \quad c_2 = \frac{21}{7} = \frac{3}{2}
\]

\[
\Leftrightarrow \quad c_1 = \frac{9}{2} - \frac{3}{2} = 3 \quad \text{and} \quad c_2 = \frac{3}{2}
\]

\[
\Leftrightarrow \quad y(x) = \frac{1}{2}e^{3x} + 3e^{-2x} + \frac{3}{2}e^{5x}.
\]
20.3 a. For \( y_p \): Let \( y(x) = y_p(x) = A \cos(2x) + B \sin(2x) \). Then

\[
10 \cos(2x) + 15 \sin(2x) = y'' + 9y
\]

\[
= [A \cos(2x) + B \sin(2x)]'' + 9[A \cos(2x) + B \sin(2x)]
\]

\[
= [-4A \cos(2x) - 4B \sin(2x)] + [9A \cos(2x) + 9B \sin(2x)]
\]

\[
= 5A \cos(2x) + 5B \sin(2x)
\]

Cutting out the middle yields

\[
10 \cos(2x) + 15 \sin(2x) = 5A \cos(2x) + 5B \sin(2x)
\]

Comparing the terms on either side then gives us

\[
\cos(2x) \text{ terms: } 10 = 5A
\]

and

\[
\sin(2x) \text{ terms: } 15 = 5B
\]

Solving for \( A \) and \( B \), we get \( A = \frac{10}{5} = 2 \) and \( B = \frac{15}{5} = 3 \). Thus,

\[
y_p(x) = 2 \cos(2x) + 3 \sin(2x)
\]

For \( y_h \):

\[
y'' + 9y = 0
\]

\[
\Longleftrightarrow 0 = r^2 + 9 \implies r = \pm \sqrt{-9} = \pm 3i
\]

\[
\Longleftrightarrow y_h(x) = c_1 \cos(3x) + c_2 \sin(3x)
\]

The general solution to the nonhomogeneous equation is then

\[
y(x) = y_p(x) + y_h(x)
\]

\[
= 2 \cos(2x) + 3 \sin(2x) + c_1 \cos(3x) + c_2 \sin(3x)
\]

20.3 c. For \( y_p \): Let \( y(x) = y_p(x) = A \cos\left(\frac{x}{3}\right) + B \sin\left(\frac{x}{3}\right) \). Then

\[
26 \cos\left(\frac{x}{3}\right) - 12 \sin\left(\frac{x}{3}\right) = y'' + 3y'
\]

\[
= \left[ A \cos\left(\frac{x}{3}\right) + B \sin\left(\frac{x}{3}\right) \right]'' + 3 \left[ A \cos\left(\frac{x}{3}\right) + B \sin\left(\frac{x}{3}\right) \right]
\]

\[
= \left[ -\frac{A}{9} \cos\left(\frac{x}{3}\right) - \frac{B}{9} \sin(2) \right]
\]

\[
+ 3 \left[ -\frac{A}{3} \sin\left(\frac{x}{3}\right) + \frac{B}{3} \cos\left(\frac{x}{3}\right) \right]
\]

\[
= \left[ -\frac{A}{9} + B \right] \cos\left(\frac{x}{3}\right) + \left[ -A - \frac{B}{9} \right] \sin\left(\frac{x}{3}\right)
\]
Comparing terms:

\[ \cos \left( \frac{x}{3} \right) \text{ terms:} \quad 26 = - \frac{A}{9} + B \]

and

\[ \sin \left( \frac{x}{3} \right) \text{ terms:} \quad -12 = -A - \frac{B}{9} \]

Solving for \( A \) and \( B \), and plugging the results back into the formula for \( y_p \):

\[ 26 = - \frac{A}{9} + B \quad \text{and} \quad -12 = -A - \frac{B}{9} \]

\[ \Leftrightarrow \quad A = 9[B - 26] \quad \text{and} \quad -12 = -9[B - 26] - \frac{B}{9} \]

\[ \Leftrightarrow \quad A = 9[B - 26] \quad \text{and} \quad B = \frac{9}{82}[12 + 9 \cdot 26] = 27 \]

\[ \Leftrightarrow \quad A = 9[27 - 26] = 9 \quad \text{and} \quad B = 27 \]

\[ \rightarrow \quad y_p(x) = 9 \cos \left( \frac{x}{3} \right) + 27 \sin \left( \frac{x}{3} \right) . \]

For \( y_h \):

\[ y'' + 3y' = 0 \]

\[ \Leftrightarrow \quad 0 = r^2 + 3r = r(r + 3) \]

\[ \Leftrightarrow \quad r = 0 \quad \text{and} \quad r = -3 \]

\[ \rightarrow \quad y_h(x) = c_1 + c_2 e^{-3x} . \]

The general solution to the nonhomogeneous equation is then

\[ y(x) = y_p(x) + y_h(x) = 9 \cos \left( \frac{x}{3} \right) + 27 \sin \left( \frac{x}{3} \right) + c_1 + c_2 e^{-3x} . \]

20.5 a. For \( y_p \): Since \( g(x) \), the right side of the differential equation, is a constant, we’ll let \( y(x) = y_p(x) = A \). Plugging that into the differential equation, we then have

\[ -200 = y'' - 3y' - 10y = [A]' - [A]' - 10A = 0 - 0 - 10A \]

So, \( A = \frac{-200}{-10} = 20 \) and

\[ y_p(x) = 20 . \]

For \( y_h \):

\[ y'' - 3y' - 10y = 0 \]

\[ \Leftrightarrow \quad 0 = r^2 - 3r - 10 = (r + 2)(r - 5) \]

\[ \Leftrightarrow \quad r = -2 \quad \text{and} \quad r = 5 \]

\[ \rightarrow \quad y_h(x) = c_1 e^{-2x} + c_2 e^{5x} . \]
So the general solution to the nonhomogeneous differential equation is

\[ y(x) = y_p(x) + y_h(x) = 20 + c_1e^{-2x} + c_2e^{3x} . \]

20.5. For \( y_p \): Since \( g(x) \), the right side of the differential equation, is a second degree polynomial, we’ll let \( y(x) = y_p(x) = Ax^2 + Bx + C \). Plugging that into the differential equation, we then have

\[
18x^2 + 3x + 4 = y'' - 6y' + 9y
= \left[ Ax^2 + Bx + C \right]'' - 6 \left[ Ax^2 + Bx + C \right]'
+ 9 \left[ Ax^2 + Bx + C \right]
= 2A - 6[2Ax + B] + 9 \left[ Ax^2 + Bx + C \right]
\]

Cutting out the middle and comparing terms, we get the system

\[
\begin{align*}
x^2 \text{ terms: } & \quad 18 = 9A \\
x \text{ terms: } & \quad 3 = -12A + 9B \\
\text{constant terms: } & \quad 4 = 2A - 6B + 9C
\end{align*}
\]

Consequently,

\[
\begin{align*}
A &= \frac{18}{9} = 2 , \\
B &= \frac{3 + 12A}{9} = \frac{3 + 12 \cdot 2}{9} = 3 , \\
C &= \frac{4 - 2A + 6B}{9} = \frac{4 - 2 \cdot 2 + 6 \cdot 3}{9} = \frac{15}{3} = 2 ,
\end{align*}
\]

and the particular solution is

\[ y_p(x) = Ax^2 + Bx + C = 2x^2 + 3x + 2 . \]

For \( y_h \):

\[ y'' - 6y' + 9y = 0 \]

\[ \iff 0 = r^2 - 6r + 9 = (r - 3)^2 \implies r = 3 \]

\[ \iff y_h(x) = c_1e^{3x} + c_2xe^{3x} . \]

So the general solution to the nonhomogeneous differential equation is

\[ y(x) = y_p(x) + y_h(x) = 2x^2 + 3x + 2 + c_1e^{3x} + c_2xe^{3x} . \]

20.7. For \( y_p \): Since \( g(x) = 25x \cos(2x) \), we’ll let

\[ y(x) = y_p(x) = [Ax + B] \cos(2x) + [Cx + D] \sin(2x) . \]
Computing the derivatives, we get
\[
y'(x) = [A] \cos(2x) + [Ax + B](-2 \sin(2x)) \\
+ [C] \sin(2x) + [Cx + D](2 \cos(2x)) \\
= [2Cx + A + 2D] \cos(2x) + [-2Ax - 2B + C] \sin(2x)
\]
and
\[
y''(x) = [2C] \cos(2x) + [2Cx + A + 2D](-2 \sin(2x)) \\
+ [-2A] \sin(2x) + [-2Ax - 2B + C](2 \cos(2x)) \\
= [-4Ax - 4B + 4C] \cos(2x) + [-4Cx - 4A - 4D] \sin(2x)
\]
Plugging these into the differential equation, we then have
\[
25x \cos(2x) = y'' + 9y \\
= ([4Ax - 4B + 4C] \cos(2x) + [-4Cx - 4A - 4D] \sin(2x)) \\
+ 9([Ax + B] \cos(2x) + [Cx + D] \sin(2x)) \\
= [5Ax + 5B + 4C] \cos(2x) + [5Cx - 4A + 5D] \sin(2x)
\]
Cutting out the middle and comparing terms, we get the system
\[
\begin{align*}
x \cos(2x) \text{ terms:} & \quad 25 = 5A \\
\cos(2x) \text{ terms:} & \quad 0 = 5B + 4C \\
x \sin(2x) \text{ terms:} & \quad 0 = 5C \\
\sin(2x) \text{ terms:} & \quad 0 = -4A + 5D
\end{align*}
\]
Hence,
\[
A = \frac{25}{5} = 5, \quad C = 0, \quad B = -\frac{4}{5}C = 0 \quad \text{and} \quad D = \frac{4}{5}A = 4
\]
and particular solution (⋆) ends up being
\[
y_p(x) = [5x + 0] \cos(2x) + [0x + 4] \sin(x) = 5x \cos(2x) + 4 \sin(2x)
\]
For \(y_h\): From several of the previous problems, we already know that the general solution to the corresponding homogeneous equation \(y' + 9y = 0\) is
\[
y_h(x) = c_1 \cos(3x) + c_2 \sin(3x)
\]
Consequently, the general solution to the given nonhomogeneous differential equation is
\[
y(x) = y_p(x) + y_h(x) \\
= 5x \cos(2x) + 4 \sin(2x) + c_1 \cos(3x) + c_2 \sin(3x)
\]
\[\textbf{20.7 c.}\] For \(y_p\): Since \(g(x) = 54x^2 e^{3x}\), we’ll let
\[
y(x) = y_p(x) = \left[Ax^2 + Bx + C\right] e^{3x} \quad (⋆)
\]
Computing the derivatives, we get
\[
y'(x) = [2Ax + B]e^{3x} + \left[ Ax^2 + Bx + C \right](3e^{3x})
\]
\[
= \left[ 3Ax^2 + (2A + 3B)x + (B + 3C) \right]e^{3x}
\]
and
\[
y''(x) = [6Ax + (2A + 3B)]e^{3x} + \left[ 3Ax^2 + (2A + 3B)x + (B + 3C) \right](3e^{3x})
\]
\[
= \left[ 9Ax^2 + (12A + 9B)x + (2A + 6B + 9C) \right]e^{3x}.
\]
Plugging these into the differential equation, we then have
\[
54x^2e^{3x} = y'' + 9y
\]
\[
= \left[ 9Ax^2 + (12A + 9B)x + (2A + 6B + 9C) \right]e^{3x}
\]
\[
+ 9 \left[ Ax^2 + Bx + C \right]e^{3x}
\]
\[
= \left[ 18Ax^2 + (12A + 18B)x + (2A + 6B + 18C) \right]e^{3x}.
\]
Cutting out the middle and comparing terms, we get:
\[
\begin{align*}
x^2e^{3x} & \text{ terms: } 54 = 18A \\
x e^{3x} & \text{ terms: } 0 = 12A + 18B \\
e^{3x} & \text{ terms: } 0 = 2A + 6B + 18C
\end{align*}
\]
Hence,
\[
A = \frac{54}{18} = 3, \quad B = -\frac{12}{18}A = -\frac{2}{3} \cdot 3 = -2, \quad C = -\frac{1}{18}[2A + 6B] = -\frac{1}{18}[2 \cdot 3 - 6 \cdot 2] = \frac{1}{3},
\]
and particular solution (⋆) ends up being
\[
y_p(x) = \left[ 3x^2 - 2x + \frac{1}{3} \right]e^{3x}.
\]
For \( y_h \): From several of the previous problems, we already know that the general solution to the corresponding homogeneous equation \( y' + 9y = 0 \) is
\[
y_h(x) = c_1 \cos(3x) + c_2 \sin(3x)
\]
Consequently, the general solution to the given nonhomogeneous differential equation is
\[
y(x) = y_p(x) + y_h(x) = \left[ 3x^2 - 2x + \frac{1}{3} \right]e^{3x} + c_1 \cos(3x) + c_2 \sin(3x).
\]
\[20.7 \text{ e.} \quad \text{For } y_p : \text{ We’ll let}
\]
\[
y(x) = y_p(x) = [Ax + B] \cos(2x) + [Cx + D] \sin(2x) \quad (⋆)
\]
Computing the derivatives, we get
\[
y'(x) = [A] \cos(2x) + [Ax + B][-2 \sin(2x)] + [C] \sin(2x) + [Cx + D]2 \cos(2x)
\]
\[
= [2Cx + A + 2D] \cos(2x) + [-2Ax - 2B + C] \sin(2x)
\]
and
\[
y''(x) = [2C] \cos(2x) + [2Cx + A + 2D][-2 \sin(2x)] + [-2A] \sin(2x) + [-2Ax - 2B + C]2 \cos(2x)
\]
\[
= [-4Ax - 4B + 4C] \cos(2x) + [-4Cx - 4A - 4D] \sin(2x)
\]
Plugging that into the differential equation, we then have
\[
[-6x - 8] \cos(2x) + [8x - 11] \sin(2x)
\]
\[
y'' - 2y' + y
\]
\[
= ([-4Ax - 4B + 4C] \cos(2x) + [-4Cx - 4A - 4D] \sin(2x))
\]
\[
- 2((2Cx + A + 2D] \cos(2x) + [-2Ax - 2B + C] \sin(2x))
\]
\[
+ ([Ax + B] \cos(2x) + [Cx + D] \sin(x))
\]
\[
= [(-3A + 4C)x + (-2A - 3B + 4C - 4D)] \cos(2x)
\]
\[
+ [(4A - 3C)x + (-4A + 4B - 2C - 3D)] \sin(2x)
\].

Cutting out the middle and comparing terms, we get the system

<table>
<thead>
<tr>
<th>Term Type</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x \cos(2x))</td>
<td>(-6 = -3A + 4C)</td>
</tr>
<tr>
<td>(\cos(2x))</td>
<td>(-8 = -2A - 3B + 4C - 4D)</td>
</tr>
<tr>
<td>(x \sin(2x))</td>
<td>(8 = 4A - 3C)</td>
</tr>
<tr>
<td>(\sin(2x))</td>
<td>(-11 = -4A + 4B - 2C - 3D)</td>
</tr>
</tbody>
</table>

Since there are two equations just involving \(A\) and \(C\), we will solve that pair first:

\[
\begin{align*}
-6 &= -3A + 4C & 8 &= 4A - 3C \\
\Rightarrow A &= \frac{6 + 4C}{3} & & 8 &= 4 \left[ \frac{6 + 4C}{3} \right] - 3C = 8 + \frac{7C}{3} \\
\Rightarrow A &= \frac{6 + 4C}{3} & C &= \frac{3(8 - 8)}{7} = 0 \\
\Rightarrow A &= \frac{6 + 4 \cdot 0}{3} = 2 & C &= 0 \\
\end{align*}
\]

With these values for \(A\) and \(C\), the other equations in the system for the coefficients become

\[
-8 = -2 \cdot 2 - 3B + 4 \cdot 0 - 4D \quad \Rightarrow \quad 4 = 3B + 4D
\]
and
\[
-11 = -4 \cdot 2 + 4B - 2 \cdot 0 - 3D \quad \Rightarrow \quad -3 = 4B - 3D
\]
Solving for $B$ and $D$:

$$4 = 3B + 4D \quad \text{and} \quad -3 = 4B - 3D$$

$$\iff B = \frac{4 - 4D}{3} \quad \text{and} \quad -3 = 4 \left[ \frac{4 - 4D}{3} \right] - 3D = \frac{16}{3} - \frac{25D}{3}$$

$$\iff B = \frac{4 - 4D}{3} \quad \text{and} \quad D = \frac{3}{25} \left[ \frac{3 + 16}{3} \right] = 1$$

So particular solution (*) becomes

$$y_p(x) = [2x + 0] \cos(2x) + [0x + 1] \sin(2x) = 2x \cos(2x) + \sin(2x)$$.

For $y_h$:

$$y'' - 2y' + y = 0$$

$$\iff 0 = r^2 - 2r + 1 = (r - 1)^2 \implies r = 1$$

$$\iff y_h(x) = c_1 e^x + c_2 xe^x$$.

Hence, the general solution to the given nonhomogeneous differential equation is

$$y(x) = y_p(x) + y_h(x) = 2x \cos(2x) + \sin(2x) + c_1 e^x + c_2 xe^x$$.

20.9 a. For these problems, the “first guess” will not work. For this reason we start by finding the general solution $y_h$ to the corresponding homogeneous differential equation:

$$y'' - 3y' - 10y = 0$$

$$\iff 0 = r^2 - 3r - 10 = (r + 2)(r - 5)$$

$$\iff r = -2 \quad \text{and} \quad r = 5$$

$$\iff y_h(x) = c_1 e^{-2x} + c_2 e^{5x}$$.

For $y_p$: Since $g(x) = -3e^{-2x}$,

the “first guess” for $y_p(x)$ is $A e^{-2x}$.

But that is a term in $y_h$. So we must consider the “second guess”;

“second guess” for $y_p(x) = x \times “first guess” = x \left[ A e^{-2x} \right] = Axe^{-2x}$.

Since this is not a solution to the corresponding homogeneous equation, it is an appropriate “guess”. Accordingly, we will let

$$y(x) = y_p(x) = Axe^{-2x} \iff y'(x) = Ae^{-2x} - 2Ax e^{-2x}$$

$$\iff y''(x) = -4A e^{-2x} + 4x A e^{-2x}$$.
Using this with the given differential equation yields
\[-3e^{-2x} = y'' - 3y' - 10y\]
\[= \left[-4Ae^{-2x} + 4xe^{-2x}\right] - 3\left[Ae^{-2x} - 2Ax e^{-2x}\right] - 10\left[Axe^{-2x}\right]\]
\[= [4 + 6 - 10]Axe^{-2x} + [-4 - 3]Ae^{-2x}\]
\[= [0]Axe^{-2x} + [-7]Ae^{-2x} = -7Ae^{-2x} .\]
Thus, \(A = \frac{-3}{-7} = \frac{3}{7}\), and
\[y_p(x) = Axe^{-2x} = \frac{3}{7}xe^{-2x},\]
with the general solution being
\[y(x) = y_p(x) + y_h(x) = \frac{3}{7}xe^{-2x} + c_1e^{-2x} + c_2e^{5x} .\]

20.9 c. For \(y_h\):
\[y'' + 4y' = 0\]
\[\iff\]
\[0 = r^2 + 4r = r(r + 4)\]
\[\iff\]
\[r = 0 \quad \text{and} \quad r = -4\]
\[y_h(x) = c_1 + c_2e^{-4x} .\]
For \(y_p\):
“first guess” = \(Ax^2 + Bx + C\),
which we must reject since one term, the constant \(C\), is also a solution to the corresponding homogeneous differential equation. So we consider
“second guess” = \(x\left[Ax^2 + Bx + C\right] = Ax^3 + Bx^2 + Cx\),
No terms in this can be viewed as a term in \(y_h\). So, in finding \(y_p\) we will let
\[y(x) = y_p(x) = Ax^3 + Bx^2 + Cx\]
\[\iff\]
\[y'(x) = 3Ax^2 + 2Bx + C \implies y''(x) = 6Ax + 2B .\]
Plugging into the differential equation,
\[x^2 = y'' + 4y'\]
\[= [6Ax + 2B] + 4\left[3Ax^2 + 2Bx + C\right]\]
leads to the system

\[
\begin{align*}
\text{x}^2 \text{ terms:} & \quad 1 = 12A \\
x \text{ terms:} & \quad 0 = 6A + 8B \\
\text{constant terms:} & \quad 0 = 2B + 4C
\end{align*}
\]

Hence,

\[
A = \frac{1}{12}, \quad B = -\frac{6}{8}A = -\frac{1}{16}, \quad C = -\frac{2}{4}B = \frac{1}{32},
\]

\[
y_p(x) = Ax^3 + Bx^2 + Cx = \frac{1}{12}x^3 - \frac{1}{16}x^2 + \frac{1}{32}x,
\]

and the general solution is

\[
y(x) = \frac{1}{12}x^3 - \frac{1}{16}x^2 + \frac{1}{32}x + c_1 + c_2e^{-4x}.
\]

20.9 e. For \(y_h\):

\[
y'' - 6y' + 9y = 0
\]

\[
\iff \quad 0 = r^2 - 6r + 9 = (r - 3)^2 \implies r = 3
\]

\[
y_h(x) = c_1e^{3x} + c_2xe^{3x}.
\]

For \(y_p\):

“First guess” = \(Ae^{3x}\),

which we reject since it is a term in \(y_h\) (with \(A = c_1\)).

“Second guess” = \(x[Ae^{3x}] = Axe^{3x}\),

which we also reject since it is also a term in \(y_h\). But

“Third guess” = \(x[Axe^{3x}] = Ax^2e^{3x}\)

is not a solution to the corresponding homogeneous equation. Thus, to find \(y_p\) we let

\[
y(x) = y_p(x) = Ax^2e^{3x} \implies y'(x) = 2Axe^{3x} + 3Ax^2e^{3x}
\]

\[
\iff \quad y''(x) = 2Ae^{3x} + 12Axe^{3x} + 9Ax^2e^{3x}.
\]

Plugging into the differential equation, we get

\[
10e^{3x} = y'' - 6y' + 9
\]

\[
= \left[2Ae^{3x} + 12Axe^{3x} + 9Ax^2e^{3x}\right] - 6\left[2Axe^{3x} + 3Ax^2e^{3x}\right]
\]

\[
+ 9\left[Ax^2e^{3x}\right]
\]

\[
= 2Ae^{3x} + [12 - 12]Axe^{3x} + [9 - 18 + 9]Ax^2e^{3x} = 2Ae^{3x}.
\]
So \( A = \frac{10}{2} = 5 \),

\[ y_p(x) = Ax^2e^{3x} = 5x^2e^{3x} \]

and the general solution is

\[ y(x) = 5x^2e^{3x} + c_1e^{3x} + c_2xe^{3x} \]

20.10 a. For \( y_h \):

\[ y'' - 3y' - 10y = 0 \]

\[ 0 = r^2 - 3r - 10 = (r + 2)(r - 5) \]

\[ y_h(x) = c_1e^{-2x} + c_2e^{5x} \]

For \( y_p \):

\[
\text{“First guess”} = \left[ Ax^2 + Bx + C \right]e^{2x} = Ax^2e^{2x} + Bxe^{2x} + Ce^{2x}.
\]

Since no term in this is also a term in \( y_h \), this guess is what we will use,

\[ y(x) = y_p(x) = \left[ Ax^2 + Bx + C \right]e^{2x} \]

Computing the derivatives:

\[ y'(x) = [2Ax + B]e^{2x} + \left[ Ax^2 + Bx + C \right] \left( 2e^{2x} \right) = \left[ 2Ax^2 + (2A + 2B)x + (B + 2C) \right]e^{2x} \]

and

\[ y''(x) = [4Ax + (2A + 2B)]e^{2x} + \left[ 2Ax^2 + (2A + 2B)x + (B + 2C) \right] \left( 2e^{2x} \right) = \left[ 4Ax^2 + (8A + 4B)x + (2A + 4B + 4C) \right]e^{2x} \]

Plugging into the differential equation:

\[
\left[ 72x^2 - 1 \right]e^{2x} = y'' - 3y' - 10y
\]

\[ = \left[ [4Ax^2 + (8A + 4B)x + (2A + 4B + 4C)]e^{2x} \right] - 3\left[ [2Ax^2 + (2A + 2B)x + (B + 2C)]e^{2x} \right] - 10\left[ [Ax^2 + Bx + C]e^{2x} \right] \]

\[ = \left[ -12Ax^2 + (2A - 12B)x + (2A + B - 12C) \right]e^{2x} \]

Cutting out the middle and comparing terms, we get the system

- \( x^2e^{2x} \) terms: 72 = -12A
- \( xe^{2x} \) terms: 0 = 2A - 12B
- \( e^{2x} \) terms: -1 = 2A + B - 12C
Hence,
\[
A = \frac{72}{-12} = -6 \quad B = \frac{2}{12} A = -1 \quad C = \frac{1}{12} [1 + 2A + B] = -1
\]
particular solution (**) is
\[
y_p(x) = \left[-6x^2 - x - 1\right]e^{3x}
\]
and the general solution is
\[
y(x) = \left[-6x^2 - x - 1\right]e^{3x} + c_1 e^{-2x} + c_2 e^{5x}
\]

**20.10 c.** For \(y_h:\)
\[
y'' - 10y' + 25y = 0
\]
\[
\iff 0 = r^2 - 10r + 25 = (r - 5)^2 \implies r = 5
\]
\[
y_h(x) = c_1 e^{5x} + c_2 xe^{5x}
\]
For \(y_p:\)

"First guess" = \(Ae^{5x}\),
which we reject since it is a term in \(y_h\) (with \(A = c_1\)).

"Second guess" = \(xe^{5x}\),
which we also reject since it is also a term in \(y_h\). But

"third guess" = \(xe^{5x}\)
is not a solution to the corresponding homogeneous equation. Thus, to find \(y_p\) we let
\[
y(x) = y_p(x) = Ax^2 e^{5x} \implies y'(x) = 2Ax e^{5x} + 5Ax^2 e^{5x}
\]
\[
y''(x) = 2A e^{5x} + 20Ax e^{5x} + 25Ax^2 e^{5x}
\]
Plugging into the differential equation,
\[
6e^{5x} = y'' - 10y' + 25y
\]
\[
= \left[2A e^{5x} + 20Ax e^{5x} + 25Ax^2 e^{5x}\right] - 10 \left[2Ax e^{5x} + 5Ax^2 e^{5x}\right]
\]
\[
+ 25 \left[Ax^2 e^{5x}\right]
\]
\[
= 2A e^{5x} + [20 - 20]Ax e^{5x} + [25 - 50 - 25]Ax^2 e^{5x} = 2A e^{5x}
\]
So \(A = \frac{6}{2} = 3\),
\[
y_p(x) = Ax^2 e^{5x} = 3x^2 e^{5x}
\]
and the general solution is
\[
y(x) = 3x^2 e^{5x} + c_1 e^{5x} + c_2 xe^{5x}
\]
For \( y_h \):

\[ y'' + 4y' + 5y = 0 \]

\[ \iff \quad r^2 + 4r + 5 = 0 \quad \iff \quad r = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 5}}{2} = -2 \pm 1i \]

\[ \iff \quad y_h(x) = c_1e^{-2x} \cos(x) + c_2e^{-2x} \sin(x) \quad . \]

For \( y_p \),

"First guess" = \( A \cos(3x) + B \sin(3x) \) .

Since no term in this is also a term in \( y_h \), this is what we will use for \( y = y_p \):

\[ 24 \sin(3x) = y'' + 4y' + 5y \]

\[ = [A \cos(3x) + B \sin(3x)]'' + 4[A \cos(3x) + B \sin(3x)]' \]
\[ + 5[A \cos(3x) + B \sin(3x)] \]
\[ = [-9A \cos(3x) - 9B \sin(3x)] + 4[-3A \sin(3x) + 3B \cos(3x)] \]
\[ + 5[A \cos(3x) + B \sin(3x)] \]
\[ = [-4A + 12B] \cos(3x) + [-12A - 4B] \sin(3x) \quad . \]

Comparing terms, we get

\[
\begin{align*}
\text{cos}(3x) \text{ terms:} & \quad 0 = -4A + 12B \\
\text{sin}(3x) \text{ terms:} & \quad 24 = -12A - 4B 
\end{align*}
\]

At this point, you should be able to figure out that the solution to this system is \( A = -\frac{9}{5} \) and \( B = -\frac{3}{5} \). Hence,

\[ y_p(x) = -\frac{9}{5} \cos(3x) - \frac{3}{5} \sin(3x) \]

and the general solution is

\[ y(x) = -\frac{9}{5} \cos(3x) - \frac{3}{5} \sin(3x) + c_1e^{-2x} \cos(x) + c_2e^{-2x} \sin(x) \quad . \]

For \( y_h \):

\[ y'' - 4y' + 5y = 0 \]

\[ \iff \quad r^2 + 4r + 5 = 0 \quad \iff \quad r = \frac{4 \pm \sqrt{4^2 - 4 \cdot 5}}{2} = -2 \pm 1i \]

\[ \iff \quad y_h(x) = c_1e^{2x} \cos(x) + c_2e^{2x} \sin(x) \quad . \]

For \( y_p \),

"First guess" = \( Ae^{2x} \cos(x) + Be^{2x} \sin(x) \) ,

which we reject since both terms are also terms in \( y_h \). Continuing:

"Second guess" = \( x \left[ Ae^{2x} \cos(x) + Be^{2x} \sin(x) \right] \]
\[ = Axe^{2x} \cos(x) + Bxe^{2x} \sin(x) \quad . \]
Neither term in this guess is a term in \( y_h \), so this is what we will use for \( y = y_p \). For convenience in computing the derivatives, it may be best to write this as
\[
y(x) = y_p(x) = xe^{2x}[A \cos(x) + B \sin(x)]
\]
Computing the derivatives:
\[
y'(x) = \left(e^{2x} + 2xe^{2x}\right)[A \cos(x) + B \sin(x)] + xe^{2x}[-A \sin(x) + B \cos(x)]
\]
and
\[
y''(x) = 2e^{2x}\left((A + [2A + B]x) \cos(x) + (B + [2B - A]x) \sin(x)\right)
\]
\[+ e^{2x}\left((0 + [2A + B]) \cos(x) + (0 + [2B - A]) \sin(x)\right)
\]
\[+ e^{2x}\left((A + [2A + B]x)(- \sin(x)) + (B + [2B - A]x) \cos(x)\right)
\]
\[= e^{2x}\left([[4A + 2B] + [3A + 4B]x] \cos(x)
\]
\[+ [[4B - 2A] + [3B - 4A]x] \sin(x)\right) .
\]
Plugging into the differential equation:
\[
e^{2x} \sin(x) = y'' + 4y' + 5y
\]
\[= e^{2x}\left([[4A + 2B] + [3A + 4B]x] \cos(x)
\]
\[+ [[4B - 2A] + [3B - 4A]x] \sin(x)\right)
\]
\[= e^{2x}\left([2B + 0x] \cos(x) + [-2A + 0x] \sin(x)\right) .
\]
Comparing terms, we get
\[
e^{2x} \cos(x) \text{ terms: } 0 = 2B
\]
\[
e^{2x} \sin(x) \text{ terms: } 1 = -2A
\]
Hence, \( A = -\frac{1}{2} \), \( B = 0 \),
\[
y_p(x) = -\frac{1}{2}xe^{2x} \cos(x) + 0xe^{2x} \sin(x) = -\frac{1}{2}xe^{2x} \cos(x)
\]
and the general solution is
\[
y(x) = -\frac{1}{2}xe^{2x} \cos(x) + c_1e^{2x} \cos(x) + c_2e^{2x} \sin(x)
\].

20.10 i. From several problems above, we know
\[
y_h(x) = c_1e^{2x} \cos(x) + c_2e^{2x} \sin(x)
\].
For \( y_p \):
\[\text{“First guess” } = A
\].
Since no term in this is also a term in $y_h$, this is what we will use for $y = y_p$:

$$100 = y'' - 4y' + 5y = [A]' - 4[A]' + 5A = 5A.$$ 

So $A = \frac{100}{5} = 20$, 

$$y_p(x) = 20$$

and the general solution is 

$$y(x) = 20 + c_1e^{2x}\cos(x) + c_2e^{2x}\sin(x).$$

20.10 k. From several problems above, we know 

$$y_h(x) = c_1e^{2x}\cos(x) + c_2e^{2x}\sin(x).$$

For $y_p$:

"First guess" = $Ax^2 + Bx + C$.

Since no term in this is also a term in $y_h$, this is what we will use for $y = y_p$:

$$10x^2 + 4x + 8 = y'' - 4y' + 5y$$

$$= \left[ Ax^2 + Bx + C \right]' - 4\left[ Ax^2 + Bx + C \right]'$$

$$+ 5\left[ Ax^2 + Bx + C \right]$$

$$= [2A] - 4[2Ax + B] + 5\left[ Ax^2 + Bx + C \right]$$

$$= 5Ax^2 + [-8A + 5B]x + [2A - 4B + 5C].$$

Comparing terms:

$x^2$ terms: $10 = 5A$

$x$ terms: $4 = -8A + 5B$

Constant terms: $8 = 2A - 4B + 5C$

So,

$$A = \frac{10}{5} = 2, \quad B = \frac{4 + 8A}{5} = 4, \quad C = \frac{8 - 2A + 4B}{5} = 4,$$

$$y_p(x) = 2x^2 + 4x + 4$$

and the general solution is 

$$y(x) = 2x^2 + 4x + 4 + c_1e^{2x}\cos(x) + c_2e^{2x}\sin(x).$$

20.10 m. For $y_h$:

$$y'' + y = 0$$

$$\iff r^2 + 1 = 0 \quad \implies r = \pm\sqrt{-1} = \pm i$$

$$\implies y_h(x) = c_1\cos(x) + c_2\sin(x).$$
For $y_p$, 

“First guess” $= A \cos(x) + B \sin(x)$,

which we reject since both terms are also terms in $y_h$. Continuing:

“Second guess” $= x [A \cos(x) + B \sin(x)] = Ax \cos(x) + Bxe \sin(x)$.

Neither term in this guess is a term in $y_h$, so this is what we will use for $y_p$. Computing the derivatives:

$y(x) = y_p(x) = x[A \cos(x) + B \sin(x)]$

$\Rightarrow y'(x) = [A \cos(x) + B \sin(x)] + x[-A \sin(x) + B \cos(x)]$

$\Rightarrow y''(x) = 2[-A \sin(x) + B \cos(x)] - x[A \cos(x) + B \sin(x)]$.

Plugging into the differential equation:

$6 \cos(x) - 3 \sin(x) = y'' + y$

$= 2[-A \sin(x) + B \cos(x)] - x[A \cos(x) + B \sin(x)]$

$+ x[A \cos(x) + B \sin(x)]$

$= 2B \cos(x) - 2A \sin(x)$.

Comparing terms, we get

\[\begin{align*}
\cos(x) \text{ terms:} & \quad 6 = 2B \\
\sin(x) \text{ terms:} & \quad -3 = -2A
\end{align*}\]

Hence, $A = \frac{-3}{-2} = \frac{3}{2}$, $B = \frac{6}{2} = 3$,

\[y_p(x) = \frac{3}{2}x \cos(x) + 3x \sin(x),\]

and the general solution is

\[y(x) = \frac{3}{2}x \cos(x) + 3x \sin(x) + c_1 \cos(x) + c_2 \sin(x).\]

20.11 a. From several problems above, we know

$y_h(x) = c_1 e^{2x} \cos(x) + c_2 e^{2x} \sin(x)$.

For $y_p$:

“First guess” $= \left[A_0x^3 + A_1x^2 + A_2x + A_3\right] e^{-x} \sin(x)$

$+ \left[B_0x^3 + B_1x^2 + B_2x + B_3\right] e^{-x} \cos(x)$.

Since no term in this is also a term in $y_h$, this guess is what we use for $y_p$. 

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**Method of Undetermined Coefficients**
20.11 c. For \( y_h \):

\[
y'' - 5y' + 6y = 0
\]

\[
0 = r^2 - 5r + 6 = (r - 2)(r - 3)
\]

\[
y_h(x) = c_1e^{2x} + c_2e^{3x}.
\]

For \( y_p \):

“First guess” = \([Ax^2 + Bx + C]e^{-7x} = Ax^2e^{-7x} + Bxe^{-7x} + Ce^{-7x}\).

Since no term in this is also a term in \( y_h \), this guess is what we use for \( y_p \).

20.11 e. From previous problems, we know

\[
y_h(x) = c_1e^{2x} + c_2e^{3x}.
\]

For \( y_p \):

“First guess” = \( Ae^{-8x}\).

Since no term in this is also a term in \( y_h \), this guess is what we use for \( y_p \).

20.11 g. From previous problems, we know

\[
y_h(x) = c_1e^{2x} + c_2e^{3x}.
\]

For \( y_p \):

“First guess” = \([Ax^2 + Bx + C]e^{3x} = Ax^2e^{3x} + Bxe^{3x} + Ce^{3x}\).

which we reject because the last term is also a term in \( y_h \). Continuing.

“Second guess” = \( x[Ax^2 + Bx + C]e^{3x} = Ax^3e^{3x} + Bx^2e^{3x} + Cxe^{3x}\).

Since no term in this is also a term in \( y_h \), this guess is what we use for \( y_p \).

20.11 i. From previous problems, we know

\[
y_h(x) = c_1e^{2x} + c_2e^{3x}.
\]

For \( y_p \):

“First guess” = \([Ax^2 + Bx + C]e^{3x} \cos(2x) + [Dx^2 + Ex + F]e^{3x} \sin(2x)\).

Since no term in this is also a term in \( y_h \), this guess is what we use for \( y_p \).
20.11 k. For \( y_h \):

\[
y'' - 4y' + 20y = 0
\]

\[
\iff r^2 - 4r + 20 = 0 \implies r = \frac{4 \pm \sqrt{(-4)^2 - 4 \cdot 20}}{2} = 2 \pm 4i
\]

\[
y_h(x) = c_1 e^{2x} \cos(4x) + c_2 e^{2x} \sin(4x)
\]

For \( y_p \):

“First guess” = \( Ae^{2x} \cos(4x) + Be^{2x} \sin(4x) \)

which we reject because each term is also a term in \( y_h \). Continuing,

“Second guess” = \( x \left[ Ae^{2x} \cos(4x) + Be^{2x} \sin(4x) \right] \)

\[
= Ax e^{2x} \cos(4x) + Bx e^{2x} \sin(4x)
\]

Since no term in this is also a term in \( y_h \), this guess is what we use for \( y_p \).

20.11 m. For \( y_h \):

\[
y'' - 10y' + 25y = 0
\]

\[
\iff 0 = r^2 - 10r + 25 = (r - 5)^2 \implies r = 5
\]

\[
y_h(x) = c_1 e^{5x} + c_2 xe^{5x}
\]

For \( y_p \):

“First guess” = \( x \left[ Ax^2 + Bx + C \right] e^{5x} = Ax^2 e^{5x} + Bxe^{5x} + Ce^{5x} \)

which we reject because two terms also terms in \( y_h \). Continuing,

“Second guess” = \( x \left[ Ax^3 + Bx^2 + Cx \right] e^{5x} = Ax^3 e^{5x} + Bx^2 e^{5x} + Cxe^{5x} \)

which we also reject because one term is also a term in \( y_h \). Continuing,

“Third guess” = \( x^2 \left[ Ax^4 + Bx^3 + Cx^2 \right] e^{5x} = Ax^4 e^{5x} + Bx^3 e^{5x} + Cx^2 e^{5x} \)

Since no term in this is also a term in \( y_h \), this guess is what we use for \( y_p \).

20.12 a. For \( y_p \):

“First guess” = \( Ae^{-2x} \).
Since no term in this is also a term in \(y_h\), this guess is what we will use for \(y = y_p\). Plugging it into the differential equation, we obtain

\[
12e^{-2x} = y^{(4)} - 4y^{(3)} = \left[ Ae^{-2x} \right]^{(4)} - 4 \left[ Ae^{-2x} \right]^{(3)}
\]

\[
= (-2)^4 Ae^{-2x} - 4(-2)^3 Ae^{-2x} = 48Ae^{-2x}.
\]

So \(A = \frac{12}{48} = \frac{1}{4}\), and \(y_p(x) = \frac{1}{4}e^{-2x}\).

**20.12 c.** For \(y_p\):

“First guess” = \(Ae^{4x}\).

which we reject because it is a term in \(y_h\). Continuing.

“Second guess” = \(x \left[ Ae^{4x} \right] = Axe^{4x}\).

Since no term in this is also a term in \(y_h\), this guess is what we will use for \(y = y_p\). Doing so and computing the necessary derivatives:

\[
y(x) = y_p(x) = Axe^{4x} \quad \implies y'(x) = Ae^{4x} + 4Axe^{4x}
\]

\[
\implies y''(x) = 8Ae^{4x} + 16Axe^{4x} \quad \implies y^{(3)}(x) = 48Ae^{4x} + 64Axe^{4x}
\]

\[
\implies y^{(4)}(x) = 256Ae^{4x} + 256Axe^{4x}.
\]

Plugging these into the differential equation, we obtain

\[
32e^{4x} = y^{(4)} - 4y^{(3)}
\]

\[
= \left[ 256Ae^{4x} + 256Axe^{4x} \right] - 4 \left[ 48Ae^{4x} + 64Axe^{4x} \right]
\]

\[
= 64Ae^{4x} + 0Axe^{4x} = 64Ae^{4x}.
\]

So \(A = \frac{32}{64} = \frac{1}{2}\), and \(y_p(x) = \frac{1}{2}xe^{4x}\).

**20.12 e.** For \(y_p\):

“First guess” = \(Ax^2 + Bx + C\).

Since no term in this is also a term in \(y_h\), this guess is what we will use for \(y = y_p\). Plugging it into the differential equation, we obtain

\[
x^2 = y^{(3)} - y'' + y' - y
\]

\[
= \left[ Ax^2 + Bx + C \right]^{(3)} - \left[ Ax^2 + Bx + C \right]''
\]

\[
+ \left[ Ax^2 + Bx + C \right]' - \left[ Ax^2 + Bx + C \right]
\]

\[
= [0] - [2A] + [2Ax + B] + [-Ax^2 - Bx - C]
\]

\[
= -Ax^2 + [2A - B]x + [2A + B - C].
\]
Comparing terms, we get

- $x^2$ terms: $1 = -A$
- $x$ terms: $0 = 2A - B$
- Constant terms: $0 = 2A + B - C$

Hence, $A = -1$, $B = -2A = 2$, $C = 2A + B = 0$, and $y_p(x) = -x^2 - 2x$.

20.12 g. For $y_p$:

“First guess” $= Ae^x$,

which we reject because it is a term in $y_h$. Continuing,

“Second guess” $= x \left[Ae^x\right] = Axe^x$.

Since no term in this is also a term in $y_h$, this guess is what we will use for $y = y_p$. Doing so and computing the necessary derivatives:

$y(x) = y_p(x) = Axe^x \implies y'(x) = Ae^x + Axe^x$

$\implies y''(x) = 2Ae^x + Axe^x \implies y'''(x) = 3Ae^x + Axe^x$.

Plugging these into the differential equation, we obtain

$6e^x = y''' - y'' + y' - y$

$= [3Ae^x + Axe^x] - [2Ae^x + Axe^x] + [Ae^x + Axe^x] - [Ae^x]$

$= 2Ae^x + 0Axe^x = 2Ae^x$.

So $A = \frac{6}{2} = 3$, and $y_p(x) = 3xe^x$.

20.13 a. For $y_p$:

“First guess” $= \left[Ax^2 + Bx + C\right]e^{3x} = Ax^2e^{3x} + Bxe^{3x} + Ce^{3x}$.

Since no term in this is also a term in $y_h$, this guess is what we use for $y_p$.

20.13 c. For $y_p$:

“First guess” $= \left[Ax^2 + Bx + C\right]e^{3x}\cos(3x)$

$+ \left[Dx^2 + Ex + F\right]e^{3x}\sin(3x)$.

Since no term in this is also a term in $y_h$, this guess is what we use for $y_p$. 
20.13 e. For $y_p$:

“First guess” = \[ [Ax + B] \cos(x) + [Cx + D] \sin(x) \]
\[ = Ax \cos(x) + B \cos(x) + Cx \sin(x) + D \sin(x) \]
which we reject because it has two terms that are also in $y_h$. Continuing,

“Second guess” = \[ x ([Ax + B] \cos(x) + [Cx + D] \sin(x)) \]
\[ = \left[ Ax^2 + Bx \right] \cos(x) + [Cx^2 + Dx] \sin(x) \]
\[ = Ax^2 \cos(x) + Bx \cos(x) + Cx^2 \sin(x) + Dx \sin(x) \]

Since no term in this is also a term in $y_h$, this guess is what we use for $y_p$.

20.13 g. For $y_p$:

“First guess” = \[ [Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F] e^{2x} \]
\[ = Ax^5 e^{2x} + Bx^4 e^{2x} + Cx^3 e^{2x} + Dx^2 e^{2x} + Ex e^{2x} + Fe^{2x} \]

Since no term in this is also a term in $y_h$, this guess is what we use for $y_p$.

20.14 a. From the answer to exercise 20.1 b, we know that one solution to

\[ y'' - 6y' + 9y = 27e^{6x} = g_1(x) \]

is

\[ y_{p1} = 3e^{6x} \]

and from the answer to exercise 20.3 b, we know that one solution to

\[ y'' - 6y' + 9y = 25 \sin(6x) = g_2(x) \]

is

\[ y_{p2}(x) = \frac{4}{9} \cos(6x) - \frac{1}{3} \sin(6x) \]

So, by superposition, it follows that one solution to

\[ y'' - 6y' + 9y = 27e^{6x} + 25 \sin(6x) = g_1(x) + g_2(x) \]

is

\[ y_p(x) = y_{p1}(x) + y_{p2}(x) = 3e^{6x} + \frac{4}{9} \cos(6x) - \frac{1}{3} \sin(6x) \]

20.14 c. Using the identity

\[ \sin^2(x) = \frac{1}{2} - \frac{1}{2} \cos(2x) \]

we can rewrite the differential equation as

\[ y'' - 4y' + 5y = g_1(x) + g_2(x) \]

where

\[ g_1(x) = \frac{1}{2} \quad \text{and} \quad g_2(x) = -\frac{1}{2} \cos(2x) \]
Our solution is \( y_p = y_{p1} + y_{p2} \) where \( y_{p1} \) and \( y_{p2} \) are, respectively, particular solutions to
\[
y'' - 4y' + 5y = g_1(x) \quad \text{and} \quad y'' - 4y' + 5y = g_2(x)
\]
which we will find by the method of guess.

First, we must find (again) \( y_h \):
\[
y'' - 4y' + 5y = 0
\]
\[arr \quad r^2 - 4r + 5 = 0 \quad \rarr \quad r = \frac{4 \pm \sqrt{(-4)^2 - 4 \cdot 5}}{2} = 2 \pm 1i
\]
\[
\rarr \quad y_h(x) = c_1 e^{2x} \cos(x) + c_2 e^{2x} \sin(x)
\]

For \( y_{p1} \): Since \( g_1 \) is a constant,

“First guess” = \( A \).

Since this is not a term in \( y_h \), this guess is what we will use for \( y = y_{p1} \). Plugging it into the differential equation:
\[
\frac{5}{2} = g_1(x) = y'' - 4y' + 5y = [A]'' - [A]' + 5A = 5A
\]
So \( A = \frac{5}{5.2} = \frac{1}{2} \) , and
\[
y_{p1}(x) = \frac{1}{2}
\]

For \( y_{p2} \): Since \( g_2(x) = \frac{1}{2} \cos(2x) \),

“First guess” = \( A \cos(2x) + B \sin(2x) \).

Since no term is also a term in \( y_h \), this guess is what we will use for \( y = y_{p2} \). Plugging it into the differential equation:
\[
-\frac{1}{2} \cos(2x) = g_2(x)
\]
\[
= y'' - 4y' + 5y
\]
\[
= [A \cos(2x) + B \sin(2x)]'' - 4[A \cos(2x) + B \sin(2x)]'
\]
\[
+ 5[A \cos(2x) + B \sin(2x)]
\]
\[
= [-4A \cos(2x) - 4B \sin(2x)] - 4[-2A \sin(2x) + 2B \cos(2x)A]
\]
\[
+ 5[A \cos(2x) + B \sin(2x)]
\]
\[
= [A - 8B] \cos(2x) + [8A + B] \sin(2x)
\]

Comparing terms, we get
\[
\text{cos}(x) \text{ terms:} \quad -\frac{1}{2} = A - 8B
\]
\[
\text{sin}(x) \text{ terms:} \quad 0 = 8A + B
\]

Solving this yields \( A = -\frac{1}{26} \) and \( B = \frac{4}{13} \). Hence,
\[
y_{p2}(x) = -\frac{1}{26} \cos(2x) + \frac{4}{13} \sin(2x)
\]
Finally, combining the above yields

\[ y_p(x) = y_{p1} + y_{p2} = \frac{1}{2} - \frac{1}{26} \cos(2x) + \frac{4}{13} \sin(2x) \, . \]