Chapter 16: Springs: Part I

16.1 a.
$$\kappa = \frac{|F_0|}{|y_0|} = \frac{2\ (\text{kg}\cdot\text{meter/sec}^2)}{1/2\ (\text{meter})} = 4 \quad \left(\frac{\text{kg}}{\text{sec}^2}\right) \quad .$$

16.1 b.
$$\omega_0 = \sqrt{\frac{\kappa}{m}} = \sqrt{\frac{4}{16}} = \frac{1}{2} \quad \left(\text{sec}^{-1}\right) \quad ,$$
$$p_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{1/2} = 4\pi \quad (\text{sec})$$
and
$$\nu_0 = \frac{1}{p_0} = \frac{1}{4\pi} \quad \left(\text{sec}^{-1}\right) \quad .$$

16.1 c. For each of the following, because this is an undamped system with $\omega_0 = 1/2$,
$$y(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) = c_1\cos\left(\frac{1}{2}t\right) + c_2\sin\left(\frac{1}{2}t\right) \quad ,$$
and
$$y'(t) = -\frac{1}{2}c_1\sin\left(\frac{1}{2}t\right) + \frac{1}{2}c_2\cos\left(\frac{1}{2}t\right) \quad .$$

Since we will be solving several initial-value problems with this y at $t = 0$, let's go ahead and observe that
$$y(0) = c_1\cos\left(\frac{1}{2}\cdot 0\right) + c_2\sin\left(\frac{1}{2}\cdot 0\right) = c_1 \qquad (\star.1)$$
and
$$y'(0) = -\frac{1}{2}c_1\sin\left(\frac{1}{2}\cdot 0\right) + \frac{1}{2}c_2\cos\left(\frac{1}{2}\cdot 0\right) = \frac{1}{2}c_2 \quad . \qquad (\star.2)$$

16.1 c i. Applying the initial conditions to equation set ($\star$), above, yields
$$2 = y(0) = c_1 \qquad \text{and} \qquad 0 = y'(0) = \frac{1}{2}c_2 \quad .$$

So $c_1 = 2$ and $c_2 = 2\cdot 0 = 0$. Hence,
$$y(t) = c_1\cos\left(\frac{1}{2}t\right) + c_2\sin\left(\frac{1}{2}t\right) = 2\cos\left(\frac{1}{2}t\right) \quad ,$$
$$A = \sqrt{(c_1)^2 + (c_2)^2} = \sqrt{2^2 + 0^2} = 2 \quad ,$$
and we have both
$$\cos(\phi) = \frac{c_1}{A} = \frac{2}{2} = 1 \qquad \text{and} \qquad \sin(\phi) = \frac{c_2}{A} = \frac{0}{2} = 0 \quad , \qquad (\star)$$
which, clearly, means $\phi = 0$ (in this case, we only needed to know that $\cos(\phi) = 1$ and $0 \le \phi < 2\pi$). Thus, we also have
$$y(x) = A\cos(\omega_0 t - \phi) = 2\cos\left(\frac{1}{2}t\right) \quad .$$

16.1 c ii. Applying the initial conditions to equation set (⋆), above, yields

$$0 = y(0) = c_1 \quad \text{and} \quad 2 = y'(0) = \frac{1}{2}c_2 \quad .$$

So $c_1 = 0$ and $c_2 = 2 \cdot 2 = 4$. Hence,

$$y = c_1 \cos\left(\frac{1}{2}t\right) + c_2 \sin\left(\frac{1}{2}t\right) = 4 \sin\left(\frac{1}{2}t\right) \quad ,$$

$$A = \sqrt{(c_1)^2 + (c_2)^2} = \sqrt{0^2 + (4)^2} = 4 \quad ,$$

and we have both

$$\cos(\phi) = \frac{c_1}{A} = \frac{0}{4} = 0 \quad \text{and} \quad \sin(\phi) = \frac{c_2}{A} = \frac{4}{4} = 1 \quad , \qquad (\star)$$

which, clearly, means $\phi = \frac{\pi}{2}$ (in this case, we only needed to know that $\sin(\phi) = 1$ and $0 \leq \phi < 2\pi$). Thus, we also have

$$y(x) = A\cos(\omega_0 t - \phi) = 4\cos\left(\frac{1}{2}t - \frac{\pi}{2}\right) \quad .$$

16.1 c iii. Applying the initial conditions to equation set (⋆), above, yields

$$0 = y(0) = c_1 \quad \text{and} \quad -2 = y'(0) = \frac{1}{2}c_2 \quad .$$

So $c_1 = 0$ and $c_2 = -2 \cdot 2 = -4$. Hence,

$$y = c_1 \cos\left(\frac{1}{2}t\right) + c_2 \sin\left(\frac{1}{2}t\right) = -4 \sin\left(\frac{1}{2}t\right) \quad ,$$

$$A = \sqrt{(c_1)^2 + (c_2)^2} = \sqrt{0^2 + (-4)^2} = 4 \quad ,$$

and we have both

$$\cos(\phi) = \frac{c_1}{A} = \frac{0}{4} = 0 \quad \text{and} \quad \sin(\phi) = \frac{c_2}{A} = \frac{-4}{4} = -1 \quad , \qquad (\star)$$

which, clearly, means $\phi = \frac{3\pi}{2}$ (in this case, we only needed to know that $\sin(\phi) = -1$ and $0 \leq \phi < 2\pi$). Thus, we also have

$$y(x) = A\cos(\omega_0 t - \phi) = 4\cos\left(\frac{1}{2}t - \frac{3\pi}{2}\right) \quad .$$

16.1 c iv. Applying the initial conditions to equation set (⋆), above, yields

$$2 = y(0) = c_1 \quad \text{and} \quad \sqrt{3} = y'(0) = \frac{1}{2}c_2 \quad .$$

So $c_1 = 2$ and $c_2 = 2\sqrt{3}$. Hence,

$$y = c_1 \cos\left(\frac{1}{2}t\right) + c_2 \sin\left(2\frac{1}{2}t\right) = 2\cos\left(\frac{1}{2}t\right) + 2\sqrt{3}\sin\left(\frac{1}{2}t\right) \quad ,$$

$$A = \sqrt{(c_1)^2 + (c_2)^2} = \sqrt{2^2 + \left(\sqrt{3}\right)^2} = 4 \quad ,$$

and we have both

$$\cos(\phi) = \frac{c_1}{A} = \frac{2}{4} = \frac{1}{2} \qquad \text{and} \qquad \sin(\phi) = \frac{c_2}{A} = \frac{2\sqrt{3}}{4} = \frac{1}{2}\sqrt{3} \quad . \qquad (\star\star)$$

Now, keeping in mind that $0 \leq \phi < 2\pi$,

$$\cos(\phi) = \frac{1}{2} \quad \rightarrowtail \quad \phi = \frac{\pi}{3} \quad \text{or} \quad \phi = \frac{5\pi}{3} \quad ,$$

and

$$\sin(\phi) = \frac{1}{2}\sqrt{3} \quad \rightarrowtail \quad \phi = \frac{\pi}{3} \quad \text{or} \quad \phi = \frac{2\pi}{3} \quad .$$

Hence, for $(\star\star)$ to hold, we must have $\phi = \frac{\pi}{3}$. Thus, we also have

$$y(x) = A\cos(\omega_0 t - \phi) = 4\cos\left(\frac{1}{2}t - \frac{\pi}{3}\right) \quad .$$

16.3. For the given system,

$$y(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)$$

and

$$y'(t) = -c_1\omega_0\sin(\omega_0 t) + c_2\omega_0\cos(\omega_0 t) \quad .$$

Plugging in the given initial conditions, we have

$$y_0 = y(0) = c_1\cos(\omega_0 0) + c_2\sin(\omega_0 0) = c_1$$

and

$$v_0 = y'(0) = -c_1\omega_0\sin(\omega_0 0) + c_2\omega_0\cos(\omega_0 0) = c_2\omega_0 \quad .$$

So $c_1 = y_0$ and $c_2 = \frac{v_0}{\omega_0}$. Thus,

$$A = \sqrt{(c_1)^2 + (c_2)^2} = \sqrt{{y_0}^2 + \left[\frac{v_0}{\omega_0}\right]^2}$$

and

$$\tan(\phi) = \frac{\sin(\phi)}{\cos(\phi)} = \frac{c_2/A}{c_1/A} = \frac{c_2}{c_1} = \frac{v_0/\omega_0}{y_0} = \frac{v_0}{y_0\omega_0} \quad .$$

16.4. These are based on the fact that $\omega_0 = \sqrt{\frac{\kappa}{m}}$. Solving this for κ , we have $\kappa = (\omega_0)^2 m$. Since, for this set of problems, $p_0 = 3$ and $\omega_0 = \frac{2\pi}{p_0}$, we have

$$\kappa = (\omega_0)^2 m = \left(\frac{2\pi}{p_0}\right)^2 m = \left(\frac{2\pi}{3}\right)^2 m = \frac{4\pi^2}{9}m \quad .$$

16.4 a. $\kappa = \frac{4\pi^2}{9}m = \frac{4\pi^2}{9}\cdot 1 = \frac{4\pi^2}{9}$.

16.4 b. $\kappa = \frac{4\pi^2}{9}m = \frac{4\pi^2}{9}\cdot 2 = \frac{8\pi^2}{9}$.

16.4 c. $\kappa = \frac{4\pi^2}{9}m = \frac{4\pi^2}{9} \cdot \frac{1}{2} = \frac{2\pi^2}{9}$.

16.6 a. We have

$$\gamma = 4 \quad \text{and} \quad 2\sqrt{\kappa m} = 2\sqrt{37 \cdot 4} > 2\sqrt{36 \cdot 4} = 24 > 4 = \gamma \quad .$$

So $0 < \gamma < 2\sqrt{\kappa m}$, which means we classify this system as underdamped.

16.6 b. Since $r_\pm = \alpha \pm i\omega$ are the solutions to the characteristic equation of system's differential equation:

$$mr^2 + \gamma r + \kappa = 0 \quad \rightarrowtail \quad 4r^2 + 4r + 37 = 0$$

$$\hookrightarrow \quad r = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 4 \cdot 37}}{2 \cdot 4} = -\frac{1}{2} \pm 3i$$

$$\hookrightarrow \quad \alpha = -\frac{1}{2} \quad \text{and} \quad \omega = 3$$

$$\hookrightarrow \quad p = \frac{2\pi}{\omega} = \frac{2\pi}{3} \quad \text{and} \quad \nu = \frac{\omega}{2\pi} = \frac{3}{2\pi} \quad .$$

16.6 c. For each of the following, because this is an underdamped system with $\alpha = {}^1\!/_2$ and $\omega = 3$,

$$y(t) = e^{-\alpha t}\left[c_1 \cos(\omega t) + c_2 \sin(\omega t)\right] = e^{-t/2}\left[c_1 \cos(3t) + c_2 \sin(3t)\right] \quad ,$$

and

$$\begin{aligned} y'(t) &= -\frac{1}{2}e^{-t/2}\left[c_1 \cos(3t) + c_2 \sin(3t)\right] \\ &\quad + e^{-t/2}\left[-3c_1 \sin(3t) + 3c_2 \cos(3t)\right] \\ &= e^{-t/2}\left\{c_1\left[-\frac{1}{2}\cos(3t) - \sin(3t)\right] + c_2\left[3\cos(3t) - \frac{1}{2}\sin(3t)\right]\right\} \quad . \end{aligned}$$

Since we will be solving several initial-value problems with this y at $t = 0$, let's go ahead and observe that

$$y(0) = e^{-0/2}\left[c_1 \cos(3 \cdot 0) + c_2 \sin(3 \cdot 0)\right] = c_1 \qquad (\star.1)$$

and

$$\begin{aligned} y'(0) &= e^{-0/2}\left\{c_1\left[-\frac{1}{2}\cos(3 \cdot 0) - \sin(3 \cdot 0)\right] + c_2\left[3\cos(3 \cdot 0) - \frac{1}{2}\sin(3 \cdot 0)\right]\right\} \\ &= -\frac{1}{2}c_1 + 3c_2 \quad . \qquad (\star.2) \end{aligned}$$

16.6 c i. Applying the initial conditions to equation set ($\star$), above, yields

$$1 = y(0) = c_1 \quad \text{and} \quad 0 = y'(0) = -\frac{1}{2}c_1 + 3c_2 \quad .$$

So $c_1 = 1$ and $c_2 = \frac{1}{2 \cdot 3}c_1 = \frac{1}{6} \cdot 1 = \frac{1}{6}$. Hence,

$$y = e^{-t/2}\left[c_1 \cos(3t) + c_2 \sin(3t)\right] = e^{-t/2}\left[\cos(3t) + \frac{1}{6}\sin(3t)\right]$$

and

$$A = \sqrt{(c_1)^2 + (c_2)^2} = \sqrt{1^2 + \left(\frac{1}{6}\right)^2} = \frac{1}{6}\sqrt{37} \quad .$$

16.6 c ii. Applying the initial conditions to equation set (⋆), above, yields

$$4 = y(0) = c_1 \qquad \text{and} \qquad -2 = y'(0) = -\frac{1}{2}c_1 + 3c_2 \quad .$$

So $c_1 = 4$ and $c_2 = \frac{1}{3}\left[-2 + \frac{1}{2}c_1\right] = \frac{1}{3}\left[-2 + \frac{1}{2} \cdot 4\right] = 0$. Hence,

$$y = e^{-t/2}\left[c_1 \cos(3t) + c_2 \sin(3t)\right] = 4e^{-t/2}\cos(3t)$$

and

$$A = \sqrt{(c_1)^2 + (c_2)^2} = \sqrt{4^2 + 0^2} = 4 \quad .$$

16.6 c iii. Applying the initial conditions to equation set (⋆), above, yields

$$0 = y(0) = c_1 \qquad \text{and} \qquad 1 = y'(0) = -\frac{1}{2}c_1 + 3c_2 \quad .$$

So $c_1 = 0$ and $c_2 = \frac{1}{3}\left[1 + \frac{1}{2}c_1\right] = \frac{1}{3}\left[1 + \frac{1}{2} \cdot 0\right] = \frac{1}{3}$. Hence,

$$y = e^{-t/2}\left[c_1 \cos(3t) + c_2 \sin(3t)\right] = \frac{1}{3}e^{-t/2}\sin(3t)$$

and

$$A = \sqrt{(c_1)^2 + (c_2)^2} = \sqrt{0^2 + \left(\frac{1}{3}\right)^2} = \frac{1}{3} \quad .$$

16.6 c iv. Applying the initial conditions to equation set (⋆), above, yields

$$2 = y(0) = c_1 \qquad \text{and} \qquad 2 = y'(0) = -\frac{1}{2}c_1 + 3c_2 \quad .$$

So $c_1 = 2$ and $c_2 = \frac{1}{3}\left[2 + \frac{1}{2}c_1\right] = \frac{1}{3}\left[1 + \frac{1}{2} \cdot 2\right] = \frac{2}{3}$. Hence,

$$y(t) = e^{-t/2}\left[c_1 \cos(3t) + c_2 \sin(3t)\right] = e^{-t/2}\left[2\cos(3t) + \frac{2}{3}\sin(3t)\right]$$

and

$$A = \sqrt{(c_1)^2 + (c_2)^2} = \sqrt{2^2 + \left(\frac{2}{3}\right)^2} = \frac{2}{3}\sqrt{10} \quad .$$

16.8 a. We have

$$\gamma = 4 \qquad \text{and} \qquad 2\sqrt{\kappa m} = 2\sqrt{4 \cdot 1} = 4 = \gamma \quad .$$

So $0 < \gamma = 2\sqrt{\kappa m}$, which means we classify this system as critically damped.

16.8 b. Since this is a critically damped system,

$$y(t) = c_1 e^{-\alpha t} + c_2 t e^{-\alpha t} \qquad \text{where} \qquad \alpha = \sqrt{\frac{\kappa}{m}} = \sqrt{\frac{4}{1}} = 2 \quad .$$

So,

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t} \qquad \text{and} \qquad y'(t) = -2c_1 e^{-2t} + c_2 [1 - 2t] e^{-2t} \quad .$$

Applying the initial conditions, we have

$$2 = y(0) = c_1 e^{-2\cdot 0} + c_2 \cdot 0 e^{-2\cdot 0} = c_1$$

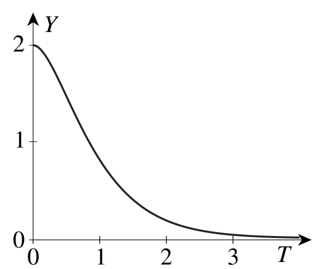
and

$$0 = y'(0) = -2c_1 e^{-2\cdot 0} + c_2 [1 - 2\cdot 0] e^{-2\cdot 0} = -2c_1 + c_2$$

So $c_1 = 2$, $c_2 = 2c_2 = 2 \cdot 2 = 4$, and

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t} = 2e^{-2t} + 2te^{-2t} \quad .$$

The graph of this is



16.8 c. From the exercise above, we know

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t} \qquad \text{and} \qquad y'(t) = -2c_1 e^{-2t} + c_2 [1 - 2t] e^{-2t} \quad .$$

Applying the initial conditions, we have

$$0 = y(0) = c_1 e^{-2\cdot 0} + c_2 \cdot 0 e^{-2\cdot 0} = c_1$$

and

$$2 = y'(0) = -2c_1 e^{-2\cdot 0} + c_2 [1 - 2\cdot 0] e^{-2\cdot 0} = -2c_1 + c_2$$

So $c_1 = 0$, $c_2 = 2 + 2c_2 = 2 + 2 \cdot 0 = 2$, and

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t} = 2te^{-2t} \quad .$$

The graph of this is