Chapter 17: Arbitrary Homogeneous Linear Equations with Constant Coefficients

17.1 a. Writing out and factoring the characteristic equation:

\[ 0 = r^4 - 4r^3 = r^3(r - 4) = (r - 0)^3(r - 4) \]

\[ \iff r = 0 \text{ (mult. 3)} \quad \text{and} \quad r = 4 \text{ (mult. 3)} \]

So the corresponding fundamental set of solutions is

\[ \left\{ e^{0x}, xe^{0x}, x^2e^{0x}, e^{4x} \right\} = \left\{ 1, x, x^2, e^{4x} \right\} , \]

and the general solution is

\[ y(x) = c_1 \cdot 1 + c_2x + c_3x^2 + c_4e^{4x} . \]

17.1 c. Starting with the observation that the powers in the characteristic polynomial are all even:

\[ 0 = r^4 - 34r^2 + 225 = \left( r^2 \right)^2 - 34 \left( r^2 \right) + 225 \]

\[ \iff r^2 = \frac{-(-34) \pm \sqrt{(-34)^2 - 4(225)}}{2} = 17 \pm 8 \]

\[ \iff r^2 = 9 \quad \text{and} \quad r^2 = 25 \]

\[ \iff r = \pm\sqrt{9} = \pm 3 \quad \text{and} \quad r = \pm\sqrt{25} = \pm 5 . \]

Since we have four different roots for our fourth-degree polynomial, each root must be simple (i.e., each has multiplicity 1), and so,

\[ y(x) = c_1e^{3x} + c_2e^{-3x} + c_3e^{5x} + c_4e^{-5x} . \]

17.1 e.

\[ 0 = r^4 - 18r^2 + 81 \]

\[ = \left( r^2 \right)^2 - 18 \left( r^2 \right) + 81 \]

\[ = \left( r^2 - 9 \right)^2 \]

\[ = \left( (r - 3)(r + 3) \right)^2 = (r - 3)^2(r - [-3])^2 . \]

So,

\[ y(x) = c_1e^{3x} + c_2xe^{3x} + c_3e^{-3x} + c_4xe^{-3x} \]

\[ = [c_1 + c_2x]e^{3x} + [c_3 + c_4x]e^{-3x} . \]

17.2 a. Here, \( p(x) = r^3 - r^2 + r - 1 \).

Trying \( r = 1 \):

\[ p(1) = 1^3 - 1^2 + 1 - 1 = 1 - 1 + 1 - 1 = 0 . \]
So one root is \( r = 1 \), and one factor of \( p \) is \( r - 1 \). Dividing \( p(r) \) by \( r - 1 \):
\[
\begin{align*}
\frac{r^2 + 1}{r - 1} \cdot \frac{r^3 - r^2 + r - 1}{r^3 + r^2} = \frac{r - 1}{-r + 1} = 0
\end{align*}
\]
Hence, \( p(r) = (r - 1)(r^2 + 1) \).

Using this to find the solutions to the characteristic equation and the corresponding general solution to the differential equation:
\[
0 = r^3 - r^2 + r - 1 = (r - 1)(r^2 + 1)
\]
\[\mapsto r = 1 \quad \text{or} \quad r = \pm \sqrt{-1} = \pm i
\]
\[\mapsto y(x) = c_1 e^x + c_2 \cos(x) + c_3 \sin(x)
\]

17.2 c. Here, \( p(x) = r^3 - 8r^2 + 37r - 50 \).

Trying \( r = 1 \):
\[
p(1) = 1^3 - 8 \cdot 1^2 + 37 \cdot 1 - 50 = 1 - 8 + 37 - 50 = -20
\]
Since \( p(1) \neq 0 \), \( r = 1 \) is not a root of \( p \).

Trying \( r = 2 \):
\[
p(1) = 2^3 - 8 \cdot 2^2 + 37 \cdot 2 - 50 = 8 - 32 + 74 - 50 = 0.
\]
So one root is \( r = 2 \), and one factor of \( p \) is \( r - 2 \). Dividing \( p(r) \) by \( r - 2 \):
\[
\begin{align*}
\frac{r^2 - 6r + 25}{r - 2} \cdot \frac{r^3 - 8r^2 + 37r - 50}{r^3 + 2r^2} = \frac{-6r^2 + 37}{6r^2 - 12r} = \frac{25r - 50}{-25r + 50} = 0
\end{align*}
\]
Hence, \( p(r) = (r - 2)(r^2 - 6r + 25) \).

Using this to find the solutions to the characteristic equation and the corresponding general solution to the differential equation:
\[
0 = r^3 - 8r^2 + 37r - 50 = (r - 2)(r^2 - 6r + 25)
\]
\[\mapsto r = 2 \quad \text{and} \quad r = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 25}}{2} = 3 \pm 4i
\]
\[\mapsto y(x) = c_1 e^{2x} + c_2 e^{3x} \cos(4x) + c_3 e^{3x} \sin(4x)
\]
17.3 a. Finding the general solution:

\[ 0 = r^3 + 4r = r \left( r^2 + 4 \right) \]

\[ \iff \quad r = 0 \quad \text{and} \quad r = \pm \sqrt{4} = \pm 2i \]

\[ \iff \quad y(x) = c_1 e^{0x} + c_2 \cos(2x) + c_3 \sin(2x) . \]

Simplifying and differentiating, we have

\[ y(x) = c_1 + c_2 \cos(2x) + c_3 \sin(2x) , \quad (\ast) \]

\[ y'(x) = 0 - 2c_2 \sin(2x) + 2c_3 \cos(2x) , \]

and

\[ y''(x) = 0 - 4c_2 \cos(2x) - 4c_3 \sin(2x) . \]

Applying the initial conditions:

\[ 4 = y(0) = c_1 + c_2 \cos(2 \cdot 0) + c_3 \sin(2 \cdot 0) = c_1 + c_2 , \]

\[ 6 = y'(0) = 0 - 2c_2 \sin(2 \cdot 0) + 2c_3 \cos(2 \cdot 0) = 2c_3 , \]

and

\[ 8 = y''(0) = 0 - 4c_2 \cos(2 \cdot 0) - 4c_3 \sin(2 \cdot 0) = -4c_2 . \]

Hence,

\[ c_2 = \frac{8}{-4} = -2 , \quad c_3 = \frac{6}{2} = 3 \]

and

\[ c_1 = 4 - c_2 = 4 - (-2) = 6 . \]

With these values, formula (\ast) becomes

\[ y(x) = 6 - 2 \cos(2x) + 3 \sin(2x) . \]

17.3 c. Finding the general solution:

\[ 0 = r^4 + 26r^2 + 25 = \left( r^2 \right)^2 + 26 \left( r^2 \right) + 25 \]

\[ \iff \quad 0 = \left( r^2 + 1 \right) \left( r^2 + 25 \right) \]

\[ \iff \quad r = \pm \sqrt{-1} = \pm i \quad \text{and} \quad r = \pm \sqrt{25} = \pm 5i \]

\[ \iff \quad y(x) = c_1 \cos(x) + c_2 \sin(x) + c_3 \cos(5x) + c_4 \sin(5x) . \quad (\ast) \]

Computing the derivatives that will be needed:

\[ y'(x) = -c_1 \sin(x) + c_2 \cos(x) - 5c_3 \sin(5x) + 5c_4 \cos(5x) , \]

\[ y''(x) = -c_1 \cos(x) - c_2 \sin(x) - 25c_3 \cos(5x) - 25c_4 \sin(5x) , \]
and

\[ y'''(x) = c_1 \sin(x) - c_2 \cos(x) + 125c_3 \sin(5x) - 125c_4 \cos(5x) \]

Applying the initial conditions:

\[ 6 = y(0) = c_1 \cos(0) + c_2 \sin(0) + c_3 \cos(5 \cdot 0) + c_4 \sin(5 \cdot 0) = c_1 + c_3 \]

\[ -28 = y'(0) = -c_1 \sin(0) + c_2 \cos(0) - 5c_3 \sin(5 \cdot 0) + 5c_4 \cos(5 \cdot 0) = c_2 + 5c_4 \]

\[ -102 = y''(0) = -c_1 \cos(0) - c_2 \sin(0) - 25c_3 \cos(5 \cdot 0) - 25c_4 \sin(5 \cdot 0) = -c_1 - 25c_3 \]

and

\[ 622 = y'''(0) = c_1 \sin(0) - c_2 \cos(0) + 125c_3 \sin(5 \cdot 0) - 125c_4 \cos(5x) = c_2 - 125c_4 \]

Solving the above for \( c_1 \) and \( c_3 \):

\[ 6 = c_1 + c_3 \quad \text{and} \quad -102 = -c_1 - 25c_3 \]

\[ \leftrightarrow \quad c_1 = 6 - c_3 \quad \text{and} \quad -102 = -[6 - c_3] - 25c_3 \]

\[ c_1 = 6 - c_3 \quad \text{and} \quad c_3 = \frac{102 - 6}{24} = 4 \]

\[ c_1 = 6 - 4 = 2 \quad \text{and} \quad c_3 = 4 \]

Then for \( c_2 \) and \( c_4 \):

\[ -28 = c_2 + 5c_4 \quad \text{and} \quad 622 = c_2 - 125c_4 \]

\[ \leftrightarrow \quad c_2 = -28 - 5c_4 \quad \text{and} \quad 622 = [-28 - 5c_4] - 125c_4 \]

\[ c_2 = -28 - 5c_4 \quad \text{and} \quad c_4 = \frac{622 + 28}{-130} = -5 \]

\[ c_2 = -28 + 5(-5) = -3 \quad \text{and} \quad c_4 = -5 \]

Plugging these values back into formula (\( \ast \)) for \( y \) then gives our answer,

\[ y(x) = 2\cos(x) - 3\sin(x) + 4\cos(5x) - 5\sin(5x) \]
17.4 a. \( r^3 - 8 = 0 \quad \Rightarrow \quad r = \sqrt[3]{8} = 2 \).

So \( r = 2 \) is one root of the characteristic polynomial, and \( r - 2 \) is one factor. Dividing out that factor,

\[
\begin{array}{c|c|c}
 & r^2 + 2r + 4 & -8 \\
r - 2) & r^3 & -8 \\
& -r^3 + 2r^2 & \\
& 2r^2 & 4r - 8 \\
& -2r^2 + 4r & -4r + 8 \\
& & 0 \\
\end{array}
\]

Continuing:

\[
0 = r^3 - 8 = (r - 2) \left( r^2 + 2r + 4 \right)
\]

\( \Rightarrow \quad r = 2 \quad \text{and} \quad r^2 + 2r + 4 = 0 \)

\( \Rightarrow \quad r = 2 \quad \text{and} \quad r = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 4}}{2} = -1 \pm \sqrt{3} \)

\( \Rightarrow \quad y(x) = c_1 e^{2x} + c_2 e^{-x} \cos(\sqrt{3}x) + c_3 e^{-x} \sin(\sqrt{3}x) \).

17.4 c. The characteristic polynomial is

\[
p(r) = r^6 - 3r^4 + 3r^2 - 1 = \left( r^2 \right)^3 - 3 \left( r^2 \right)^2 + 3 \left( r^2 \right) - 1 \]

Testing \( r = \pm 1 \), we have

\[
p(1) = \left( 1^2 \right)^3 - 3 \left( 1^2 \right)^2 + 3 \left( 1^2 \right) - 1 = 1 - 3 + 3 - 1 = 0 \]

and, since \((-1)^2 = 1\),

\[
p(-1) = (1)^3 - 3(1)^2 + 3(1) - 1 = 1 - 3 + 3 - 1 = 0 \]

So, both \( r = 1 \) and \( r = -1 \) roots, and both \( r - 1 \) and \( r + 1 \) are factors. Dividing out \((r - 1)(r + 1) = r^2 - 1\) from the characteristic polynomial gives

\[
\begin{array}{c|c|c|c|c|c}
 & r^4 - 2r^2 + 1 & \ \\
r^2 - 1) & r^6 - 3r^4 + 3r^2 - 1 & -r^6 + r^4 & -2r^4 + 3r^2 & 2r^4 - 2r^2 & r^2 - 1 \\
& & -r^6 + r^4 & -2r^4 + 3r^2 & 2r^4 - 2r^2 & -r^2 + 1 \\
& & & & & 0 \\
\end{array}
\]
Using this, we can write out the characteristic equation as follows:

\[ 0 = r^6 - 3r^4 + 3r^2 - 1 \]
\[ = (r + 1)(r - 1) \left( r^4 - 2r^2 + 1 \right) \]
\[ = (r + 1)(r - 1) \left( r^2 - 1 \right)^2 \]
\[ = (r + 1)(r - 1) \left( (r + 1)(r - 1) \right)^2 = (r + 1)^3(r - 1)^3. \]

This means \( r = -1 \) and \( r = 1 \) are each roots of multiplicity 3. Thus,

\[ y(x) = \left[ c_1 + c_2 x + c_3 x^2 \right] e^{-x} + \left[ c_4 + c_5 x + c_6 x^2 \right] e^x. \]