

**Chapter 2: Integration and Differential Equations**

**2.2 a.** Since the equation is in the form  $\frac{dy}{dx} = f(x)$ , it is directly integrable.

**2.2 c.** Algebraically solving the equation for the highest derivative gives

$$\frac{dy}{dx} = e^{2x} - 4y .$$

Since the righthand side involves  $y$ , it is not a formula of  $x$  only. Hence the differential equation is not directly integrable.

**2.2 e.** Algebraically solving the equation for the highest derivative gives

$$\frac{dy}{dx} = 2\frac{x}{y} .$$

The righthand side involves  $y$ , and is not a formula of  $x$  only. So the differential equation is not directly integrable.

**2.2 g.** Algebraically solving the equation for the highest derivative gives

$$\frac{d^2y}{dx^2} = \frac{1}{x^2} .$$

Since the righthand side is a formula of  $x$  only (no  $y$ 's), the differential equation is not directly integrable.

**2.2 i.** Algebraically solving the equation for the highest derivative gives

$$\frac{d^2y}{dx^2} = e^{-x^2} - 3\frac{dy}{dx} - 8y .$$

Since the righthand side is not a formula of  $x$  only, the differential equation is not directly integrable.

**2.3 a.**  $y(x) = \int \frac{dy}{dx} dx = \int 4x^3 dx = x^4 + c .$

**2.3 c.** First, we must solve for the derivative,

$$\begin{aligned} x\frac{dy}{dx} + \sqrt{x} &= 2 \quad \rightsquigarrow \quad x\frac{dy}{dx} = 2 - \sqrt{x} \\ \Leftrightarrow \quad \frac{dy}{dx} &= \frac{2}{x} - \frac{\sqrt{x}}{x} = 2\frac{1}{x} - x^{-1/2} . \end{aligned}$$

Integrating this gives the solution,

$$y(x) = \int \frac{dy}{dx} dx = \int \left[ 2\frac{1}{x} - x^{-1/2} \right] dx = 2\ln|x| + 2x^{1/2} + c .$$

**2.3 e.** Using the substitution  $u = x^2$  (hence  $du = 2x dx$ ),

$$\begin{aligned} y(x) &= \int x \cos(x^2) dx = \frac{1}{2} \int \cos(u) du \\ &= \frac{1}{2} \sin(u) + c = \frac{1}{2} \sin(x^2) + c . \end{aligned}$$

**2.3 g.** Dividing through by  $x^2 - 9$  yields

$$\frac{dy}{dx} = \frac{x}{x^2 - 9} .$$

This can be integrated using the substitution  $u = x^2 - 9$ :

$$\begin{aligned} y(x) &= \int \frac{x}{x^2 - 9} dx = \frac{1}{2} \int \frac{1}{x^2 - 9} 2x dx \\ &= \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln |u| + c = \frac{1}{2} \ln |x^2 - 9| + c . \end{aligned}$$

**2.3 i.**  $1 = x^2 - 9 \frac{dy}{dx} \rightsquigarrow 9 \frac{dy}{dx} = x^2 - 1 \rightsquigarrow \frac{dy}{dx} = \frac{1}{9} x^2 - \frac{1}{9}$

$$\hookrightarrow y(x) = \int \left[ \frac{1}{9} x^2 - \frac{1}{9} \right] dx = \frac{1}{27} x^3 - \frac{1}{9} x + c .$$

**2.3 k.**  $\frac{d^2 y}{dx^2} - 3 = x \rightsquigarrow \frac{d^2 y}{dx^2} = x + 3$

$$\hookrightarrow \frac{dy}{dx} = \int \frac{d^2 y}{dx^2} dx = \int [x + 3] dx = \frac{1}{2} x^2 + 3x + c_1$$

$$\hookrightarrow y(x) = \int \left[ \frac{1}{2} x^2 + 3x + c_1 \right] dx = \frac{1}{6} x^3 + \frac{3}{2} x^2 + c_1 x + c_2 .$$

**2.4 a.** We first find the general solution to the differential equation:

$$y(x) = \int \frac{dy}{dx} dx = \int [4x + 10e^{2x}] dx = 2x^2 + 5e^{2x} + c .$$

Then use the initial condition to determine the value of  $c$ :

$$4 = y(0) = 2 \cdot 0^2 + 5e^{2 \cdot 0} + c = 0 + 5 + c \rightsquigarrow c = 4 - 5 = -1 .$$

So the solution to the initial-value problem is given by  $y(x) = 2x^2 + 5e^{2x} + c$  with  $c = -1$ ; that is,  $y(x) = 2x^2 + 5e^{2x} - 1$ . Moreover, this solution is valid for all values of  $x$  since all functions in the differential equation and solution are continuous on  $(-\infty, \infty)$ .

**2.4 c.** Finding the general solution to the differential equation:

$$\begin{aligned} y(x) &= \int \frac{x-1}{x+1} dx = \int \frac{x+1-2}{x+1} dx \\ &= \int \left[ \underbrace{\frac{x+1}{x+1}}_{=1} - 2 \frac{1}{x+1} \right] dx = x - 2 \ln|x+1| + c . \end{aligned}$$

Applying the initial condition to find  $c$ :

$$8 = y(0) = 8 - 2 \ln|0+1| + c = 8 - 2 \cdot 0 + c \quad \rightsquigarrow \quad c = 8 .$$

So,  $y(x) = x - 2 \ln|x+1| + c$  with  $c = 8$ ; that is,  $y(x) = x - 2 \ln|x+1| + 8$ . And since the derivative in the differential equation “blows up” at  $x = -1$  and the initial condition is given at  $x = 0 > -1$ , the solution is only valid for all values of  $x$  greater than  $-1$ .

**2.4 e.** Solving for the derivative yields the initial-value problem

$$\frac{dy}{dx} = \frac{\sin(x)}{\cos(x)} \quad \text{with} \quad y(0) = 3 .$$

The largest interval containing  $x = 0$  on which this derivative does not “blow up” is  $(-\pi/2, \pi/2)$  (since  $(-\pi/2, \pi/2)$  is the largest interval containing  $x = 0$  on which  $\cos(x)$  is nonzero). So our solution will only be valid on  $(-\pi/2, \pi/2)$ . Integrating the above:

$$\begin{aligned} y(x) &= \int \frac{\sin(x)}{\cos(x)} dx \\ &= - \int \frac{1}{\cos(x)} \frac{d}{dx}[\cos(x)] dx = - \ln|\cos(x)| + c . \end{aligned}$$

Then applying the initial condition, and writing down the final result:

$$\begin{aligned} 3 &= y(0) = - \ln|\cos(0)| + c = - \ln(1) + c = 0 + c \\ \hookrightarrow \quad c &= 3 \quad \rightsquigarrow \quad y(x) = - \ln|\cos(x)| + 3 . \end{aligned}$$

**2.4 g.** Solving for the highest derivative:

$$x \frac{d^2 y}{dx^2} + 2 = \sqrt{x} \quad \rightsquigarrow \quad \frac{d^2 y}{dx^2} = \frac{1}{x} [\sqrt{x} - 2] = x^{-1/2} - \frac{2}{x} .$$

Clearly, the righthand side requires  $x > 0$ . Integrating and applying the second initial condition:

$$\begin{aligned} \frac{dy}{dx} &= \int \left[ x^{-1/2} - \frac{2}{x} \right] dx = 2x^{1/2} - 2 \ln|x| + c_1 . \\ \hookrightarrow \quad 6 &= y'(1) = 2 \cdot 1^{1/2} - 2 \ln|1| + c_1 = 2 - 2 \cdot 0 + c_1 . \\ \hookrightarrow \quad c_1 &= 6 - 2 = 4 \quad \text{and, thus} \quad \frac{dy}{dx} = 2x^{1/2} - 2 \ln|x| + 4 . \end{aligned}$$

Integrating this last equation (possibly using integration by parts to compute the integral of  $\ln|x|$ ):

$$\begin{aligned} y(x) &= \int [2x^{1/2} - 2\ln|x| + 4] dx \\ &= \frac{4}{3}x^{3/2} - 2[x\ln|x| - x] + 4x + c_2 \\ &= \frac{4}{3}x^{3/2} - 2x\ln|x| + 6x + c_2 . \end{aligned}$$

Then, applying the first initial condition:

$$\begin{aligned} 8 &= y(1) = \frac{4}{3} \cdot 1^{3/2} - 2 \cdot 1 \ln|1| + 6 \cdot 1 + c_2 = \frac{4}{3} + 6 + c_2 \\ \hookrightarrow c_2 &= \frac{2}{3} \quad \text{and, thus} \quad y(x) = \frac{4}{3}x^{3/2} - 2x\ln|x| + 6x + \frac{2}{3} . \end{aligned}$$

**2.5 a.**

$$\begin{aligned} y(x) - y(0) &= \int_0^x \frac{dy}{dx} ds = \int_0^x \sin\left(\frac{s}{2}\right) ds = -2\cos\left(\frac{s}{2}\right) \Big|_0^x \\ &= -2\cos\left(\frac{x}{2}\right) + 2\cos\left(\frac{0}{2}\right) = -2\cos\left(\frac{x}{2}\right) + 2 \\ \hookrightarrow y(x) &= -2\cos\left(\frac{x}{2}\right) + 2 + y(0) . \end{aligned}$$

**2.5 b i.** Plugging the initial value into the above formula:

$$\begin{aligned} y(x) &= -2\cos\left(\frac{x}{2}\right) + 2 + y(0) \\ &= -2\cos\left(\frac{x}{2}\right) + 2 + 0 = -2\cos\left(\frac{x}{2}\right) + 2 . \end{aligned}$$

$$\text{So } y(\pi) = -2\cos\left(\frac{\pi}{2}\right) + 2 = -2 \cdot 0 + 2 = 2 .$$

**2.5 b ii.**

$$\begin{aligned} y(x) &= -2\cos\left(\frac{x}{2}\right) + 2 + y(0) \\ &= -2\cos\left(\frac{x}{2}\right) + 2 + 3 = -2\cos\left(\frac{x}{2}\right) + 5 . \end{aligned}$$

$$\text{So } y(\pi) = -2\cos\left(\frac{\pi}{2}\right) + 5 = -2 \cdot 0 + 5 = 5 .$$

**2.5 b iii.**

$$\begin{aligned} y(2\pi) &= -2\cos\left(\frac{2\pi}{2}\right) + 2 + y(0) \\ &= -2\cos(\pi) + 2 + 3 = -2(-1) + 5 = 7 . \end{aligned}$$

**2.7 a.**

$$\begin{aligned} y(x) - y(0) &= \int_{-}^x \frac{dy}{ds} ds \rightsquigarrow y(x) - 3 = \int_0^x s e^{-s^2} ds \\ \hookrightarrow y(x) &= -\frac{1}{2}e^{-s^2} \Big|_0^x + 3 = -\frac{1}{2}e^{-x^2} + \frac{1}{2}e^0 + 3 = -\frac{1}{2}e^{-x^2} + \frac{7}{2} . \end{aligned}$$

2.7 c. 
$$y(x) - y(1) = \int_1^x \frac{dy}{ds} ds \quad \rightsquigarrow \quad y(x) - 0 = \int_1^x \frac{1}{s^2+1} ds$$

$$\hookrightarrow y(x) = \arctan(s) \Big|_1^x = \arctan(x) - \arctan(1) = \arctan(x) - \frac{\pi}{4} .$$

2.7 e. 
$$x \frac{dy}{dx} = \sin(x) \quad \rightsquigarrow \quad \frac{dy}{dx} = \frac{\sin(x)}{x}$$

$$\hookrightarrow y(x) - 4 = \int_0^x \frac{\sin(s)}{s} dx = \text{Si}(x) \quad \rightsquigarrow \quad y(x) = \text{Si}(x) + 4 .$$

2.9 a. For the graph of  $\text{step}(x)$ , see the page 33 of the text.

Since  $y(0) = 0$ , 
$$y(x) = \int_0^x \text{step}(s) ds + y(0) = \int_0^x \text{step}(s) ds .$$

If  $x < 0$  and  $x < s < 0$ , then  $\text{step}(s) = 0$ . Thus,

$$y(x) = \int_0^x \text{step}(s) ds = \int_0^x 0 ds = 0 \quad \text{if } x < 0 .$$

If  $0 \leq x$  and  $0 \leq s \leq x$ , then  $\text{step}(s) = 1$ . Thus,

$$y(x) = \int_0^x \text{step}(s) ds = \int_0^x 1 ds = x \quad \text{if } 0 \leq x .$$

In summary,

$$y(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leq x \end{cases} = \text{ramp}(x) .$$

2.9 c. Since  $y(0) = 0$ , 
$$y(x) = \int_0^x f(s) ds + y(0) = \int_0^x f(s) ds .$$

If  $x < 1$  and  $x < s < 1$ , then  $f(s) = 0$ . Thus,

$$y(x) = \int_0^x \text{step}(s) ds = \int_0^x 0 ds = 0 \quad \text{if } x < 1 .$$

If  $1 \leq x < 2$ , then

$$\begin{aligned} y(x) &= \int_0^x f(s) ds = \int_0^1 \underbrace{f(s)}_{=0} ds + \int_1^x \underbrace{f(s)}_{=1} ds \\ &= \int_0^1 0 ds + \int_1^x 1 ds = 0 + x - 1 . \end{aligned}$$

If  $2 \leq x$ , then

$$\begin{aligned} y(x) &= \int_0^x f(s) ds \\ &= \int_0^1 \underbrace{f(s)}_{=0} ds + \int_1^2 \underbrace{f(s)}_{=1} ds + \int_2^x \underbrace{f(s)}_{=0} ds \\ &= \int_0^1 0 ds + \int_1^2 1 ds + \int_2^x 0 ds = 0 + 1 + 0 . \end{aligned}$$

So,

$$y(x) = \begin{cases} 0 & \text{if } x < 1 \\ x - 1 & \text{if } 1 \leq x \leq 2 \\ 1 & \text{if } 2 < x \end{cases} .$$

**2.9 e.** Since  $y(0) = 0$ ,  $y(x) = \int_0^x \text{stair}(s) ds + y(0) = \int_0^x \text{stair}(s) ds$  .

$$\text{If } x < 0, \text{ then } y(x) = \int_0^x \underbrace{\text{stair}(s)}_{=0} ds = \int_0^x 0 ds = 0 .$$

$$\text{If } 0 \leq x < 1, \text{ then } y(x) = \int_0^x \underbrace{\text{stair}(s)}_{=1} ds = \int_0^x 1 ds = x .$$

If  $1 \leq x < 2$ , then

$$\begin{aligned} y(x) &= \int_0^1 \underbrace{\text{stair}(s)}_{=1} ds + \int_1^x \underbrace{\text{stair}(s)}_{=2} ds \\ &= \int_0^1 1 ds + \int_1^x 2 ds = 1 + (2x - 2) = 2x - 1 = 2\left(x - \frac{1}{2}\right) . \end{aligned}$$

If  $2 \leq x < 3$ , then

$$\begin{aligned} y(x) &= \int_0^1 1 ds + \int_1^2 2 ds + \int_2^x 3 ds \\ &= 1 + (4 - 2) + (3x - 6) = 3x - 3 = 3(x - 1) . \end{aligned}$$

If  $3 \leq x < 4$ , then

$$\begin{aligned} y(x) &= \int_0^1 1 ds + \int_1^2 2 ds + \int_2^3 3 ds + \int_3^x 4 ds \\ &= 1 + 2 + 3 + (4x - 12) = 4x - 6 = 4\left(x - \frac{3}{2}\right) . \end{aligned}$$

So,

$$y(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leq x < 1 \\ 2\left(x - \frac{1}{2}\right) & \text{if } 1 \leq x < 2 \\ 3\left(x - \frac{2}{2}\right) & \text{if } 2 \leq x < 3 \\ 4\left(x - \frac{3}{2}\right) & \text{if } 3 \leq x < 4 \end{cases} .$$