Chapter 20: Method of Undetermined Coefficients

20.1 a. For y_p : Let $y(x) = y_p(x) = Ae^{2x}$. Then

$$52e^{2x} = y'' + 9y = \left[Ae^{2x}\right]'' + 9Ae^{2x} = 4Ae^{2x} + 9Ae^{2x} = 13Ae^{2x} \quad .$$

Solving for A , we get $A = \frac{52}{13} = 4$. Thus,

$$y_p(x) = 4e^{2x} \quad .$$

For y_h :

$$y'' + 9y = 0$$

$$\hookrightarrow \qquad r^2 + 9 = 0 \quad \rightarrowtail \quad r = \pm\sqrt{-9} = \pm 3i$$

$$\hookrightarrow \qquad y_h(x) = c_1 \cos(3x) + c_2 \sin(3x) \quad .$$

The general solution to the nonhomogeneous equation is then

$$y(x) = y_p(x) + y_h(x) = 4e^{2x} + c_1 \cos(3x) + c_2 \sin(3x) \quad .$$

20.1 c. For y_p : Let $y(x) = y_p(x) = Ae^{-4x}$. Then

$$\begin{aligned} 30e^{-4x} &= y'' + 4y' - 5y \\ &= \left[Ae^{-4x}\right]'' + 4\left[Ae^{-4x}\right]' - 5Ae^{-4x} \\ &= 16Ae^{-4x} - 16Ae^{-4x} - 5Ae^{-4x} = -5Ae^{-4x} \quad . \end{aligned}$$

Solving for A , we get $A = \frac{30}{-5} = -6$. Thus,

$$y_p(x) = -6e^{-4x} \quad .$$

For y_h :

$$y'' + 4y' - 5y = 0$$

$$\hookrightarrow \qquad 0 = r^2 + 4r - 5 = (r-1)(r+5)$$

$$\hookrightarrow \qquad r = 1 \qquad \text{and} \qquad r = -5$$

$$\hookrightarrow \qquad y_h(x) = c_1 e^x + c_2 e^{-5x} \quad .$$

The general solution to the nonhomogeneous equation is then

$$y(x) = y_p(x) + y_h(x) = -6e^{-4x} + c_1 e^x + c_2 e^{-5x} \quad .$$

20.2. For y_p : Let $y(x) = y_p(x) = Ae^{3x}$. Then

$$\begin{aligned} -5e^{3x} &= y'' - 3y' - 10y \\ &= \left[Ae^{3x}\right]'' - 3\left[Ae^{3x}\right]' - 10Ae^{3x} \\ &= 9Ae^{3x} - 9Ae^{3x} - 10Ae^{3x} = -10Ae^{3x} \quad . \end{aligned}$$

Solving for A , we get $A = \frac{-5}{-10} = \frac{1}{2}$. Thus,

$$y_p(x) = \tfrac{1}{2}e^{3x} \quad .$$

For y_h :

$$y'' - 3y' - 10y = 0$$

$$\hookrightarrow \qquad 0 = r^2 - 3r - 10 = (r+2)(r-5)$$

$$\hookrightarrow \qquad r = -2 \quad \text{and} \quad r = 5$$

$$\hookrightarrow \qquad y_h(x) = c_1e^{-2x} + c_2e^{5x} \quad .$$

The general solution to the nonhomogeneous equation is then

$$y(x) = y_p(x) + y_h(x) = \tfrac{1}{2}e^{3x} + c_1e^{-2x} + c_2e^{5x} \quad , \qquad (\star)$$

and the derivative is

$$y'(x) = \tfrac{3}{2}e^{3x} - 2c_1e^{-2x} + 5c_2e^{5x} \quad .$$

Applying the initial conditions, we have

$$5 = y(0) = \tfrac{1}{2}e^{3\cdot 0} + c_1e^{-2\cdot 0} + c_2e^{5\cdot 0} = \tfrac{1}{2} + c_1 + c_2$$

and

$$3 = y'(0) = \tfrac{3}{2}e^{3\cdot 0} - 2c_1e^{-2\cdot 0} + 5c_2e^{5\cdot 0} = \tfrac{3}{2} - 2c_1 + 5c_2 \quad .$$

Solving for the constants and plugging them back into formula $(\star)$ for y :

$$5 = \tfrac{1}{2} + c_1 + c_2 \quad \text{and} \quad 3 = \tfrac{3}{2} - 2c_1 + 5c_2$$

$$\hookrightarrow \qquad c_1 = \tfrac{9}{2} - c_2 \quad \text{and} \quad \tfrac{3}{2} = -2\left[\tfrac{9}{2} - c_2\right] + 5c_2$$

$$\hookrightarrow \qquad c_1 = \tfrac{9}{2} - c_2 \quad \text{and} \quad c_2 = \tfrac{21}{7\cdot 2} = \tfrac{3}{2}$$

$$\hookrightarrow \qquad c_1 = \tfrac{9}{2} - \tfrac{3}{2} = 3 \quad \text{and} \quad c_2 = \tfrac{3}{2}$$

$$\hookrightarrow \qquad y(x) = \tfrac{1}{2}e^{3x} + 3e^{-2x} + \tfrac{3}{2}e^{5x} \quad .$$

20.3 a. For y_p : Let $y(x) = y_p(x) = A\cos(2x) + B\sin(2x)$. Then

$$\begin{aligned}10\cos(2x) + 15\sin(2x) &= y'' + 9y \\ &= [A\cos(2x) + B\sin(2x)]'' \\ &\quad + 9[A\cos(2x) + B\sin(2)] \\ &= [-4A\cos(2x) - 4B\sin(2x)] \\ &\quad + [9A\cos(2x) + 9B\sin(2x)] \\ &= 5A\cos(2x) + 5B\sin(2x) \quad .\end{aligned}$$

Cutting out the middle yields

$$10\cos(2x) + 15\sin(2x) = 5A\cos(2x) + 5B\sin(2x) \quad .$$

Comparing the terms on either side then gives us

$$\cos(2x) \text{ terms:} \qquad 10 = 5A$$

and

$$\sin(2x) \text{ terms:} \qquad 15 = 5B \quad .$$

Solving for A and B , we get $A = \frac{10}{5} = 2$ and $B = \frac{15}{5} = 3$. Thus,

$$y_p(x) = 2\cos(2x) + 3\sin(2x) \quad .$$

For y_h :

$$y'' + 9y = 0$$

$$\hookrightarrow \qquad 0 = r^2 + 9 \quad \rightarrowtail \quad r = \pm\sqrt{-9} = \pm 3i$$

$$\hookrightarrow \qquad y_h(x) = c_1\cos(3x) + c_2\sin(3x) \quad .$$

The general solution to the nonhomogeneous equation is then

$$\begin{aligned}y(x) &= y_p(x) + y_h(x) \\ &= 2\cos(2x) + 3\sin(2x) + c_1\cos(3x) + c_2\sin(3x) \quad .\end{aligned}$$

20.3 c. For y_p : Let $y(x) = y_p(x) = A\cos\left(\frac{x}{3}\right) + B\sin\left(\frac{x}{3}\right)$. Then

$$\begin{aligned}26\cos\left(\frac{x}{3}\right) - 12\sin\left(\frac{x}{3}\right) &= y'' + 3y' \\ &= \left[A\cos\left(\frac{x}{3}\right) + B\sin\left(\frac{x}{3}\right)\right]'' \\ &\quad + 3\left[A\cos\left(\frac{x}{3}\right) + B\sin\left(\frac{x}{3}\right)\right]' \\ &= \left[-\frac{A}{9}\cos\left(\frac{x}{3}\right) - \frac{B}{9}\sin(2)\right] \\ &\quad + 3\left[-\frac{A}{3}\sin\left(\frac{x}{3}\right) + \frac{B}{3}\cos\left(\frac{x}{3}\right)\right] \\ &= \left[-\frac{A}{9} + B\right]\cos\left(\frac{x}{3}\right) + \left[-A - \frac{B}{9}\right]\sin\left(\frac{x}{3}\right) \quad .\end{aligned}$$

Comparing terms:

$$\cos\left(\frac{x}{3}\right) \text{ terms:} \qquad 26 \;=\; -\frac{A}{9} \;+\; B$$

and

$$\sin\left(\frac{x}{3}\right) \text{ terms:} \qquad -12 \;=\; -A \;-\; \frac{B}{9} \quad .$$

Solving for A and B, and plugging the results back into the formula for y_p:

$$26 \;=\; -\frac{A}{9} \;+\; B \qquad \text{and} \qquad -12 \;=\; -A \;-\; \frac{B}{9}$$

$$\hookrightarrow \qquad A \;=\; 9[B-26] \qquad \text{and} \qquad -12 \;=\; -9[B-26] \;-\; \frac{B}{9}$$

$$\hookrightarrow \qquad A \;=\; 9[B-26] \qquad \text{and} \qquad B \;=\; \frac{9}{82}[12+9\cdot 26] \;=\; 27$$

$$\hookrightarrow \qquad A \;=\; 9[27-26] \;=\; 9 \qquad \text{and} \qquad B \;=\;=\; 27$$

$$\hookrightarrow \qquad y_p(x) \;=\; 9\cos\left(\frac{x}{3}\right) \;+\; 27\sin\left(\frac{x}{3}\right) \quad .$$

For y_h:

$$y'' \;+\; 3y' \;=\; 0$$

$$\hookrightarrow \qquad 0 \;=\; r^2 \;+\; 3r \;=\; r(r+3)$$

$$\hookrightarrow \qquad r=0 \qquad \text{and} \qquad r \;=\; -3$$

$$\hookrightarrow \qquad y_h(x) \;=\; c_1 \;+\; c_2e^{-3x} \quad .$$

The general solution to the nonhomogeneous equation is then

$$y(x) \;=\; y_p(x) \;+\; y_h(x) \;=\; 9\cos\left(\frac{x}{3}\right) \;+\; 27\sin\left(\frac{x}{3}\right) \;+\; c_1 \;+\; c_2e^{-3x} \quad .$$

20.5 a. For y_p: Since $g(x)$, the right side of the differential equation, is a constant, we'll let $y(x) \;=\; y_p(x) \;=\; A$. Plugging that into the differential equation, we then have

$$-200 \;=\; y'' \;-\; 3y' \;-\; 10y \;=\; [A]'' \;-\; [A]' \;-\; 10A \;=\; 0 \;-\; 0 \;-\; 10A \quad .$$

So, $A = \frac{-200}{-10} = 20$ and

$$y_p(x) \;=\; 20 \quad .$$

For y_h:

$$y'' \;-\; 3y' \;-\; 10y \;=\; 0$$

$$\hookrightarrow \qquad 0 \;=\; r^2 \;-\; 3r \;-\; 10 \;=\; (r+2)(r-5)$$

$$\hookrightarrow \qquad r \;=\; -2 \qquad \text{and} \qquad r \;=\; 5$$

$$\hookrightarrow \qquad y_h(x) \;=\; c_1e^{-2x} \;+\; c_2e^{5x} \quad .$$

So the general solution to the nonhomogeneous differential equation is

$$y(x) = y_p(x) + y_h(x) = 20 + c_1 e^{-2x} + c_2 e^{5x} \quad .$$

20.5 c. For y_p : Since $g(x)$, the right side of the differential equation, is a second degree polynomial, we'll let $y(x) = y_p(x) = Ax^2 + Bx + C$. Plugging that into the differential equation, we then have

$$\begin{aligned} 18x^2 + 3x + 4 &= y'' - 6y' + 9y \\ &= \left[Ax^2 + Bx + C\right]'' - 6\left[Ax^2 + Bx + C\right]' \\ &\quad + 9\left[Ax^2 + Bx + C\right] \\ &= 2A - 6[2Ax + B] + 9\left[Ax^2 + Bx + C\right] \\ &= [9A]x^2 + [-12A + 9B]x + [2A - 6B + 9C] \quad . \end{aligned}$$

Cutting out the middle and comparing terms, we get the system

$$\begin{aligned} x^2 \text{ terms:} &\qquad 18 = 9A \\ x \text{ terms:} &\qquad 3 = -12A + 9B \\ \text{constant terms:} &\qquad 4 = 2A - 6B + 9C \end{aligned} \quad .$$

Consequently,

$$A = \frac{18}{9} = 2 \quad ,$$

$$B = \frac{3 + 12A}{9} = \frac{3 + 12 \cdot 2}{9} = 3 \quad ,$$

$$C = \frac{4 - 2A + 6B}{9} = \frac{4 - 2 \cdot 2 + 6 \cdot 3}{9} = \frac{15}{-3} = 2 \quad ,$$

and the particular solution is

$$y_p(x) = Ax^2 + Bx + C = 2x^2 + 3x + 2 \quad .$$

For y_h :

$$y'' - 6y' + 9y = 0$$

$$\hookrightarrow \qquad 0 = r^2 - 6r + 9 = (r-3)^2 \quad \rightarrowtail \quad r = 3$$

$$\hookrightarrow \qquad y_h(x) = c_1 e^{3x} + c_2 x e^{3x} \quad .$$

So the general solution to the nonhomogeneous differential equation is

$$y(x) = y_p(x) + y_h(x) = 2x^2 + 3x + 2 + c_1 e^{3x} + c_2 x e^{3x} \quad .$$

20.7 a. For y_p : Since $g(x) = 25x\cos(2x)$, we'll let

$$y(x) = y_p(x) = [Ax + B]\cos(2x) + [Cx + D]\sin(2x) \quad . \qquad (\star)$$

Computing the derivatives, we get

$$\begin{aligned} y'(x) &= [A]\cos(2x) + [Ax+B](-2\sin(2x)) \\ &\quad + [C]\sin(2x) + [Cx+D](2\cos(2x)) \\ &= [2Cx+A+2D]\cos(2x) + [-2Ax-2B+C]\sin(2x) \quad , \end{aligned}$$

and

$$\begin{aligned} y''(x) &= [2C]\cos(2x) + [2Cx+A+2D](-2\sin(2x)) \\ &\quad + [-2A]\sin(2x) + [-2Ax-2B+C](2\cos(2x)) \\ &= [-4Ax-4B+4C]\cos(2x) + [-4Cx-4A-4D]\sin(2x) \quad . \end{aligned}$$

Plugging these into the differential equation, we then have

$$\begin{aligned} 25x\cos(2x) &= y'' + 9y \\ &= \big([-4Ax-4B+4C]\cos(2x) + [-4Cx-4A-4D]\sin(2x)\big) \\ &\quad + 9\big([Ax+B]\cos(2x) + [Cx+D]\sin(2x)\big) \\ &= [5Ax+5B+4C]\cos(2x) + [5Cx-4A+5D]\sin(2x) \quad . \end{aligned}$$

Cutting out the middle and comparing terms, we get the system

$x\cos(2x)$ terms:	$25 = 5A$
$\cos(2x)$ terms:	$0 = 5B + 4C$
$x\sin(2x)$ terms:	$0 = 5C$
$\sin(2x)$ terms:	$0 = -4A + 5D$

Hence,

$$A = \frac{25}{5} = 5 \quad , \quad C = 0 \quad , \quad B = -\frac{4}{5}C = 0 \quad \text{and} \quad D = \frac{4}{5}A = 4 \quad ,$$

and particular solution (⋆) ends up being

$$y_p(x) = [5x+0]\cos(2x) + [0x+4]\sin(x) = 5x\cos(2x) + 4\sin(2x) \quad .$$

For y_h : From several of the previous problems, we already know that the general solution to the corresponding homogeneous equation $y'+9y=0$ is

$$y_h(x) = c_1\cos(3x) + c_2\sin(3x)$$

Consequently, the general solution to the given nonhomogeneous differential equation is

$$\begin{aligned} y(x) &= y_p(x) + y_h(x) \\ &= 5x\cos(2x) + 4\sin(2x) + c_1\cos(3x) + c_2\sin(3x) \quad . \end{aligned}$$

20.7 c. For y_p : Since $g(x) = 54x^2e^{3x}$, we'll let

$$y(x) = y_p(x) = \Big[Ax^2 + Bx + C\Big]e^{3x} \quad . \qquad (\star)$$

Computing the derivatives, we get

$$y'(x) = [2Ax + B]e^{3x} + \left[Ax^2 + Bx + C\right]\left(3e^{3x}\right)$$
$$= \left[3Ax^2 + (2A + 3B)x + (B + 3C)\right]e^{3x}$$

and

$$y''(x) = [6Ax + (2A + 3B)]e^{3x} + \left[3Ax^2 + (2A + 3B)x + (B + 3C)\right]\left(3e^{3x}\right)$$
$$= \left[9Ax^2 + (12A + 9B)x + (2A + 6B + 9C)\right]e^{3x} \quad .$$

Plugging these into the differential equation, we then have

$$54x^2e^{3x} = y'' + 9y$$
$$= \left[9Ax^2 + (12A + 9B)x + (2A + 6B + 9C)\right]e^{3x}$$
$$+ 9\left[Ax^2 + Bx + C\right]e^{3x}$$
$$= \left[18Ax^2 + (12A + 18B)x + (2A + 6B + 18C)\right]e^{3x} \quad .$$

Cutting out the middle and comparing terms, we get:

x^2e^{3x} terms: $\quad 54 = 18A$

xe^{3x} terms: $\quad 0 = 12A + 18B$

e^{3x} terms: $\quad 0 = 2A + 6B + 18C$

Hence,

$$A = \frac{54}{18} = 3 \quad , \quad B = -\frac{12}{18}A = -\frac{2}{3}\cdot 3 = -2 \quad ,$$
$$C = -\frac{1}{18}[2A + 6B] = -\frac{1}{18}[2\cdot 3 - 6\cdot 2] = \frac{1}{3} \quad ,$$

and particular solution ($\star$) ends up being

$$y_p(x) = \left[3x^2 - 2x + \frac{1}{3}\right]e^{3x} \quad .$$

For y_h : From several of the previous problems, we already know that the general solution to the corresponding homogeneous equation $y' + 9y = 0$ is

$$y_h(x) = c_1\cos(3x) + c_2\sin(3x)$$

Consequently, the general solution to the given nonhomogeneous differential equation is

$$y(x) = y_p(x) + y_h(x) = \left[3x^2 - 2x + \frac{1}{3}\right]e^{3x} + c_1\cos(3x) + c_2\sin(3x) \quad .$$

20.7 e. For y_p : We'll let

$$y(x) = y_p(x) = [Ax + B]\cos(2x) + [Cx + D]\sin(2x) \quad . \qquad (\star)$$

Computing the derivatives, we get

$$\begin{aligned} y'(x) &= [A]\cos(2x) + [Ax+B](-2\sin(2x)) \\ &\quad + [C]\sin(2x) + [Cx+D](2\cos(2x)) \\ &= [2Cx + A + 2D]\cos(2x) + [-2Ax - 2B + C]\sin(2x) \end{aligned}$$

and

$$\begin{aligned} y''(x) &= [2C]\cos(2x) + [2Cx + A + 2D](-2\sin(2x)) \\ &\quad + [-2A]\sin(2x) + [-2Ax - 2B + C](2\cos(2x)) \\ &= [-4Ax - 4B + 4C]\cos(2x) + [-4Cx - 4A - 4D]\sin(2x) \end{aligned}$$

Plugging that into the differential equation, we then have

$$\begin{aligned} &[-6x - 8]\cos(2x) + [8x - 11]\sin(2x) \\ &\quad = y'' - 2y' + y \\ &\quad = \big([-4Ax - 4B + 4C]\cos(2x) + [-4Cx - 4A - 4D]\sin(2x)\big) \\ &\qquad - 2\big([2Cx + A + 2D]\cos(2x) + [-2Ax - 2B + C]\sin(2x)\big) \\ &\qquad + \big([Ax + B]\cos(2x) + [Cx + D]\sin(x)\big) \\ &\quad = [(-3A + 4C)x + (-2A - 3B + 4C - 4D)]\cos(2x) \\ &\qquad + [(4A - 3C)x + (-4A + 4B - 2C - 3D)]\sin(2x) \quad . \end{aligned}$$

Cutting out the middle and comparing terms, we get the system

$$\begin{aligned} x\cos(2x) \text{ terms:} &\qquad -6 = -3A + 4C \\ \cos(2x) \text{ terms:} &\qquad -8 = -2A - 3B + 4C - 4D \\ x\sin(2x) \text{ terms:} &\qquad 8 = 4A - 3C \\ \sin(2x) \text{ terms:} &\qquad -11 = -4A + 4B - 2C - 3D \end{aligned}$$

Since there are two equations just involving A and C, we will solve that pair first:

$$-6 = -3A + 4C \qquad \text{and} \qquad 8 = 4A - 3C$$

$$\hookrightarrow \quad A = \frac{6+4C}{3} \qquad \text{and} \qquad 8 = 4\left[\frac{6+4C}{3}\right] - 3C = 8 + \frac{7C}{3}$$

$$\hookrightarrow \quad A = \frac{6+4C}{3} \qquad \text{and} \qquad C = \frac{3(8-8)}{7} = 0$$

$$\hookrightarrow \quad A = \frac{6+4\cdot 0}{3} = 2 \qquad \text{and} \qquad C = = 0 \quad .$$

With these values for A and C, the other equations in the system for the coefficients become

$$-8 = -2\cdot 2 - 3B + 4\cdot 0 - 4D \quad \rightarrowtail \quad 4 = 3B + 4D$$

and

$$-11 = -4\cdot 2 + 4B - 2\cdot 0 - 3D \quad \rightarrowtail \quad -3 = 4B - 3D \quad .$$

Solving for B and D:

$$4 = 3B + 4D \quad \text{and} \quad -3 = 4B - 3D$$

$$\hookrightarrow \quad B = \frac{4-4D}{3} \quad \text{and} \quad -3 = 4\left[\frac{4-4D}{3}\right] - 3D = \frac{16}{3} - \frac{25D}{3}$$

$$\hookrightarrow \quad B = \frac{4-4D}{3} \quad \text{and} \quad D = \frac{3}{25}\left[3 + \frac{16}{3}\right] = 1$$

$$\hookrightarrow \quad B = \frac{4 - 4\cdot 1}{3} = 0 \quad \text{and} \quad D = 1 \quad .$$

So particular solution ($\star$) becomes

$$y_p(x) = [2x+0]\cos(2x) + [0x+1]\sin(2x) = 2x\cos(2x) + \sin(2x) \quad .$$

For y_h:

$$y'' - 2y' + y = 0$$

$$\hookrightarrow \quad 0 = r^2 - 2r + 1 = (r-1)^2 \quad \rightarrowtail \quad r = 1$$

$$\hookrightarrow \quad y_h(x) = c_1e^x + c_2xe^x \quad .$$

Hence, the general solution to the given nonhomogeneous differential equation is

$$y(x) = y_p(x) + y_h(x) = 2x\cos(2x) + \sin(2x) + c_1e^x + c_2xe^x \quad .$$

20.9 a. For these problems, the "first guess" will not work. For this reason we start by finding the general solution y_h to the corresponding homogeneous differential equation:

$$y'' - 3y' - 10y = 0$$

$$\hookrightarrow \quad 0 = r^2 - 3r - 10 = (r+2)(r-5)$$

$$\hookrightarrow \quad r = -2 \quad \text{and} \quad r = 5$$

$$\hookrightarrow \quad y_h(x) = c_1e^{-2x} + c_2e^{5x} \quad .$$

For y_p: Since $g(x) = -3e^{-2x}$,

$$\text{the "first guess" for } y_p(x) \text{ is } Ae^{-2x} \quad .$$

But that is a term in y_h. So we must consider the "second guess",

$$\text{"second guess" for } y_p(x) = x \times \text{"first guess"} = x\left[Ae^{-2x}\right] = Axe^{-2x} \quad .$$

Since this is not a solution to the corresponding homogeneous equation, it is an appropriate "guess". Accordingly, we will let

$$y(x) = y_p(x) = Axe^{-2x} \quad \rightarrowtail \quad y'(x) = Ae^{-2x} - 2Axe^{-2x}$$

$$\hookrightarrow \quad y''(x) = -4Ae^{-2x} + 4xAe^{-2x} \quad .$$

Using this with the given differential equation yields

$$\begin{aligned}
-3e^{-2x} &= y'' - 3y' - 10y \\
&= \left[-4Ae^{-2x} + 4xAe^{-2x}\right] - 3\left[Ae^{-2x} - 2Axe^{-2x}\right] - 10\left[Axe^{-2x}\right] \\
&= [4+6-10]Axe^{-2x} + [-4-3]Ae^{-2x} \\
&= [0]Axe^{-2x} + [-7]Ae^{-2x} = -7Ae^{-2x} \quad .
\end{aligned}$$

Thus, $A = \frac{-3}{-7} = \frac{3}{7}$, and

$$y_p(x) = Axe^{-2x} = \frac{3}{7}xe^{-2x} \quad ,$$

with the general solution being

$$y(x) = y_p(x) + y_h(x) = \frac{3}{7}xe^{-2x} + c_1 e^{-2x} + c_2 e^{5x} \quad .$$

20.9 c. For y_h :

$$y'' + 4y' = 0$$

$$\hookrightarrow \quad 0 = r^2 + 4r = r(r+4)$$

$$\hookrightarrow \quad r = 0 \quad \text{and} \quad r = -4$$

$$\hookrightarrow \quad y_h(x) = c_1 + c_2 e^{-4x} \quad .$$

For y_p :

$$\text{“first guess”} = Ax^2 + Bx + C \quad ,$$

which we must reject since one term, the constant C , is also a solution to the corresponding homogeneous differential equation. So we consider

$$\text{“second guess”} = x\left[Ax^2 + Bx + C\right] = Ax^3 + Bx^2 + Cx \quad .$$

No terms in this can be viewed as a term in y_h . So, in finding y_p we will let

$$y(x) = y_p(x) = Ax^3 + Bx^2 + Cx$$

$$\hookrightarrow \quad y'(x) = 3Ax^2 + 2Bx + C \quad \rightarrowtail \quad y''(x) = 6Ax + 2B \quad .$$

Plugging into the differential equation,

$$\begin{aligned}
x^2 &= y'' + 4y' \\
&= [6Ax + 2B] + 4\left[3Ax^2 + 2Bx + C\right] \\
&= [12A]x^2 + [6A+8B]x + [2B+4C] \quad ,
\end{aligned}$$

leads to the system

$$\begin{aligned} x^2 \text{ terms:} &\quad 1 = 12A \\ x \text{ terms:} &\quad 0 = 6A + 8B \\ \text{constant terms:} &\quad 0 = 2B + 4C \end{aligned} \quad .$$

Hence,

$$A = \frac{1}{12} \quad , \quad B = -\frac{6}{8}A = -\frac{1}{16} \quad , \quad C = -\frac{2}{4}B = \frac{1}{32} \quad ,$$

$$y_p(x) = Ax^3 + Bx^2 + Cx = \frac{1}{12}x^3 - \frac{1}{16}x^2 + \frac{1}{32}x \quad ,$$

and the general solution is

$$y(x) = \frac{1}{12}x^3 - \frac{1}{16}x^2 + \frac{1}{32}x + c_1 + c_2e^{-4x} \quad .$$

20.9 e. For y_h :

$$y'' - 6y' + 9y = 0$$

$$\hookrightarrow \quad 0 = r^2 - 6r + 9 = (r-3)^2 \quad \rightarrowtail \quad r = 3$$

$$\hookrightarrow \quad y_h(x) = c_1e^{3x} + c_2xe^{3x} \quad .$$

For y_p :

$$\text{“First guess”} = Ae^{3x} \quad ,$$

which we reject since it is a term in y_h (with $A = c_1$).

$$\text{“Second guess”} = x\left[Ae^{3x}\right] = Axe^{3x} \quad ,$$

which we also reject since it is also a term in y_h . But

$$\text{“Third guess”} = x\left[Axe^{3x}\right] = Ax^2e^{3x}$$

is not a solution to the corresponding homogeneous equation. Thus, to find y_p we let

$$y(x) = y_p(x) = Ax^2e^{3x} \quad \rightarrowtail \quad y'(x) = 2Axe^{3x} + 3Ax^2e^{3x}$$

$$\hookrightarrow \quad y''(x) = 2Ae^{3x} + 12Axe^{3x} + 9Ax^2e^{3x} \quad .$$

Plugging into the differential equation, we get

$$\begin{aligned} 10e^{3x} &= y'' - 6y' + 9 \\ &= \left[2Ae^{3x} + 12Axe^{3x} + 9Ax^2e^{3x}\right] - 6\left[2Axe^{3x} + 3Ax^2e^{3x}\right] \\ &\quad + 9\left[Ax^2e^{3x}\right] \\ &= 2Ae^{3x} + [12-12]Axe^{3x} + [9-18+9]Ax^2e^{3x} = 2Ae^{3x} \quad . \end{aligned}$$

So $A = \frac{10}{2} = 5$,

$$y_p(x) = Ax^2e^{3x} = 5x^2e^{3x} \quad ,$$

and the general solution is

$$y(x) = 5x^2e^{3x} + c_1e^{3x} + c_2xe^{3x} \quad .$$

20.10 a. For y_h :

$$y'' - 3y' - 10y = 0$$

$$\hookrightarrow \qquad 0 = r^2 - 3r - 10 = (r+2)(r-5)$$

$$\hookrightarrow \qquad y_h(x) = c_1e^{-2x} + c_2e^{5x} \quad .$$

For y_p :

$$\text{“First guess”} = \left[Ax^2 + Bx + C\right]e^{2x} = Ax^2e^{2x} + Bxe^{2x} + Ce^{2x} \quad .$$

Since no term in this is also a term in y_h , this guess is what we will use,

$$y(x) = y_p(x) = \left[Ax^2 + Bx + C\right]e^{2x} \quad .$$

Computing the derivatives:

$$\begin{aligned} y'(x) &= [2Ax + B]e^{2x} + \left[Ax^2 + Bx + C\right]\left(2e^{2x}\right) \\ &= \left[2Ax^2 + (2A + 2B)x + (B + 2C)\right]e^{2x} \end{aligned}$$

and

$$\begin{aligned} y''(x) &= [4Ax + (2A + 2B)]e^{2x} + \left[2Ax^2 + (2A + 2B)x + (B + 2C)\right]\left(2e^{2x}\right) \\ &= \left[4Ax^2 + (8A + 4B)x + (2A + 4B + 4C)\right]e^{2x} \quad . \end{aligned}$$

Plugging into the differential equation:

$$\begin{aligned} \left[72x^2 - 1\right]e^{2x} &= y'' - 3y' - 10y \\ &= \left[\left[4Ax^2 + (8A + 4B)x + (2A + 4B + 4C)\right]e^{2x}\right] \\ &\quad - 3\left[\left[2Ax^2 + (2A + 2B)x + (B + 2C)\right]e^{2x}\right] \\ &\quad - 10\left[\left[Ax^2 + Bx + C\right]e^{2x}\right] \\ &= \left[-12Ax^2 + (2A - 12B)x + (2A + B - 12C)\right]e^{2x} \quad . \end{aligned}$$

Cutting out the middle and comparing terms, we get the system

$$\begin{aligned} x^2e^{2x} \text{ terms:} &\qquad 72 = -12A \\ xe^{2x} \text{ terms:} &\qquad 0 = 2A - 12B \\ e^{2x} \text{ terms:} &\qquad -1 = 2A + B - 12C \end{aligned}$$

Hence,

$$A = \frac{72}{-12} = -6 \quad , \quad B = \frac{2}{12}A = -1 \quad , \quad C = \frac{1}{12}[1 + 2A + B] = -1 \quad ,$$

particular solution (⋆) is

$$y_p(x) = \left[-6x^2 - x - 1\right]e^{3x} \quad ,$$

and the general solution is

$$y(x) = \left[-6x^2 - x - 1\right]e^{3x} + c_1 e^{-2x} + c_2 e^{5x} \quad .$$

20.10 c. For y_h :

$$y'' - 10y' + 25y = 0$$

$$\hookrightarrow \qquad 0 = r^2 - 10r + 25 = (r-5)^2 \quad \rightarrowtail \quad r = 5$$

$$\hookrightarrow \qquad y_h(x) = c_1 e^{5x} + c_2 x e^{5x} \quad .$$

For y_p :

$$\text{“First guess”} = Ae^{5x} \quad ,$$

which we reject since it is a term in y_h (with $A = c_1$).

$$\text{“Second guess”} = x\left[Ae^{5x}\right] = Axe^{3x} \quad ,$$

which we also reject since it is also a term in y_h . But

$$\text{“third guess”} = x\left[Axe^{5x}\right] = Ax^2e^{5x}$$

is not a solution to the corresponding homogeneous equation. Thus, to find y_p we let

$$y(x) = y_p(x) = Ax^2e^{5x} \quad \rightarrowtail \quad y'(x) = 2Axe^{5x} + 5Ax^2e^{5x}$$

$$\hookrightarrow \qquad y''(x) = 2Ae^{5x} + 20Axe^{5x} + 25Ax^2e^{5x} \quad .$$

Plugging into the differential equation,

$$\begin{aligned} 6e^{5x} &= y'' - 10y' + 25y \\ &= \left[2Ae^{5x} + 20Axe^{5x} + 25Ax^2e^{5x}\right] - 10\left[2Axe^{5x} + 5Ax^2e^{5x}\right] \\ &\quad + 25\left[Ax^2e^{5x}\right] \\ &= 2Ae^{5x} + [20-20]Axe^{5x} + [25-50-25]Ax^2e^{5x} = 2Ae^{5x} \quad . \end{aligned}$$

So $A = \frac{6}{2} = 3$,

$$y_p(x) = Ax^2e^{5x} = 3x^2e^{5x} \quad ,$$

and the general solution is

$$y(x) = 3x^2e^{5x} + c_1e^{5x} + c_2xe^{5x} \quad .$$

20.10 e. For y_h :

$$y'' + 4y' + 5y = 0$$

$$\hookrightarrow \quad r^2 + 4r + 5 = 0 \quad \rightarrowtail \quad r = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 5}}{2} = -2 \pm 1i$$

$$\hookrightarrow \quad y_h(x) = c_1 e^{-2x} \cos(x) + c_2 e^{-2x} \sin(x) \quad .$$

For y_p ,

$$\text{“First guess”} = A\cos(3x) + B\sin(3x) \quad .$$

Since no term in this is also a term in y_h , this is what we will use for $y = y_p$:

$$\begin{aligned}
24\sin(3x) &= y'' + 4y' + 5y \\
&= [A\cos(3x) + B\sin(3x)]'' + 4[A\cos(3x) + B\sin(3x)]' \\
&\quad + 5[A\cos(3x) + B\sin(3x)] \\
&= [-9A\cos(3x) - 9B\sin(3x)] + 4[-3A\sin(3x) + 3B\cos(3x)] \\
&\quad + 5[A\cos(3x) + B\sin(3x)] \\
&= [-4A + 12B]\cos(3x) + [-12A - 4B]\sin(3x) \quad .
\end{aligned}$$

Comparing terms, we get

$$\begin{aligned}
&\cos(3x) \text{ terms:} \qquad & 0 &= -4A + 12B \\
&\sin(3x) \text{ terms:} \qquad & 24 &= -12A - 4B
\end{aligned}$$

At this point, you should be able to figure out that the solution to this system is $A = -\frac{9}{5}$ and $B = -\frac{3}{5}$. Hence,

$$y_p(x) = -\frac{9}{5}\cos(3x) - \frac{3}{5}\sin(3x)$$

and the general solution is

$$y(x) = -\frac{9}{5}\cos(3x) - \frac{3}{5}\sin(3x) + c_1 e^{-2x}\cos(x) + c_2 e^{-2x}\sin(x) \quad .$$

20.10 g. For y_h :

$$y'' - 4y' + 5y = 0$$

$$\hookrightarrow \quad r^2 + 4r + 5 = 0 \quad \rightarrowtail \quad r = \frac{4 \pm \sqrt{4^2 - 4 \cdot 5}}{2} = -2 \pm 1i$$

$$\hookrightarrow \quad y_h(x) = c_1 e^{2x} \cos(x) + c_2 e^{2x} \sin(x) \quad .$$

For y_p ,

$$\text{“First guess”} = Ae^{2x}\cos(x) + Be^{2x}\sin(x) \quad ,$$

which we reject since both terms are also terms in y_h . Continuing:

$$\begin{aligned}
\text{“Second guess”} &= x\left[Ae^{2x}\cos(x) + Be^{2x}\sin(x)\right] \\
&= Axe^{2x}\cos(x) + Bxe^{2x}\sin(x) \quad .
\end{aligned}$$

Neither term in this guess is a term in y_h, so this is what we will use for $y = y_p$. For convenience in computing the derivatives, it may be best to write this as

$$y(x) = y_p(x) = xe^{2x}[A\cos(x) + B\sin(x)] \quad .$$

Computing the derivatives:

$$\begin{aligned} y'(x) &= \left(e^{2x} + 2xe^{2x}\right)[A\cos(x) + B\sin(x)] + xe^{2x}[-A\sin(x) + B\cos(x)] \\ &= e^{2x}\big[\big(A + [2A+B]x\big)\cos(x) + \big(B + [2B-A]x\big)\sin(x)\big] \end{aligned}$$

and

$$\begin{aligned} y''(x) &= 2e^{2x}\big[\big(A + [2A+B]x\big)\cos(x) + \big(B + [2B-A]x\big)\sin(x)\big] \\ &\quad + e^{2x}\big[\big(0 + [2A+B]\big)\cos(x) + \big(0 + [2B-A]\big)\sin(x)\big] \\ &\quad + e^{2x}\big[\big(A + [2A+B]x\big)(-\sin(x)) + \big(B + [2B-A]x\big)\cos(x)\big] \\ &= e^{2x}\big[\big([4A+2B] + [3A+4B]x\big)\cos(x) \\ &\qquad + \big([4B-2A] + [3B-4A]x\big)\sin(x)\big] \quad . \end{aligned}$$

Plugging into the differential equation:

$$\begin{aligned} e^{2x}\sin(x) &= y'' + 4y' + 5y \\ &= e^{2x}\big[\big([4A+2B] + [3A+4B]x\big)\cos(x) \\ &\qquad + \big([4B-2A] + [3B-4A]x\big)\sin(x)\big] \\ &\quad - 4e^{2x}\big[\big(A + [2A+B]x\big)\cos(x) + \big(B + [2B-A]x\big)\sin(x)\big] \\ &\quad + 5e^{2x}[Ax\cos(x) + Bx\sin(x)] \\ &= e^{2x}\big[\big(2B + 0x\big)\cos(x) + \big(-2A + 0x\big)\sin(x)\big] \quad . \end{aligned}$$

Comparing terms, we get

$$\begin{aligned} &e^{2x}\cos(x) \text{ terms:} && 0 = 2B \\ &e^{2x}\sin(x) \text{ terms:} && 1 = -2A \end{aligned}$$

Hence, $A = -\frac{1}{2}$, $B = 0$,

$$y_p(x) = -\frac{1}{2}xe^{2x}\cos(x) + 0xe^{2x}\sin(x) = -\frac{1}{2}xe^{2x}\cos(x)$$

and the general solution is

$$y(x) = -\frac{1}{2}xe^{2x}\cos(x) + c_1e^{2x}\cos(x) + c_2e^{2x}\sin(x) \quad .$$

20.10 i. From several problems above, we know

$$y_h(x) = c_1e^{2x}\cos(x) + c_2e^{2x}\sin(x) \quad .$$

For y_p:

$$\text{“First guess”} = A \quad .$$

Since no term in this is also a term in y_h, this is what we will use for $y = y_p$:

$$100 = y'' - 4y' + 5y = [A]'' - 4[A]' + 5A = 5A \quad .$$

So $A = \frac{100}{5} = 20$,

$$y_p(x) = 20$$

and the general solution is

$$y(x) = 20 + c_1 e^{2x}\cos(x) + c_2 e^{2x}\sin(x) \quad .$$

20.10 k. From several problems above, we know

$$y_h(x) = c_1 e^{2x}\cos(x) + c_2 e^{2x}\sin(x) \quad .$$

For y_p:

$$\text{"First guess"} = Ax^2 + Bx + C \quad .$$

Since no term in this is also a term in y_h, this is what we will use for $y = y_p$:

$$\begin{aligned} 10x^2 + 4x + 8 &= y'' - 4y' + 5y \\ &= \left[Ax^2 + Bx + C\right]'' - 4\left[Ax^2 + Bx + C\right]' \\ &\quad + 5\left[Ax^2 + Bx + C\right] \\ &= [2A] - 4[2Ax + B] + 5\left[Ax^2 + Bx + C\right] \\ &= 5Ax^2 + [-8A + 5B]x + [2A - 4B + 5C] \quad . \end{aligned}$$

Comparing terms:

$$\begin{aligned} x^2 \text{ terms:} &\qquad 10 = 5A \\ x \text{ terms:} &\qquad 4 = -8A + 5B \\ \text{constant terms:} &\qquad 8 = 2A - 4B + 5C \end{aligned}$$

So,

$$A = \frac{10}{5} = 2 \quad , \quad B = \frac{4 + 8A}{5} = 4 \quad , \quad C = \frac{8 - 2A + 4B}{5} = 4 \quad ,$$

$$y_p(x) = 2x^2 + 4x + 4 \quad ,$$

and the general solution is

$$y(x) = 2x^2 + 4x + 4 + c_1 e^{2x}\cos(x) + c_2 e^{2x}\sin(x) \quad .$$

20.10 m. For y_h:

$$y'' + y = 0$$

$$\hookrightarrow \qquad r^2 + 1 = 0 \quad \rightarrowtail \quad r = \pm\sqrt{-1} = \pm 1i$$

$$\hookrightarrow \qquad y_h(x) = c_1\cos(x) + c_2\sin(x) \quad .$$

For y_p ,

$$\text{“First guess”} \;=\; A\cos(x) \;+\; B\sin(x) \quad ,$$

which we reject since both terms are also terms in y_h . Continuing:

$$\text{“Second guess”} \;=\; x\,[A\cos(x) + B\sin(x)] \;=\; Ax\cos(x) \;+\; Bxe\sin(x) \quad .$$

Neither term in this guess is a term in y_h , so this is what we will use for y_p . Computing the derivatives:

$$y(x) \;=\; y_p(x) \;=\; x[A\cos(x) \;+\; B\sin(x)]$$

$$\hookrightarrow \qquad y'(x) \;=\; [A\cos(x) \;+\; B\sin(x)] \;+\; x[-A\sin(x) \;+\; B\cos(x)]$$

$$\hookrightarrow \quad y''(x) \;=\; 2[-A\sin(x) \;+\; B\cos(x)] \;-\; x[A\cos(x) \;+\; B\sin(x)] \quad .$$

Plugging into the differential equation:

$$\begin{aligned} 6\cos(x) \;-\; 3\sin(x) \;&=\; y'' \;+\; y \\ &=\; 2[-A\sin(x) \;+\; B\cos(x)] \;-\; x[A\cos(x) \;+\; B\sin(x)] \\ &\qquad +\; x[A\cos(x) \;+\; B\sin(x)] \\ &=\; 2B\cos(x) \;-\; 2A\sin(x) \quad . \end{aligned}$$

Comparing terms, we get

$$\begin{aligned} &\cos(x)\text{ terms:} && 6 \;=\; 2B \\ &\sin(x)\text{ terms:} && -3 \;=\; -2A \quad . \end{aligned}$$

Hence, $A = \frac{-3}{-2} = \frac{3}{2}$, $B = \frac{6}{2} = 3$,

$$y_p(x) \;=\; \frac{3}{2}x\cos(x) \;+\; 3x\sin(x) \quad ,$$

and the general solution is

$$y(x) \;=\; \frac{3}{2}x\cos(x) \;+\; 3x\sin(x) \;+\; c_1\cos(x) \;+\; c_2\sin(x) \quad .$$

20.11 a. From several problems above, we know

$$y_h(x) \;=\; c_1e^{2x}\cos(x) \;+\; c_2e^{2x}\sin(x) \quad .$$

For y_p :

$$\begin{aligned} \text{“First guess”} \;=\; &\left[A_0x^3 + A_1x^2 + A_2x + A_3\right]e^{-x}\sin(x) \\ &+\; \left[B_0x^3 + B_1x^2 + B_2x + B_3\right]e^{-x}\cos(x) \quad . \end{aligned}$$

Since no term in this is also a term in y_h , this guess is what we use for y_p .

20.11 c. For y_h :

$$y'' - 5y' + 6y = 0$$

$$\hookrightarrow \qquad 0 = r^2 - 5r + 6 = (r-2)(r-3)$$

$$\hookrightarrow \qquad y_h(x) = c_1 e^{2x} + c_2 e^{3x} \quad .$$

For y_p :

$$\text{“First guess”} = \left[Ax^2 + Bx + C\right]e^{-7x} = Ax^2e^{-7x} + Bxe^{-7x} + Ce^{-7x} \quad .$$

Since no term in this is also a term in y_h , this guess is what we use for y_p .

20.11 e. From previous problems, we know

$$y_h(x) = c_1 e^{2x} + c_2 e^{3x} \quad .$$

For y_p :

$$\text{“First guess”} = Ae^{-8x} \quad .$$

Since no term in this is also a term in y_h , this guess is what we use for y_p .

20.11 g. From previous problems, we know

$$y_h(x) = c_1 e^{2x} + c_2 e^{3x} \quad .$$

For y_p :

$$\text{“First guess”} = \left[Ax^2 + Bx + C\right]e^{3x} = Ax^2e^{3x} + Bxe^{3x} + Ce^{3x} \quad .$$

which we reject because the last term is also a term in y_h . Continuing,

$$\text{“Second guess”} = x\left[Ax^2 + Bx + C\right]e^{3x} = Ax^3e^{3x} + Bx^2e^{3x} + Cxe^{3x} \quad .$$

Since no term in this is also a term in y_h , this guess is what we use for y_p .

20.11 i. From previous problems, we know

$$y_h(x) = c_1 e^{2x} + c_2 e^{3x} \quad .$$

For y_p :

$$\text{“First guess”} = \left[Ax^2 + Bx + C\right]e^{3x}\cos(2x) + \left[Dx^2 + Ex + F\right]e^{3x}\sin(2x) \quad .$$

Since no term in this is also a term in y_h , this guess is what we use for y_p .

20.11 k. For y_h :

$$y'' - 4y' + 20y = 0$$

$$\hookrightarrow \quad r^2 - 4r + 20 = 0 \quad \rightarrowtail \quad r = \frac{4 \pm \sqrt{(-4)^2 - 4 \cdot 20}}{2} = 2 \pm 4i$$

$$\hookrightarrow \quad y_h(x) = c_1 e^{2x} \cos(4x) + c_2 e^{2x} \sin(4x) \quad .$$

For y_p :

$$\text{“First guess”} = Ae^{2x} \cos(4x) + Be^{2x} \sin(4x) \quad ,$$

which we reject because each term is also a term in y_h . Continuing,

$$\begin{aligned} \text{“Second guess”} &= x \left[Ae^{2x} \cos(4x) + Be^{2x} \sin(4x) \right] \\ &= Axe^{2x} \cos(4x) + Bxe^{2x} \sin(4x) \quad . \end{aligned}$$

Since no term in this is also a term in y_h , this guess is what we use for y_p .

20.11 m. For y_h :

$$y'' - 10y' + 25y = 0$$

$$\hookrightarrow \quad 0 = r^2 - 10r + 25 = (r-5)^2 \quad \rightarrowtail \quad r = 5$$

$$\hookrightarrow \quad y_h(x) = c_1 e^{5x} + c_2 x e^{5x} \quad .$$

For y_p :

$$\text{“First guess”} = \left[Ax^2 + Bx + C \right] e^{5x} = Ax^2 e^{5x} + Bxe^{5x} + Ce^{5x} \quad ,$$

which we reject because two terms also terms in y_h . Continuing,

$$\begin{aligned} \text{“Second guess”} &= x \left[Ax^2 + Bx + C \right] e^{5x} \\ &= \left[Ax^3 + Bx^2 + Cx \right] e^{5x} = Ax^3 e^{5x} + Bx^2 e^{5x} + Cxe^{5x} \quad , \end{aligned}$$

which we also reject because one term is also a term in y_h . Continuing,

$$\begin{aligned} \text{“Third guess”} &= x^2 \left[Ax^2 + Bx + C \right] e^{5x} \\ &= \left[Ax^4 + Bx^3 + Cx^2 \right] e^{5x} = Ax^4 e^{5x} + Bx^3 e^{5x} + Cx^2 e^{5x} \quad , \end{aligned}$$

Since no term in this is also a term in y_h , this guess is what we use for y_p .

20.12 a. For y_p :

$$\text{“First guess”} = Ae^{-2x} \quad .$$

Since no term in this is also a term in y_h , this guess is what we will use for $y = y_p$. Plugging it into the differential equation, we obtain

$$12e^{-2x} = y^{(4)} - 4y^{(3)} = \left[Ae^{-2x}\right]^{(4)} - 4\left[Ae^{-2x}\right]^{(3)}$$
$$= (-2)^4 Ae^{-2x} - 4(-2)^3 Ae^{-2x} = 48Ae^{-2x} \quad .$$

So $A = \frac{12}{48} = \frac{1}{4}$, and $y_p(x) = \frac{1}{4}e^{-2x}$.

20.12 c. For y_p :

$$\text{“First guess”} = Ae^{4x} \quad .$$

which we reject because it is a term in y_h . Continuing,

$$\text{“Second guess”} = x\left[Ae^{4x}\right] = Axe^{4x} \quad .$$

Since no term in this is also a term in y_h , this guess is what we will use for $y = y_p$. Doing so and computing the necessary derivatives:

$$y(x) = y_p(x) = Axe^{4x} \rightarrowtail y'(x) = Ae^{4x} + 4Axe^{4x}$$
$$\hookrightarrow y''(x) = 8Ae^{4x} + 16Axe^{4x} \rightarrowtail y^{(3)}(x) = 48Ae^{4x} + 64Axe^{4x}$$
$$\hookrightarrow y^{(4)}(x) = 256Ae^{4x} + 256Axe^{4x} \quad .$$

Plugging these into the differential equation, we obtain

$$32e^{4x} = y^{(4)} - 4y^{(3)}$$
$$= \left[256Ae^{4x} + 256Axe^{4x}\right] - 4\left[48Ae^{4x} + 64Axe^{4x}\right]$$
$$= 64Ae^{4x} + 0Axe^{4x} = 64Ae^{4x} \quad .$$

So $A = \frac{32}{64} = \frac{1}{2}$, and $y_p(x) = \frac{1}{2}xe^{4x}$.

20.12 e. For y_p :

$$\text{“First guess”} = Ax^2 + Bx + C \quad .$$

Since no term in this is also a term in y_h , this guess is what we will use for $y = y_p$. Plugging it into the differential equation, we obtain

$$x^2 = y^{(3)} - y'' + y' - y$$
$$= \left[Ax^2 + Bx + C\right]^{(3)} - \left[Ax^2 + Bx + C\right]''$$
$$+ \left[Ax^2 + Bx + C\right]' - \left[Ax^2 + Bx + C\right]$$
$$= [0] - [2A] + [2Ax + B] + \left[-Ax^2 - Bx - C\right]$$
$$= -Ax^2 + [2A - B]x + [2A + B - C] \quad .$$

Comparing terms, we get

$$\begin{aligned} x^2 \text{ terms:} \quad & 1 = -A \\ x \text{ terms:} \quad & 0 = 2A - B \\ \text{constant terms:} \quad & 0 = 2A + B - C \end{aligned} \quad .$$

Hence, $A = -1$, $B = -2A = 2$, $C = 2A + B = 0$, and $y_p(x) = -x^2 - 2x$.

20.12 g. For y_p :

$$\text{“First guess”} = Ae^x \quad ,$$

which we reject because it is a term in y_h . Continuing,

$$\text{“Second guess”} = x\left[Ae^x\right] = Axe^x \quad ,$$

Since no term in this is also a term in y_h , this guess is what we will use for $y = y_p$. Doing so and computing the necessary derivatives:

$$y(x) = y_p(x) = Axe^x \rightarrowtail y'(x) = Ae^x + Axe^x$$

$$\hookrightarrow \quad y''(x) = 2Ae^x + Axe^x \rightarrowtail y^{(3)}(x) = 3Ae^x + Axe^x \quad .$$

Plugging these into the differential equation, we obtain

$$\begin{aligned} 6e^x &= y^{(3)} - y'' + y' - y \\ &= \left[3Ae^x + Axe^x\right] - \left[2Ae^x + Axe^x\right] + \left[Ae^x + Axe^x\right] - \left[Axe^x\right] \\ &= 2Ae^x + 0Axe^x = 2Ae^x \quad . \end{aligned}$$

So $A = \frac{6}{2} = 3$, and $y_p(x) = 3xe^x$.

20.13 a. For y_p :

$$\text{“First guess”} = \left[Ax^2 + Bx + C\right]e^{3x} = Ax^2e^{3x} + Bxe^{3x} + Ce^{3x} \quad .$$

Since no term in this is also a term in y_h , this guess is what we use for y_p .

20.13 c. For y_p :

$$\begin{aligned} \text{“First guess”} = &\left[Ax^2 + Bx + C\right]e^{3x}\cos(3x) \\ &+ \left[Dx^2 + Ex + F\right]e^{3x}\sin(3x) \quad . \end{aligned}$$

Since no term in this is also a term in y_h , this guess is what we use for y_p .

20.13 e. For y_p :

$$\text{“First guess”} \; = \; [Ax + B]\cos(x) \; + \; [Cx + D]\sin(x)$$
$$= \; Ax\cos(x) \; + \; B\cos(x) \; + \; Cx\sin(x) \; + \; D\sin(x) \quad ,$$

which we reject because it has two terms that are also in y_h . Continuing,

$$\text{“Second guess”} \; = \; x\big([Ax + B]\cos(x) \; + \; [Cx + D]\sin(x)\big)$$
$$= \; \left[Ax^2 + Bx\right]\cos(x) \; + \; [Cx^2 + Dx]\sin(x)$$
$$= \; Ax^2\cos(x) \; + \; Bx\cos(x) \; + \; Cx^2\sin(x) \; + \; Dx\sin(x) \quad .$$

Since no term in this is also a term in y_h , this guess is what we use for y_p .

20.13 g. For y_p :

$$\text{“First guess”} \; = \; \left[Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F\right]e^{2x}$$
$$= \; Ax^5e^{2x} \; + \; Bx^4e^{2x} \; + \; Cx^3e^{2x} \; + \; Dx^2e^{2x} \; + \; Exe^{2x} \; + \; Fe^{2x} \quad .$$

Since no term in this is also a term in y_h , this guess is what we use for y_p .

20.14 a. From the answer to exercise 20.1 b, we know that one solution to

$$y'' \; - \; 6y' \; + \; 9y \; = \; 27e^{6x} \; = \; g_1(x)$$

is

$$y_{p1} \; = \; 3e^{6x} \quad ,$$

and from the answer to exercise 20.3 b, we know that one solution to

$$y'' \; - \; 6y' \; + \; 9y \; = \; 25\sin(6x) \; = \; g_2(x)$$

is

$$y_{p2}(x) \; = \; \frac{4}{9}\cos(6x) \; - \; \frac{1}{3}\sin(6x) \quad .$$

So, by superposition, it follows that one solution to

$$y'' \; - \; 6y' \; + \; 9y \; = \; 27e^{6x} \; + \; 25\sin(6x) \; = \; g_1(x) \; + \; g_2(x)$$

is

$$y_p(x) \; = \; y_{p1}(x) \; + \; y_{p2}(x) \; = \; 3e^{6x} \; + \; \frac{4}{9}\cos(6x) \; - \; \frac{1}{3}\sin(6x) \quad .$$

20.14 c. Using the identity

$$\sin^2(x) \; = \; \frac{1}{2} \; - \; \frac{1}{2}\cos(2x)$$

we can rewrite the differential equation as

$$y'' \; - \; 4y' \; + \; 5y \; = \; g_1(x) \; + \; g_2(x)$$

where

$$g_1(x) \; = \; \frac{1}{2} \qquad \text{and} \qquad g_2(x) \; = \; -\frac{1}{2}\cos(2x) \quad .$$

Our solution is $y_p = y_{p1} + y_{p2}$ where y_{p1} and y_{p2} are, respectively, particular solutions to

$$y'' - 4y' + 5y = g_1(x) \qquad \text{and} \qquad y'' - 4y' + 5y = g_2(x) \quad ,$$

which we will find by the method of guess.

First, we must find (again) y_h :

$$y'' - 4y' + 5y = 0$$

$$\hookrightarrow \quad r^2 - 4r + 5 = 0 \quad \rightarrowtail \quad r = \frac{4 \pm \sqrt{(-4)^2 - 4 \cdot 5}}{2} = 2 \pm 1i$$

$$\hookrightarrow \qquad y_h(x) = c_1 e^{2x} \cos(x) + c_2 e^{2x} \sin(x) \quad .$$

For y_{p1} : Since g_1 is a constant,

$$\text{“First guess”} = A \quad .$$

Since this is not a term in y_h , this guess is what we will use for $y = y_{p1}$. Plugging it into the differential equation:

$$\frac{5}{2} = g_1(x) = y'' - 4y' + 5y = [A]'' - [A]' + 5A = 5A \quad .$$

So $A = \frac{5}{5 \cdot 2} = \frac{1}{2}$, and

$$y_{p1}(x) = \frac{1}{2} \quad .$$

For y_{p2} : Since $g_2(x) = -\frac{1}{2}\cos(2x)$,

$$\text{“First guess”} = A\cos(2x) + B\sin(2x) \quad .$$

Since no term is also a term in y_h , this guess is what we will use for $y = y_{p2}$. Plugging it into the differential equation:

$$\begin{aligned}
-\frac{1}{2}\cos(2x) &= g_2(x) \\
&= y'' - 4y' + 5y \\
&= [A\cos(2x) + B\sin(2x)]'' - 4[A\cos(2x) + B\sin(2x)A]' \\
&\quad + 5[A\cos(2x) + B\sin(2x)] \\
&= [-4A\cos(2x) - 4B\sin(2x)] - 4[-2A\sin(2x) + 2B\cos(2x)A] \\
&\quad + 5[A\cos(2x) + B\sin(2x)] \\
&= [A - 8B]\cos(2x) + [8A + B]\sin(2x) \quad . \quad .
\end{aligned}$$

Comparing terms, we get

$$\begin{aligned}
\cos(x) \text{ terms:} \qquad & -\frac{1}{2} = A - 8B \\
\sin(x) \text{ terms:} \qquad & 0 = 8A + B
\end{aligned}$$

Solving this yields $A = -\frac{1}{26}$ and $B = \frac{4}{13}$. Hence,

$$y_{p2}(x) = -\frac{1}{26}\cos(2x) + \frac{4}{13}\sin(2x) \quad .$$

Finally, combining the above yields

$$y_p(x) \;=\; y_{p1} \;+\; y_{p2} \;=\; \frac{1}{2} \;-\; \frac{1}{26}\cos(2x) \;+\; \frac{4}{13}\sin(2x) \quad .$$