Chapter 18: Euler Equations

18.1 a. Letting $y(x) = x^r \rightarrowtail y'(x) = rx^{r-1} \rightarrowtail y''(x) = r(r-1)x^{r-2}$,

yields

$$\begin{aligned} 0 &= x^2y'' - 5xy' + 8y \\ &= x^2r(r-1)x^{r-2} - 5xrx^{r-1} + 8x^r \\ &= \left[r^2 - r\right]x^r - 5rx^r + 8x^r \\ &= \left[r^2 - r - 5r + 8\right]x^r = \left[r^2 - 6r + 8\right]x^r \quad . \end{aligned}$$

So the indicial equation is

$$0 = r^2 - 6r + 8 \quad ,$$

which factors to

$$0 = (r-2)(r-4) \quad .$$

Thus, x^r is a solution to the differential equation if $r = 2$ or $r = 4$, and, consequently, the general solution to our differential equation is

$$y(x) = c_1x^2 + c_2x^4 \quad .$$

18.1 c. Letting $y(x) = x^r \rightarrowtail y'(x) = rx^{r-1} \rightarrowtail y''(x) = r(r-1)x^{r-2}$,

the differential equation becomes

$$\begin{aligned} 0 &= x^2y'' - 2xy' \\ &= x^2r(r-1)x^{r-2} - 2xrx^{r-1} \\ &= \left[r^2 - r\right]x^r - 2rx^r \\ &= \left[r^2 - r - 2r\right]x^r = \left[r^2 - 3r\right]x^r \quad . \end{aligned}$$

So the indicial equation is

$$0 = r^2 - 3r = r(r-3) = (r-0)(r-3) \quad ,$$

which means $r = 0$ and $r = 3$. Thus, two particular solutions to the differential equation are $x^0 = 1$ and x^3, and the general solution is

$$y(x) = c_1 \cdot 1 + c_2x^3 \quad .$$

18.1 e. Letting $y(x) = x^r \rightarrowtail y'(x) = rx^{r-1} \rightarrowtail y''(x) = r(r-1)x^{r-2}$,

the differential equation becomes

$$\begin{aligned} 0 &= x^2y'' - 5xy' + 9y \\ &= x^2r(r-1)x^{r-2} - 5xrx^{r-1} + 9x^r \\ &= \left[r^2 - r - 5r + 9\right]x^r = \left[r^2 - 6r + 9\right]x^r \quad . \end{aligned}$$

So the indicial equation is

$$0 = r^2 - 6r + 9 = (r-3)^2 \quad ,$$

which only has $r = 3$ as a solution, leading to the one solution x^3 to the differential equation. As noted in section 18.2, an appropriate second solution is obtained by either reduction of order, or, more simply, by multiplying the first solution, x^3, by $\ln|x|$. Thus, the general solution to the differential equation is

$$y(x) = c_1 x^3 + c_2 x^3 \ln|x| \quad .$$

18.1 g. Letting $\quad y(x) = x^r \quad \rightarrowtail \quad y'(x) = rx^{r-1} \quad \rightarrowtail \quad y''(x) = r(r-1)x^{r-2} \quad ,$

the differential equation becomes

$$0 = 4x^2 y'' + y = 4x^2 r(r-1)x^{r-2} + x^r = \left[4r^2 - 4r + 1\right] x^r \quad .$$

So the indicial equation is

$$0 = 4r^2 - 4r + 1 = (2r-1)^2 \quad ,$$

which only has $r = 1/2$ as a solution. Thus, the general solution to the differential equation is

$$y(x) = c_1 x^{1/2} + c_2 x^{1/2} \ln|x| \quad .$$

18.1 i. Letting $\quad y(x) = x^r \quad \rightarrowtail \quad y'(x) = rx^{r-1} \quad \rightarrowtail \quad y''(x) = r(r-1)x^{r-2} \quad ,$

the differential equation becomes

$$\begin{aligned} 0 &= x^2 y'' + 5xy' + 29y \\ &= x^2 r(r-1)x^{r-2} + 5xrx^{r-1} + 29x^r \\ &= \left[r^2 - r + 5r + 29\right] x^r = \left[r^2 + 4r + 29\right] x^r \quad . \end{aligned}$$

So the indicial equation is

$$0 = r^2 + 4r + 29 \quad ,$$

the solutions of which are

$$r_\pm = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 20}}{2} = -2 \pm 5i \quad .$$

The corresponding particular solutions (with $x > 0$) to the differential equation are then

$$\begin{aligned} y_\pm(x) = x^{r_\pm} = x^{-2\pm 5i} &= x^{-2} x^{\pm 5i} \\ &= x^{-2} e^{\ln\left(x^{\pm 5i}\right)} \\ &= x^{-2} e^{i 5 \ln|x|} \\ &= x^{-2} \left[\cos(5\ln|x|) + i \sin(5\ln|x|)\right] \\ &= \underbrace{x^{-2}\cos(5\ln|x|)}_{y_1(x)} + i\, \underbrace{x^{-2}\sin(5\ln|x|)}_{y_2(x)} \quad . \end{aligned}$$

So, in terms of real-valued functions, the general solution to the differential equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 x^{-2}\cos(5\ln|x|) + c_2 x^{-2}\sin(5\ln|x|) \quad .$$

18.1 k. Letting $y(x) = x^r \rightarrowtail y'(x) = rx^{r-1} \rightarrowtail y''(x) = r(r-1)x^{r-2}$,

the differential equation becomes

$$\begin{aligned} 0 &= 2x^2y'' + 5xy' + y \\ &= 2x^2r(r-1)x^{r-2} + 5xrx^{r-1} + x^r \\ &= \left[2r^2 - 2r + 5r + 1\right]x^r = \left[2r^2 + 3r + 1\right]x^r \quad . \end{aligned}$$

Writing out the indicial equation, and then continuing

$$2r^2 + 3r + 1 = 0$$

$$\hookrightarrow \qquad r = \frac{-3 \pm \sqrt{3^2 - 4 \cdot 2 \cdot 1}}{2 \cdot 2} = \frac{-3 \pm 1}{4}$$

$$\hookrightarrow \qquad r = -\frac{1}{2} \quad \text{and} \quad r = -1$$

$$\hookrightarrow \qquad y(x) = c_1 x^{-1/2} + c_2 x^{-1} \quad .$$

18.1 m. Letting $y(x) = x^r \rightarrowtail y'(x) = rx^{r-1} \rightarrowtail y''(x) = r(r-1)x^{r-2}$,

the differential equation becomes

$$0 = x^2y'' + xy' = x^2r(r-1)x^{r-2} + xrx^{r-1} = \left[r^2 - r + r\right]x^r = \left[r^2\right]x^r \quad .$$

Writing out the indicial equation, and then continuing

$$r^2 = 0 \rightarrowtail r = 0$$

$$\hookrightarrow \qquad y(x) = c_1 x^{-0} + c_2 x^0 \ln|x| = c_1 + c_2 \ln|x| \quad .$$

18.2 a. Letting $y(x) = x^r \rightarrowtail y'(x) = rx^{r-1} \rightarrowtail y''(x) = r(r-1)x^{r-2}$,

the differential equation becomes

$$\begin{aligned} 0 &= x^2y'' - 6xy' + 10y \\ &= x^2r(r-1)x^{r-2} - 6xrx^{r-1} + 10x^r \\ &= \left[r^2 - r - 6r + 10\right]x^r = \left[r^2 - 7r + 10\right]x^r \quad . \end{aligned}$$

Writing out the indicial equation, and then continuing

$$0 = r^2 - 7r + 10 = (r-5)(r-2)$$

$$\hookrightarrow \qquad r = 5 \quad \text{and} \quad r = 2$$

$$\hookrightarrow \qquad y(x) = c_1 x^5 + c_2 x^2 \qquad (\star)$$

$$\hookrightarrow \qquad y'(x) = 5c_1 x^4 + 2c_2 x \quad .$$

Applying the initial data:

$$-1 = y(1) = c_1 \cdot 1^5 + c_2 \cdot 1^2 = c_1 + c_2$$

and

$$7 = y'(0) = 5c_1 \cdot 1^4 + 2c_2 \cdot 1 = 5c_1 + 2c_2 \quad .$$

Solving for c_1 and c_2, and plugging the values back into formula (⋆) for y:

$$-1 = c_1 + c_2 \qquad \text{and} \qquad 7 = 5c_1 + 2c_2$$

$$\hookrightarrow \quad c_1 = -1 - c_2 \qquad \text{and} \qquad 7 = 5[-1 - c_2] + 2c_2 = -5 - 3c_2$$

$$\hookrightarrow \quad c_1 = -1 - c_2 \qquad \text{and} \qquad c_2 = \frac{7+5}{-3} = -4$$

$$\hookrightarrow \quad c_1 = -1 - [-4] = 3 \qquad \text{and} \qquad c_2 = \frac{7+5}{-3} = -4$$

$$\hookrightarrow \quad y(x) = 3x^5 - 4x^2 \quad .$$

18.2 c. Letting $y(x) = x^r \rightarrowtail y'(x) = rx^{r-1} \rightarrowtail y''(x) = r(r-1)x^{r-2}$,

the differential equation becomes

$$\begin{aligned} 0 &= x^2y'' - 11xy' + 36y \\ &= x^2r(r-1)x^{r-2} - 11xrx^{r-1} + 36x^r = \left[r^2 - 12r + 36\right]x^r \quad . \end{aligned}$$

Writing out the indicial equation, and then continuing

$$0 = r^2 - 12r + 36 = (r-6)^2$$

$$\hookrightarrow \quad r = 6 \text{ is the only root.}$$

$$\hookrightarrow \quad y(x) = c_1x^6 + c_2x^6 \ln|x| \qquad (\star)$$

$$\hookrightarrow \quad y'(x) = 6c_1x^5 + c_2\left[6x^5 \ln|x| + x^5\right] \quad .$$

Applying the initial data:

$$\frac{1}{2} = y(1) = c_1 \cdot 1^6 + c_2 \cdot 1^6 \ln|1| = c_1$$

and

$$2 = y'(0) = 6c_1 \cdot 1^5 + c_2\left[6 \cdot 1^5 \ln|1| + 1^5\right] = 6c_1 + c_2 \quad .$$

Solving for c_1 and c_2, and plugging the values back into formula (⋆) for y:

$$\frac{1}{2} = c_1 \qquad \text{and} \qquad 2 = 6c_1 + c_2$$

$$\hookrightarrow \quad c_1 = \frac{1}{2} \qquad \text{and} \qquad c_2 = 2 - 6c_1 = 2 - 6\left[\frac{1}{2}\right] = -1$$

$$\hookrightarrow \quad y(x) = \frac{1}{2}x^6 - x^6 \ln|x| \quad .$$

18.4 a. Assuming $y = x^r$ and taking three derivatives, we get

$$y' = rx^{r-1} \quad\rightarrowtail\quad y'' = r(r-1)x^{r-2} \quad\rightarrowtail\quad y''' = r(r-1)(r-2)x^{r-3} \quad .$$

Plugging these into the differential equation:

$$\begin{aligned} 0 &= x^3y''' + 2x^2y'' - 4xy' + 4y \\ &= x^3r(r-1)(r-2)x^{r-3} + 2x^2r(r-1)x^{r-2} - 4xrx^{r-1} + 4x^r \\ &= \left[r^3 - 3r^2 + 2r\right]x^r + 2\left[r^2 - r\right]x^r - 4[r]x^r + 4x^r \\ &= \left[r^3 - r^2 - 4r + 4\right]x^r \quad . \end{aligned}$$

So the indicial equation is the third-degree polynomial equation

$$\underbrace{r^3 - r^2 - 4r + 4}_{p(x)} = 0 \quad .$$

To find the solutions, we'll first test to see if $r = 1$ is one root of $p(r)$:

$$p(1) = 1^3 - 1^2 - 4 \cdot 1 + 4 = 1 - 1 - 4 + 4 = 0 \quad .$$

Hence, $r = 1$ is a root, and $r - 1$ is a factor. Dividing out this factor:

$$\begin{array}{r|l} & r^2 \qquad\quad - 4 \\ \hline r-1 & r^3 - r^2 - 4r + 4 \\ & -r^3 + r^2 \\ \hline & \qquad\quad -4r + 4 \\ & \qquad\quad 4r - 4 \\ \hline & \qquad\qquad\quad 0 \end{array} \quad .$$

This means we can factor our indicial equation as follows:

$$\begin{aligned} 0 &= r^3 - r^2 - 4r + 4 \\ &= (r-1)\left(r^2 - 4\right) = (r-1)(r-2)(r+2) \quad . \end{aligned}$$

Thus, the solutions the indicial equation are the three distinct values

$$r = 1 \quad , \quad r = 2 \quad \text{and} \quad r = -2 \quad ,$$

and the corresponding general solution to the third-order differential equation is

$$y(x) = c_1x + c_2x^2 + c_3x^{-2} \quad .$$

18.4 c. Assuming $y = x^r$ and taking three derivatives, we get

$$y' = rx^{r-1} \quad\rightarrowtail\quad y'' = r(r-1)x^{r-2} \quad\rightarrowtail\quad y''' = r(r-1)(r-2)x^{r-3} \quad ,$$

which, when plugged into the differential equation yields

$$\begin{aligned} 0 &= x^3y''' - 5x^2y'' + 14xy' - 18y \\ &= x^3r(r-1)(r-2)x^{r-3} - 5x^2r(r-1)x^{r-2} + 14xrx^{r-1} - 18x^r \end{aligned}$$

$$= \left[r^3 - 3r^2 + 2r\right]x^r \;-\; 5\left[r^2 - r\right]x^r \;+\; 14[r]x^r \;-\; 18x^r$$
$$= \left[r^3 - 8r^2 + 21r - 18\right]x^r \quad .$$

So the indicial equation is the third-degree polynomial equation

$$\underbrace{r^3 \;-\; 8r^2 \;+\; 21r \;-\; 18}_{p(x)} \;=\; 0 \quad .$$

To find the solutions, we'll compute $p(r)$ for different values of r until we find a value such that $p(r) = 0$:

$$p(1) \;=\; 1^3 \;-\; 8\cdot 1^2 \;+\; 21\cdot 1 \;-\; 18 \;=\; 1 \;-\; 8 \;+\; 21 \;-\; 18 \;=\; -4 \;\neq\; 0 \quad ,$$
$$p(2) \;=\; 2^3 \;-\; 8\cdot 2^2 \;+\; 21\cdot 2 \;-\; 18 \;=\; 8 \;-\; 32 \;+\; 42 \;-\; 18 \;=\; 0 \quad .$$

Hence, $r = 2$ is a root, and $r - 2$ is a factor. Dividing out this factor:

$$\begin{array}{r|l} & r^2 \;\; - 6r \;\; + 9 \\ \hline r-2 & r^3 - 8r^2 + 21r - 18 \\ & -r^3 + 2r^2 \\ \hline & \quad -6r^2 + 21r \\ & \quad\;\; 6r^2 - 12r \\ \hline & \qquad\quad 9r - 18 \\ & \qquad\; -9r + 18 \\ \hline & \qquad\qquad\quad 0 \end{array} \quad .$$

This means we can factor our indicial equation as follows:

$$0 \;=\; r^3 \;-\; 8r^2 \;+\; 21r \;-\; 18$$
$$=\; (r-2)\left(r^2 - -6r + 9\right) \;=\; (r-2)(r-3)^2 \quad .$$

Thus, the solutions the indicial equation are $r = 2$ (with multiplicity 1) and $r = 3$ (with multiplicity 2), and the corresponding general solution to the third-order differential equation is

$$y(x) \;=\; c_1x^2 \;+\; c_2x^3 \;+\; c_3x^3\ln|x| \quad .$$

18.4 e. Assuming $y = x^r$ and taking four derivatives, we get

$$y' \;=\; rx^{r-1} \quad\rightarrowtail\quad y'' \;=\; r(r-1)x^{r-2}$$

$$\hookrightarrow\quad y''' \;=\; r(r-1)(r-2) \quad\rightarrowtail\quad y^{(4)} \;=\; r(r-1)(r-2)(r-3)x^{r-4} \quad .$$

which, when plugged into the differential equation yields

$$0 \;=\; x^4y^{(4)} \;+\; 2x^3y''' \;+\; x^2y'' \;-\; xy' \;+\; y$$
$$=\; x^4r(r-1)(r-2)(r-3)x^{r-4} \;+\; 2x^3r(r-1)(r-2)x^{r-3}$$
$$+\; x^2r(r-1)x^{r-2} \;-\; xrx^{r-1} \;+\; x^r$$

$$\begin{aligned} &= \left[r^4 - 6r^3 + 11r^2 - 6r\right] \;+\; 2\left[r^3 - 3r^2 + 2r\right]x^r \;+\; \left[r^2 - r\right]x^r \\ &\qquad -\; [r]x^r \;+\; x^r \\ &= \left[r^4 - 4r^3 + 6r^2 - 4r + 1\right]x^r \end{aligned}$$

So the indicial equation is the fourth-degree polynomial equation

$$\underbrace{r^4 \;-\; 4r^3 \;+\; 6r^2 \;-\; 4r \;+\; 1}_{p(x)} \;=\; 0 \quad .$$

Fortunately, it's easy to see that

$$p(1) \;=\; 1^4 \;-\; 4\cdot 1^3 \;+\; 6\cdot 1^2 \;-\; 4\cdot 1 \;+\; 1 \;=\; 0 \quad ,$$

telling us that $r = 1$ is a root, and $r - 1$ is a factor. Dividing out this factor,

$$\begin{array}{r@{\,}l} & r^3 - 3r^2 + 3r - 1 \\ \cline{2-2} r - 1\big) & r^4 - 4r^3 + 6r^2 - 4r + 1 \\ & \underline{-r^4 + r^3} \\ & -3r^3 + 6r^2 \\ & \underline{3r^3 - 3r^2} \\ & 3r^2 - 4r \\ & \underline{-3r^2 + 3r} \\ & -r + 1 \\ & \underline{r - 1} \\ & 0 \end{array} \quad ,$$

we find that

$$p(r) \;=\; r^4 \;-\; 4r^3 \;+\; 6r^2 \;-\; 4r \;+\; 1 \;=\; (r-1)\big(\underbrace{r^3 - 3r^2 + 3r - 1}_{q(r)}\big) \quad .$$

It is even more easy to see that

$$q(1) \;=\; 1^3 \;-\; 3\cdot 1^2 \;+\; 3\cdot 1 \;-\; 1 \;=\; 0 \quad .$$

So $r - 1$ is also a factor of $q(r)$. Dividing out that factor,

$$\begin{array}{r@{\,}l} & r^2 - 2r + 1 \\ \cline{2-2} r - 1\big) & r^3 - 3r^2 + 3r - 1 \\ & \underline{-r^3 + r^2} \\ & -2r^2 + 3r \\ & \underline{2r^2 - 2r} \\ & r - 1 \\ & \underline{-r + 1} \\ & 0 \end{array} \quad ,$$

and we see that our indicial equation factors as follows:

$$\begin{aligned} 0 &= r^4 \;-\; 4r^3 \;+\; 6r^2 \;-\; 4r \;+\; 1 \\ &= (r-1)\big(r^3 - 3r^2 + 3r - 1\big) \end{aligned}$$

$$= (r-1)(r-1)\big(r^2 - 2r + 1\big)$$
$$= (r-1)(r-1)(r-1)(r-1) = (r-1)^4 \quad .$$

Thus,

$$y(x) = c_1 x + c_2 x \ln|x| + c_3 x(\ln|x|)^2 + c_4 x(\ln|x|)^3 \quad .$$

18.6 a. Letting $y = x^r \rightarrowtail y' = rx^{r-1} \rightarrowtail y'' = r(r-1)x^{r-2}$,

the differential equation becomes

$$\begin{aligned} 0 &= \alpha x^2 y'' + \beta x y' + \gamma y \\ &= \alpha x^2 r(r-1)x^{r-2} + \beta x r x^{r-1} + \gamma x^r \\ &= \Big[\alpha r^2 - \alpha r + \beta r + \gamma\Big] x^r \\ &= \Big[\alpha r^2 r + (\beta - \alpha) r + \gamma\Big] x^r \quad . \end{aligned}$$

So the indicial equation is

$$\alpha r^2 r + (\beta - \alpha) r + \gamma = 0 \quad . \qquad (\star)$$

18.6 b. Let $y(x) = Y(t)$ where $x = e^t$ and $t = \ln|x|$.
We will need to convert the derivatives in the given Euler equation to corresponding derivatives of Y . For convenience, let us first observe that

$$\frac{dt}{dx} = \frac{d \ln|x|}{dx} = \frac{1}{x} = e^{-t} \quad .$$

Using this and the chain rule, we then have

$$\frac{dy}{dx} = \frac{d}{dx} Y(t) = \frac{dt}{dx}\frac{d}{dt} Y(t) = e^{-t}\frac{dY}{dt}$$

and

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{dt}{dx}\frac{d}{dt}\left[e^{-t}\frac{dY}{dt}\right] \\ &= e^{-t}\left[-e^{-t}\frac{dY}{dt} + e^{-t}\frac{d^2Y}{dt^2}\right] = e^{-2t}\left[\frac{d^2Y}{dt^2} - \frac{dY}{dt}\right] \quad . \end{aligned}$$

Note that

$$xy' = x\frac{dy}{dx} = e^t \cdot e^{-t}\frac{dY}{dt} = \frac{dY}{dt}$$

and

$$x^2 y'' = x^2 \frac{d^2x}{dy^2} = \left(e^t\right)^2 e^{-2t}\left[\frac{d^2Y}{dt^2} - \frac{dY}{dt}\right] = \frac{d^2Y}{dt^2} - \frac{dY}{dt} \quad .$$

Applying the above to the given generic Euler equation gives us

$$\begin{aligned} 0 &= \alpha x^2 y'' + \beta x y' + \gamma y \\ &= \alpha\left[\frac{d^2Y}{dt^2} - \frac{dY}{dt}\right] + \beta\frac{dY}{dt} + \gamma Y(t) \quad , \end{aligned}$$

which, after a trivial bit of algebra, becomes

$$\alpha \frac{d^2Y}{dt^2} + (\beta - \alpha)\frac{dY}{dt} + \gamma Y = 0 \quad ,$$

a second-order, constant cofficient differential equation with characteristic equation

$$\alpha r^2 + (\beta - \alpha)r + \gamma = 0 \quad . \tag{$\star\star$}$$

18.6 c. Just observe that equation ($\star$), the indicial equation for the Euler equation, and equation ($\star\star$), the characteristic equation for the corresponding constant coefficient equation, are identical.