Chapter 2: Integration and Differential Equations

2.2 a. Since the equation is in the form ${}^{dy}\!/_{dx} = f(x)$, it is directly integrable.

2.2 c. Algebraically solving the equation for the highest derivative gives

$$\frac{dy}{dx} = e^{2x} - 4y \quad .$$

Since the righthand side involves y , it is not a formula of x only. Hence the differential equation is not directly integrable.

2.2 e. Algebraically solving the equation for the highest derivative gives

$$\frac{dy}{dx} = 2\frac{x}{y} \quad .$$

The righthand side involves y , and is not a formula of x only. So the differential equation is not directly integrable.

2.2 g. Algebraically solving the equation for the highest derivative gives

$$\frac{d^2y}{dx^2} = \frac{1}{x^2} \quad .$$

Since the righthand side is a formula of x only (no y's), the differential equation is not directly integrable.

2.2 i. Algebraically solving the equation for the highest derivative gives

$$\frac{d^2y}{dx^2} = e^{-x^2} - 3\frac{dy}{dx} - 8y \quad .$$

Since the righthand side is not a formula of x only, the differential equation is not directly integrable.

2.3 a. $y(x) = \int \frac{dy}{dx}\,dx = \int 4x^3\,dx = x^4 + c \quad .$

2.3 c. First, we must solve for the derivative,

$$x\frac{dy}{dx} + \sqrt{x} = 2 \quad \rightarrowtail \quad x\frac{dy}{dx} = 2 - \sqrt{x}$$

$$\hookrightarrow \quad \frac{dy}{dx} = \frac{2}{x} - \frac{\sqrt{x}}{x} = 2\frac{1}{x} - x^{-1/2} \quad .$$

Integrating this gives the solution,

$$y(x) = \int \frac{dy}{dx}\,dx = \int \left[2\frac{1}{x} - x^{-1/2}\right] dx = 2\ln|x| + 2x^{1/2} + c \quad .$$

2.3 e. Using the substitution $u = x^2$ (hence $du = 2x\,dx$),

$$y(x) = \int x \cos\left(x^2\right) dx = \frac{1}{2}\int \cos(u)\, du$$
$$= \frac{1}{2}\sin(u) + c = \frac{1}{2}\sin\left(x^2\right) + c \quad .$$

2.3 g. Dividing through by $x^2 - 9$ yields

$$\frac{dy}{dx} = \frac{x}{x^2 - 9} \quad .$$

This can be integrated using the substitution $u = x^2 - 9$:

$$y(x) = \int \frac{x}{x^2-9}\, dx = \frac{1}{2}\int \frac{1}{x^2-9} 2x\, dx$$
$$= \frac{1}{2}\int \frac{1}{u}\, du = \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln\left|x^2 - 9\right| + c \quad .$$

2.3 i.
$$1 = x^2 - 9\frac{dy}{dx} \quad \rightarrowtail \quad 9\frac{dy}{dx} = x^2 - 1 \quad \rightarrowtail \quad \frac{dy}{dx} = \frac{1}{9}x^2 - \frac{1}{9}$$

$$\hookrightarrow \quad y(x) = \int \left[\frac{1}{9}x^2 - \frac{1}{9}\right] dx = \frac{1}{27}x^3 - \frac{1}{9}x + c \quad .$$

2.3 k.
$$\frac{d^2y}{dx^2} - 3 = x \quad \rightarrowtail \quad \frac{d^2y}{dx^2} = x + 3$$

$$\hookrightarrow \quad \frac{dy}{dx} = \int \frac{d^2y}{dx^2}\, dx = \int [x+3]\, dx = \frac{1}{2}x^2 + 3x + c_1$$

$$\hookrightarrow \quad y(x) = \int \left[\frac{1}{2}x^2 + 3x + c_1\right] dx = \frac{1}{6}x^2 + \frac{3}{2}x^2 + c_1 x + c_2 \quad .$$

2.4 a. We first find the general solution to the differential equation:

$$y(x) = \int \frac{dy}{dx}\, dx = \int \left[4x + 10e^{2x}\right] dx = 2x^2 + 5e^{2x} + c \quad .$$

Then use the initial condition to determine the value of c:

$$4 = y(0) = 2 \cdot 0^2 + 5e^{2\cdot 0} + c = 0 + 5 + c \quad \rightarrowtail \quad c = 4 - 5 = -1 \quad .$$

So the solution to the initial-value problem is given by $y(x) = 2x^2 + 5e^{2x} + c$ with $c = -1$; that is, $y(x) = 2x^2 + 5e^{2x} - 1$. Moreover, this solution is valid for all values of x since all functions in the differential equation and solution are continuous on $(-\infty, \infty)$.

2.4 c. Finding the general solution to the differential equation:

$$y(x) = \int \frac{x-1}{x+1}\,dx = \int \frac{x+1-2}{x+1}\,dx$$
$$= \int \Bigg[\underbrace{\frac{x+1}{x+1}}_{=1} - 2\frac{1}{x+1}\Bigg]\,dx = x - 2\ln|x+1| + c \quad .$$

Applying the initial condition to find c :

$$8 = y(0) = 8 - 2\ln|0+1| + c = 8 - 2\cdot 0 + c \quad \rightarrowtail \quad c = 8 \quad .$$

So, $y(x) = x - 2\ln|x+1| + c$ with $c = 8$; that is, $y(x) = x - 2\ln|x| + 8$. And since the derivative in the differential equation "blows up" at $x = -1$ and the initial condition is given at $x = 0 > -1$, the solution is only valid for all values of x greater than -1 .

2.4 e. Solving for the derivative yields the initial-value problem

$$\frac{dy}{dx} = \frac{\sin(x)}{\cos(x)} \qquad \text{with} \quad y(0) = 3 \quad .$$

The largest interval containing $x = 0$ on which this derivative does not "blow up" is $(-\pi/2, \pi/2)$ (since $(-\pi/2, \pi/2)$ is the largest interval containing $x = 0$ on which $\cos(x)$ is nonzero. So our solution will only be valid on $(-\pi/2, \pi/2)$. Integrating the above:

$$y(x) = \int \frac{\sin(x)}{\cos(x)}\,dx$$
$$= -\int \frac{1}{\cos(x)}\frac{d}{dx}[\cos(x)]\,dx = -\ln|\cos(x)| + c \quad .$$

Then applying the initial condition, and writing down the final result:

$$3 = y(0) = -\ln|\cos(0)| + c = -\ln(1) + c = 0 + c$$

$$\hookrightarrow \qquad c = 3 \quad \rightarrowtail \quad y(x) = -\ln|\cos(x)| + 3 \quad .$$

2.4 g. Solving for the highest derivative:

$$x\frac{d^2y}{dx^2} + 2 = \sqrt{x} \quad \rightarrowtail \quad \frac{d^2y}{dx^2} = \frac{1}{x}\left[\sqrt{x} - 2\right] = x^{-1/2} - \frac{2}{x} \quad .$$

Clearly, the righthand side requires $x > 0$. Integrating and applying the second initial condition:

$$\frac{dy}{dx} = \int \left[x^{-1/2} - \frac{2}{x}\right] dx = 2x^{1/2} - 2\ln|x| + c_1 \quad .$$

$$\hookrightarrow \qquad 6 = y'(1) = 2\cdot 1^{1/2} - 2\ln|1| + c_1 = 2 - 2\cdot 0 + c_1 \quad .$$

$$\hookrightarrow \quad c_1 = 6 - 2 = 4 \qquad \text{and, thus} \qquad \frac{dy}{dx} = 2x^{1/2} - 2\ln|x| + 4 \quad .$$

Integrating this last equation (possibly using integration by parts to compute the integral of $\ln|x|$):

$$
\begin{aligned}
y(x) &= \int \left[2x^{1/2} - 2\ln|x| + 4\right] dx \\
&= \tfrac{4}{3}x^{3/2} \;-\; 2\left[x\ln|x| - x\right] \;+\; 4x \;+\; c_2 \\
&= \tfrac{4}{3}x^{3/2} \;-\; 2x\ln|x| \;+\; 6x \;+\; c_2 \quad .
\end{aligned}
$$

Then, applying the first initial condition:

$$8 \;=\; y(1) \;=\; \tfrac{4}{3}\cdot 1^{3/2} \;-\; 2\cdot 1\ln|1| \;+\; 6\cdot 1 \;+\; c_2 \;=\; \tfrac{4}{3} \;+\; 6 \;+\; c_2$$

$$\hookrightarrow \quad c_2 \;=\; \tfrac{2}{3} \qquad \text{and, thus} \qquad y(x) \;=\; \tfrac{4}{3}x^{3/2} \;-\; 2x\ln|x| \;+\; 6x \;+\; \tfrac{2}{3} \quad .$$

2.5 a.

$$
\begin{aligned}
y(x) \;-\; y(0) &= \int_0^x \frac{dy}{dx}\,ds \;=\; \int_0^x \sin\left(\tfrac{s}{2}\right) ds \;=\; -2\cos\left(\tfrac{s}{2}\right)\Big|_0^x \\
&= -2\cos\left(\tfrac{x}{2}\right) \;+\; 2\cos\left(\tfrac{0}{2}\right) \;=\; -2\cos\left(\tfrac{x}{2}\right) \;+\; 2
\end{aligned}
$$

$$\hookrightarrow \qquad y(x) \;=\; -2\cos\left(\tfrac{x}{2}\right) \;+\; 2 \;+\; y(0) \quad .$$

2.5 b i. Plugging the initial value into the above formula:

$$
\begin{aligned}
y(x) &= -2\cos\left(\tfrac{x}{2}\right) \;+\; 2 \;+\; y(0) \\
&= -2\cos\left(\tfrac{x}{2}\right) \;+\; 2 \;+\; 0 \;=\; -2\cos\left(\tfrac{x}{2}\right) \;+\; 2 \quad .
\end{aligned}
$$

So $y(\pi) \;=\; -2\cos\left(\tfrac{\pi}{2}\right) \;+\; 2 \;=\; -2\cdot 0 \;+\; 2 \;=\; 2 \quad .$

2.5 b ii.

$$
\begin{aligned}
y(x) &= -2\cos\left(\tfrac{x}{2}\right) \;+\; 2 \;+\; y(0) \\
&= -2\cos\left(\tfrac{x}{2}\right) \;+\; 2 \;+\; 3 \;=\; -2\cos\left(\tfrac{x}{2}\right) \;+\; 5 \quad .
\end{aligned}
$$

So $y(\pi) \;=\; -2\cos\left(\tfrac{\pi}{2}\right) \;+\; 5 \;=\; -2\cdot 0 \;+\; 5 \;=\; 5 \quad .$

2.5 b iii.

$$
\begin{aligned}
y(2\pi) &= -2\cos\left(\tfrac{2\pi}{2}\right) \;+\; 2 \;+\; y(0) \\
&= -2\cos(\pi) \;+\; 2 \;+\; 3 \;=\; -2(-1) \;+\; 5 \;=\; 7 \quad .
\end{aligned}
$$

2.7 a.

$$y(x) \;-\; y(0) \;=\; \int_{-}^{x} \frac{dy}{ds}\,ds \quad \rightarrowtail \quad y(x) \;-\; 3 \;=\; \int_0^x s\,e^{-s^2}\,ds$$

$$\hookrightarrow \quad y(x) \;=\; -\tfrac{1}{2}e^{s^2}\Big|_0^x \;+\; 3 \;=\; -\tfrac{1}{2}e^{-x^2} \;+\; \tfrac{1}{2}e^0 \;+\; 3 \;=\; -\tfrac{1}{2}e^{-x^2} \;+\; \tfrac{7}{2} \quad .$$

2.7 c.
$$y(x) - y(1) = \int_1^x \frac{dy}{ds}\,ds \quad\rightarrowtail\quad y(x) - 0 = \int_1^x \frac{1}{s^2+1}\,ds$$
$$\hookrightarrow\quad y(x) = \arctan(s)\Big|_1^x = \arctan(x) - \arctan(1) = \arctan(x) - \frac{\pi}{2} \quad .$$

2.7 e.
$$x\frac{dy}{dx} = \sin(x) \quad\rightarrowtail\quad \frac{dy}{dx} = \frac{\sin(x)}{x}$$
$$\hookrightarrow\quad y(x) - 4 = \int_0^x \frac{\sin(s)}{s}\,dx = \mathrm{Si}(x) \quad\rightarrowtail\quad y(x) = \mathrm{Si}(x) + 4 \quad .$$

2.9 a. For the graph of $\mathrm{step}(x)$, see the page 29 of the text.
Since $y(0) = 0$, $\quad y(x) = \int_0^x \mathrm{step}(s)\,ds + y(0) = \int_0^x \mathrm{step}(s)\,ds \quad .$
If $x < 0$ and $x < s < 0$, then $\mathrm{step}(s) = 0$. Thus,
$$y(x) = \int_0^x \mathrm{step}(s)\,ds = \int_0^x 0\,ds = 0 \qquad \text{if} \quad x < 0 \quad .$$
If $0 \le x$ and $0 \le s \le x$, then $\mathrm{step}(s) = 1$. Thus,
$$y(x) = \int_0^x \mathrm{step}(s)\,ds = \int_0^x 1\,ds = x \qquad \text{if} \quad 0 \ge x \quad .$$
In summary,
$$y(x) = \left\{\begin{matrix} 0 & \text{if} & x < 0 \\ x & \text{if} & 0 < x \end{matrix}\right\} = \mathrm{ramp}(x) \quad .$$

2.9 c. Since $y(0) = 0$, $\quad y(x) = \int_0^x f(s)\,ds + y(0) = \int_0^x f(s)\,ds \quad .$
If $x < 1$ and $x < s < 1$, then $f(s) = 0$. Thus,
$$y(x) = \int_0^x \mathrm{step}(s)\,ds = \int_0^x 0\,ds = 0 \qquad \text{if} \quad x < 1 \quad .$$
If $1 \le x < 2$, then
$$y(x) = \int_0^x f(s)\,ds = \int_0^1 \underbrace{f(s)}_{=0}\,ds + \int_1^x \underbrace{f(s)}_{=1}\,ds$$
$$= \int_0^1 0\,ds + \int_1^x 1\,ds = 0 + x - 1 \quad .$$
If $2 \le x$, then
$$\begin{aligned} y(x) &= \int_0^x f(s)\,ds \\ &= \int_0^1 \underbrace{f(s)}_{=0}\,ds + \int_1^2 \underbrace{f(s)}_{=1}\,ds + \int_2^x \underbrace{f(s)}_{=0}\,ds \\ &= \int_0^1 0\,ds + \int_1^2 1\,ds + \int_2^x 0\,ds = 0 + 1 + 0 \quad . \end{aligned}$$

So,

$$y(x) = \begin{cases} 0 & \text{if} \quad x < 1 \\ x - 1 & \text{if} \quad 1 \leq x \leq 2 \\ 1 & \text{if} \quad 2 < x \end{cases} \quad .$$

2.9 e. Since $y(0) = 0$, $y(x) = \int_0^x \text{stair}(s)\,ds + y(0) = \int_0^x \text{stair}(s)\,ds$.

If $x < 0$, then $y(x) = \int_0^x \underbrace{\text{stair}(s)}_{=0}\,ds = \int_0^x 0\,ds = 0$.

If $0 \leq x < 1$, then $y(x) = \int_0^x \underbrace{\text{stair}(s)}_{=1}\,ds = \int_0^x 1\,ds = x$.

If $1 \leq x < 2$, then

$$\begin{aligned} y(x) &= \int_0^1 \underbrace{\text{stair}(s)}_{=1}\,ds + \int_1^x \underbrace{\text{stair}(s)}_{=2}\,ds \\ &= \int_0^1 1\,ds + \int_1^x 2\,ds = 1 + (2x - 2) = 2x - 1 = 2\left(x - \tfrac{1}{2}\right) \quad . \end{aligned}$$

If $2 \leq x < 3$, then

$$\begin{aligned} y(x) &= \int_0^1 1\,ds + \int_1^2 2\,ds + \int_2^x 3\,ds \\ &= 1 + (4 - 2) + (3x - 6) = 3x - 3 = 3(x - 1) \quad . \end{aligned}$$

If $3 \leq x < 4$, then

$$\begin{aligned} y(x) &= \int_0^1 1\,ds + \int_1^2 2\,ds + \int_2^3 3\,ds + \int_3^x 4\,ds \\ &= 1 + 2 + 3 + (4x - 12) = 4x - 6 = 4\left(x - \tfrac{3}{2}\right) \quad . \end{aligned}$$

So,

$$y(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ x & \text{if} \quad 0 \leq x < 1 \\ 2\left(x - {}^1\!/_2\right) & \text{if} \quad 1 \leq x < 2 \\ 3\left(x - {}^2\!/_2\right) & \text{if} \quad 2 \leq x < 3 \\ 4\left(x - {}^3\!/_2\right) & \text{if} \quad 3 \leq x < 4 \end{cases} \quad .$$