Chapter 17: Arbitrary Homogeneous Linear Equations with Constant Coefficients

17.1 a. Writing out and factoring the characteristic equation:

$$0 = r^4 - 4r^3 = r^3(r-4) = (r-0)^3(r-4)$$

$$\hookrightarrow \qquad r = 0 \text{ (mult. 3)} \qquad \text{and} \qquad r = 4 \text{ (mult. 3)}$$

So the corresponding fundamental set of solutions is

$$\left\{e^{0x},\ xe^{0x},\ x^2e^{0x},\ e^{4x}\right\} = \left\{1,\ x,\ x^2,\ e^{4x}\right\} \quad ,$$

and the general solution is

$$y(x) = c_1 \cdot 1 + c_2x + c_3x^2 + c_4e^{4x} \quad .$$

17.1 c. Starting with the observation that the powers in the characteristic polynomial are all even:

$$0 = r^4 - 34r^2 + 225 = \left(r^2\right)^2 - 34\left(r^2\right) + 225$$

$$\hookrightarrow \qquad r^2 = \frac{-(-34) \pm \sqrt{(-34)^2 - 4(225)}}{2} = 17 \pm 8$$

$$\hookrightarrow \qquad r^2 = 9 \qquad \text{and} \qquad r^2 = 25$$

$$\hookrightarrow \qquad r = \pm\sqrt{9} = \pm 3 \qquad \text{and} \qquad r = \pm\sqrt{25} = \pm 5 \quad .$$

Since we have four different roots for our fourth-degree polynomial, each root must be simple (i.e., each has multiplicity 1), and so,

$$y(x) = c_1e^{3x} + c_2e^{-3x} + c_3e^{5x} + c_4e^{-5x} \quad .$$

17.1 e.

$$\begin{aligned} 0 &= r^4 - 18r^2 + 81 \\ &= \left(r^2\right)^2 - 18\left(r^2\right) + 81 \\ &= \left(r^2 - 9\right)^2 \\ &= \left((r-3)(r+3)\right)^2 = (r-3)^2(r-[-3])^2 \quad . \end{aligned}$$

So,

$$\begin{aligned} y(x) &= c_1e^{3x} + c_2xe^{3x} + c_3e^{-3x} + c_4xe^{-3x} \\ &= [c_1 + c_2x]e^{3x} + [c_3 + c_4x]e^{-3x} \quad . \end{aligned}$$

17.2 a. Here, $p(x) = r^3 - r^2 + r - 1$.

Trying $r = 1$:

$$p(1) = 1^3 - 1^2 + 1 - 1 = 1 - 1 + 1 - 1 = 0 \quad .$$

So one root is $r = 1$, and one factor of p is $r - 1$. Dividing $p(r)$ by $r - 1$:

$$\begin{array}{r l}
 & r^2 \qquad + 1 \\ \cline{2-2}
r - 1 \big) & r^3 - r^2 + r - 1 \\
 & - r^3 + r^2 \\ \cline{2-2}
 & \qquad\qquad r - 1 \\
 & \qquad\quad - r + 1 \\ \cline{2-2}
 & \qquad\qquad\quad 0
\end{array} \quad .$$

Hence, $p(r) = (r - 1)\left(r^2 + 1\right)$.

Using this to find the solutions to the characteristic equation and the corresponding general solution to the differential equation:

$$0 = r^3 - r^2 + r - 1 = (r - 1)\left(r^2 + 1\right)$$

$$\hookrightarrow \qquad r = 1 \quad \text{or} \quad r = \pm\sqrt{-1} = \pm i$$

$$\hookrightarrow \qquad y(x) = c_1 e^x + c_2 \cos(x) + c_3 \sin(x) \quad .$$

17.2 c. Here, $p(x) = r^3 - 8r^2 + 37r - 50$.

Trying $r = 1$:

$$p(1) = 1^3 - 8 \cdot 1^2 + 37 \cdot 1 - 50 = 1 - 8 + 37 - 50 = -20 \quad .$$

Since $p(1) \neq 0$, $r = 1$ is not a root of p.

Trying $r = 2$:

$$p(1) = 2^3 - 8 \cdot 2^2 + 37 \cdot 2 - 50 = 8 - 32 + 74 - 50 = 0 \quad .$$

So one root is $r = 2$, and one factor of p is $r - 2$. Dividing $p(r)$ by $r - 2$:

$$\begin{array}{r l}
 & r^2 \quad - 6r + 25 \\ \cline{2-2}
r - 2 \big) & r^3 - 8r^2 + 37r - 50 \\
 & - r^3 + 2r^2 \\ \cline{2-2}
 & \quad - 6r^2 + 37r \\
 & \qquad 6r^2 - 12r \\ \cline{2-2}
 & \qquad\qquad 25r - 50 \\
 & \qquad\quad - 25r + 50 \\ \cline{2-2}
 & \qquad\qquad\qquad 0
\end{array} \quad .$$

Hence, $p(r) = (r - 2)\left(r^2 - 6r + 25\right)$.

Using this to find the solutions to the characteristic equation and the corresponding general solution to the differential equation:

$$0 = r^3 - 8r^2 + 37r - 50 = (r - 2)\left(r^2 - 6r + 25\right)$$

$$\hookrightarrow \qquad r = 2 \quad \text{and} \quad r = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 25}}{2} = 3 \pm 4i$$

$$\hookrightarrow \qquad y(x) = c_1 e^{2x} + c_2 e^{3x} \cos(4x) + c_3 e^{3x} \sin(4x) \quad .$$

17.3 a. Finding the general solution:

$$0 = r^3 + 4r = r\left(r^2+4\right)$$

$$\hookrightarrow \qquad r = 0 \quad \text{and} \quad r = \pm\sqrt{4} = \pm 2i$$

$$\hookrightarrow \qquad y(x) = c_1 e^{0x} + c_2\cos(2x) + c_3\sin(2x) \quad .$$

Simplifying and differentiating, we have

$$y(x) = c_1 + c_2\cos(2x) + c_3\sin(2x) \quad , \qquad (\star)$$

$$y'(x) = 0 - 2c_2\sin(2x) + 2c_3\cos(2x) \quad ,$$

and

$$y''(x) = 0 - 4c_2\cos(2x) - 4c_3\sin(2x) \quad .$$

Applying the initial conditions:

$$4 = y(0) = c_1 + c_2\cos(2\cdot 0) + c_3\sin(2\cdot 0) = c_1 + c_2 \quad ,$$

$$6 = y'(0) = 0 - 2c_2\sin(2\cdot 0) + 2c_3\cos(2\cdot 0) = 2c_3 \quad ,$$

and

$$8 = y''(0) = 0 - 4c_2\cos(2\cdot 0) - 4c_3\sin(2\cdot 0) = -4c_2 \quad .$$

Hence,

$$c_2 = \frac{8}{-4} = -2 \quad , \quad c_3 = \frac{6}{2} = 3$$

and

$$c_1 = 4 - c_2 = 4 - (-2) = 6 \quad .$$

With these values, formula $(\star)$ becomes

$$y(x) = 6 - 2\cos(2x) + 3\sin(2x) \quad .$$

17.3 c. Finding the general solution:

$$0 = r^4 + 26r^2 + 25 = \left(r^2\right)^2 + 26\left(r^2\right) + 25$$

$$\hookrightarrow \qquad 0 = \left(r^2+1\right)\left(r^2+25\right)$$

$$\hookrightarrow \qquad r = \pm\sqrt{-1} = \pm i \quad \text{and} \quad r = \pm\sqrt{25} = \pm 5i$$

$$\hookrightarrow \qquad y(x) = c_1\cos(x) + c_2\sin(x) + c_3\cos(5x) + c_4\sin(5x) \quad . \qquad (\star)$$

Computing the derivatives that will be needed:

$$y'(x) = -c_1\sin(x) + c_2\cos(x) - 5c_3\sin(5x) + 5c_4\cos(5x) \quad ,$$

$$y''(x) = -c_1\cos(x) - c_2\sin(x) - 25c_3\cos(5x) - 25c_4\sin(5x) \quad ,$$

and

$$y'''(x) = c_1 \sin(x) - c_2 \cos(x) + 125c_3 \sin(5x) - 125c_4 \cos(5x)$$

Applying the initial conditions:

$$\begin{aligned} 6 &= y(0) \\ &= c_1 \cos(0) + c_2 \sin(0) + c_3 \cos(5 \cdot 0) + c_4 \sin(5 \cdot 0) \\ &= c_1 + c_3 \quad , \end{aligned}$$

$$\begin{aligned} -28 &= y'(0) \\ &= -c_1 \sin(0) + c_2 \cos(0) - 5c_3 \sin(5 \cdot 0) + 5c_4 \cos(5 \cdot 0) \\ &= c_2 + 5c_4 \quad , \end{aligned}$$

$$\begin{aligned} -102 &= y''(0) \\ &= -c_1 \cos(0) - c_2 \sin(0) - 25c_3 \cos(5 \cdot 0) - 25c_4 \sin(5 \cdot 0) \\ &= -c_1 - 25c_3 \quad , \end{aligned}$$

and

$$\begin{aligned} 622 &= y'''(0) \\ &= c_1 \sin(0) - c_2 \cos(0) + 125c_3 \sin(5 \cdot 0) - 125c_4 \cos(5x) \\ &= c_2 - 125c_4 \quad . \end{aligned}$$

Solving the above for c_1 and c_3:

$$6 = c_1 + c_3 \quad \text{and} \quad -102 = -c_1 - 25c_3$$

$$\hookrightarrow \quad c_1 = 6 - c_3 \quad \text{and} \quad -102 = -[6 - c_3] - 25c_3$$

$$\hookrightarrow \quad c_1 = 6 - c_3 \quad \text{and} \quad c_3 = \frac{102 - 6}{24} = 4$$

$$\hookrightarrow \quad c_1 = 6 - 4 = 2 \quad \text{and} \quad c_3 = 4 \quad .$$

Then for c_2 and c_4:

$$-28 = c_2 + 5c_4 \quad \text{and} \quad 622 = c_2 - 125c_4$$

$$\hookrightarrow \quad c_2 = -28 - 5c_4 \quad \text{and} \quad 622 = [-28 - 5c_4] - 125c_4$$

$$\hookrightarrow \quad c_2 = -28 - 5c_4 \quad \text{and} \quad c_4 = \frac{622 + 28}{-130} = -5$$

$$\hookrightarrow \quad c_2 = -28 - 5(-5) = -3 \quad \text{and} \quad c_4 = -5 \quad .$$

Plugging these values back into formula ($\star$) for y then gives our answer,

$$y(x) = 2\cos(x) - 3\sin(x) + 4\cos(5x) - 5\sin(5x) \quad .$$

17.4 a. $r^3 - 8 = 0 \quad \rightarrowtail \quad r = \sqrt[3]{8} = 2 \quad .$

So $r = 2$ is one root of the characteristic polynomial, and $r - 2$ is one factor. Dividing out that factor,

$$\begin{array}{r|l} & r^2 + 2r + 4 \\ \hline r - 2 & r^3 \qquad\qquad - 8 \\ & -r^3 + 2r^2 \\ \hline & 2r^2 \\ & -2r^2 + 4r \\ \hline & 4r - 8 \\ & -4r + 8 \\ \hline & 0 \end{array} \quad .$$

Continuing:

$$0 = r^3 - 8 = (r-2)\left(r^2 + 2r + 4\right)$$

$$\hookrightarrow \quad r = 2 \quad \text{and} \quad r^2 + 2r + 4 = 0$$

$$\hookrightarrow \quad r = 2 \quad \text{and} \quad r = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 4}}{2} = -1 \pm \sqrt{3}$$

$$\hookrightarrow \quad y(x) = c_1 e^{2x} + c_2 e^{-x} \cos\left(\sqrt{3}x\right) + c_3 e^{-x} \sin\left(\sqrt{3}x\right) \quad .$$

17.4 c. The characteristic polynomial is

$$p(r) = r^6 - 3r^4 + 3r^2 - 1 = \left(r^2\right)^3 - 3\left(r^2\right)^2 + 3\left(r^2\right) - 1 \quad .$$

Testing $r = \pm 1$, we have

$$p(1) = \left(1^2\right)^3 - 3\left(1^2\right)^2 + 3\left(1^2\right) - 1 = 1 - 3 + 3 - 1 = 0 \quad ,$$

and, since $(-1)^2 = 1$,

$$p(-1) = (1)^3 - 3(1)^2 + 3(1) - 1 = 1 - 3 + 3 - 1 = 0 \quad .$$

So, both $r = 1$ and $r = -1$ roots, and both $r - 1$ and $r + 1$ are factors. Dividing out $(r-1)(r+1) = r^2 - 1$ from the characteristic polynomial gives

$$\begin{array}{r|l} & r^4 - 2r^2 + 1 \\ \hline r^2 - 1 & r^6 - 3r^4 + 3r^2 - 1 \\ & -r^6 + r^4 \\ \hline & -2r^4 + 3r^2 \\ & 2r^4 - 2r^2 \\ \hline & r^2 - 1 \\ & -r^2 + 1 \\ \hline & 0 \end{array} \quad .$$

Using this, we can write out the characteristic equation as follows:

$$\begin{aligned} 0 &= r^6 - 3r^4 + 3r^2 - 1 \\ &= (r+1)(r-1)\left(r^4 - 2r^2 + 1\right) \\ &= (r+1)(r-1)\left(r^2 - 1\right)^2 \\ &= (r+1)(r-1)\left([r+1][r-1]\right)^2 = (r+1)^3(r-1)^3 \quad . \end{aligned}$$

This means $r = -1$ and $r = 1$ are each roots of multiplicity 3. Thus,

$$y(x) = \left[c_1 + c_2x + c_3x^2\right]e^{-x} + \left[c_4 + c_5x + c_6x^2\right]e^{x} \quad .$$