Chapter 16: Springs: Part I

16.1 a. \( \kappa = \frac{|F_0|}{|y_0|} = \frac{2 \text{ (kg·meter/} \text{sec}^2)}{1/2 \text{ (meter) }} = 4 \left( \frac{\text{kg}}{\text{sec}^2} \right) . \)

16.1 b. \( \omega_0 = \sqrt{\frac{\kappa}{m}} = \sqrt{\frac{4}{16}} = \frac{1}{2} \left( \text{sec}^{-1} \right) , \)

\( p_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{1/2} = 4\pi \text{ (sec)} \)

and

\( \nu_0 = \frac{1}{p_0} = \frac{1}{4\pi} \left( \text{sec}^{-1} \right) . \)

16.1 c. For each of the following, because this is an undamped system with \( \omega_0 = \frac{1}{2} , \)

\( y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) = c_1 \cos \left( \frac{1}{2} t \right) + c_2 \sin \left( \frac{1}{2} t \right) , \)

and

\( y'(t) = -\frac{1}{2} c_1 \sin \left( \frac{1}{2} t \right) + \frac{1}{2} c_2 \cos \left( \frac{1}{2} t \right) . \)

Since we will be solving several initial-value problems with this \( y \) at \( t = 0 \), let's go ahead and observe that

\( y(0) = c_1 \cos \left( \frac{1}{2} \cdot 0 \right) + c_2 \sin \left( \frac{1}{2} \cdot 0 \right) = c_1 \)  \((*) 1)\)

and

\( y'(0) = -\frac{1}{2} c_1 \sin \left( \frac{1}{2} \cdot 0 \right) + \frac{1}{2} c_2 \cos \left( \frac{1}{2} \cdot 0 \right) = \frac{1}{2} c_2 \) \((*) 2)\)

16.1 c i. Applying the initial conditions to equation set (\(*\)), above, yields

\( 2 = y(0) = c_1 \quad \text{and} \quad 0 = y'(0) = \frac{1}{2} c_2 . \)

So \( c_1 = 2 \) and \( c_2 = 2 \cdot 0 = 0 . \) Hence,

\( y(t) = c_1 \cos \left( \frac{1}{2} t \right) + c_2 \sin \left( \frac{1}{2} t \right) = 2 \cos \left( \frac{1}{2} t \right) , \)

\( A = \sqrt{(c_1)^2 + (c_2)^2} = \sqrt{2^2 + 0^2} = 2 \),

and we have both

\( \cos(\phi) = \frac{c_1}{A} = \frac{2}{2} = 1 \quad \text{and} \quad \sin(\phi) = \frac{c_2}{A} = \frac{0}{2} = 0 \) \((*)\)

which, clearly, means \( \phi = 0 \) (in this case, we only needed to know that \( \cos(\phi) = 1 \) and \( 0 \leq \phi < 2\pi \)). Thus, we also have

\( y(x) = A \cos(\omega_0 t - \phi) = 2 \cos \left( \frac{1}{2} t \right) . \)
16.1 c ii. Applying the initial conditions to equation set (⋆), above, yields

\[ 0 = y(0) = c_1 \quad \text{and} \quad 2 = y'(0) = \frac{1}{2}c_2 \ . \]

So \( c_1 = 0 \) and \( c_2 = 2 \cdot 2 = 4 \). Hence,

\[ y = c_1 \cos\left(\frac{1}{2}t\right) + c_2 \sin\left(\frac{1}{2}t\right) = 4 \sin\left(\frac{1}{2}t\right) \ , \]

\[ A = \sqrt{(c_1)^2 + (c_2)^2} = \sqrt{0^2 + (4)^2} = 4 \ , \]

and we have both

\[ \cos(\phi) = \frac{c_1}{A} = \frac{0}{4} = 0 \quad \text{and} \quad \sin(\phi) = \frac{c_2}{A} = \frac{4}{4} = 1 \ . \quad (⋆) \]

which, clearly, means \( \phi = \frac{\pi}{2} \) (in this case, we only needed to know that \( \sin(\phi) = 1 \) and \( 0 \leq \phi < 2\pi \)). Thus, we also have

\[ y(x) = A \cos(\omega_0 t - \phi) = 4 \cos\left(\frac{1}{2}t - \frac{\pi}{2}\right) \ . \]

16.1 c iii. Applying the initial conditions to equation set (⋆), above, yields

\[ 0 = y(0) = c_1 \quad \text{and} \quad -2 = y'(0) = \frac{1}{2}c_2 \ . \]

So \( c_1 = 0 \) and \( c_2 = -2 \cdot 2 = -4 \). Hence,

\[ y = c_1 \cos\left(\frac{1}{2}t\right) + c_2 \sin\left(\frac{1}{2}t\right) = -4 \sin\left(\frac{1}{2}t\right) \ , \]

\[ A = \sqrt{(c_1)^2 + (c_2)^2} = \sqrt{0^2 + (-4)^2} = 4 \ , \]

and we have both

\[ \cos(\phi) = \frac{c_1}{A} = \frac{0}{4} = 0 \quad \text{and} \quad \sin(\phi) = \frac{c_2}{A} = \frac{-4}{4} = -1 \ . \quad (⋆) \]

which, clearly, means \( \phi = \frac{3\pi}{2} \) (in this case, we only needed to know that \( \sin(\phi) = -1 \) and \( 0 \leq \phi < 2\pi \)). Thus, we also have

\[ y(x) = A \cos(\omega_0 t - \phi) = 4 \cos\left(\frac{1}{2}t - \frac{3\pi}{2}\right) \ . \]

16.1 c iv. Applying the initial conditions to equation set (⋆), above, yields

\[ 2 = y(0) = c_1 \quad \text{and} \quad \sqrt{3} = y'(0) = \frac{1}{2}c_2 \ . \]

So \( c_1 = 2 \) and \( c_2 = 2\sqrt{3} \). Hence,

\[ y = c_1 \cos\left(\frac{1}{2}t\right) + c_2 \sin\left(\frac{1}{2}t\right) = 2 \cos\left(\frac{1}{2}t\right) + 2\sqrt{3} \sin\left(\frac{1}{2}t\right) \ , \]

\[ A = \sqrt{(c_1)^2 + (c_2)^2} = \sqrt{2^2 + (\sqrt{3})^2} = 4 \ , \]
and we have both
\[
\cos(\phi) = \frac{c_1}{A} = \frac{2}{4} = \frac{1}{2} \quad \text{and} \quad \sin(\phi) = \frac{c_2}{A} = \frac{2\sqrt{3}}{4} = \frac{1}{2}\sqrt{3} . \quad (\star\star)
\]
Now, keeping in mind that \(0 \leq \phi < 2\pi\),
\[
\cos(\phi) = \frac{1}{2} \quad \implies \quad \phi = \frac{\pi}{3} \quad \text{or} \quad \phi = \frac{5\pi}{3} ,
\]
and
\[
\sin(\phi) = \frac{1}{2}\sqrt{3} \quad \implies \quad \phi = \frac{\pi}{3} \quad \text{or} \quad \phi = \frac{2\pi}{3} .
\]
Hence, for (\star\star) to hold, we must have \(\phi = \frac{\pi}{3}\). Thus, we also have
\[
y(x) = A \cos(\omega t - \phi) = 4 \cos\left(\frac{1}{2} t - \frac{\pi}{3}\right) .
\]

16.3. For the given system,
\[
y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)
\]
and
\[
y'(t) = -c_1 \omega_0 \sin(\omega_0 t) + c_2 \omega_0 \cos(\omega_0 t) .
\]
Plugging in the given initial conditions, we have
\[
y_0 = y(0) = c_1 \cos(\omega_0 0) + c_2 \sin(\omega_0 0) = c_1
\]
and
\[
v_0 = y'(0) = -c_1 \omega_0 \sin(\omega_0 0) + c_2 \omega_0 \cos(\omega_0 0) = c_2 \omega_0 .
\]
So \(c_1 = y_0\) and \(c_2 = \frac{v_0}{\omega_0}\). Thus,
\[
A = \sqrt{(c_1)^2 + (c_2)^2} = \sqrt{y_0^2 + \left[\frac{v_0}{\omega_0}\right]^2}
\]
and
\[
\tan(\phi) = \frac{\sin(\phi)}{\cos(\phi)} = \frac{c_2 / A}{c_1 / A} = \frac{c_2}{c_1} = \frac{v_0 / \omega_0}{y_0} = \frac{v_0}{y_0 \omega_0} .
\]

16.4. These are based on the fact that \(\omega_0 = \sqrt{\frac{\kappa}{m}}\). Solving this for \(\kappa\), we have \(\kappa = (\omega_0)^2 m\).
Since, for this set of problems, \(p_0 = 3\) and \(\omega_0 = \frac{2\pi}{p_0}\), we have
\[
\kappa = (\omega_0)^2 m = \left(\frac{2\pi}{p_0}\right)^2 m = \left(\frac{2\pi}{3}\right)^2 m = \frac{4\pi^2}{9} m .
\]

16.4a. \(\kappa = \frac{4\pi^2}{9} m = \frac{4\pi^2}{9} \cdot 1 = \frac{4\pi^2}{9} .
\]

16.4b. \(\kappa = \frac{4\pi^2}{9} m = \frac{4\pi^2}{9} \cdot 2 = \frac{8\pi^2}{9} .
\]
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16.4 c. \( \kappa = \frac{4\pi^2}{9} m = \frac{4\pi^2}{9} \cdot \frac{1}{2} = \frac{2\pi^2}{9} \).

16.6 a. We have
\[
y = 4 \quad \text{and} \quad 2\sqrt{\kappa m} = 2\sqrt{37 \cdot 4} > 2\sqrt{36 \cdot 4} = 24 > 4 = y.
\]
So \( 0 < y < 2\sqrt{\kappa m} \), which means we classify this system as underdamped.

16.6 b. Since \( r_{\pm} = \alpha \pm i\omega \) are the solutions to the characteristic equation of system’s differential equation:
\[
mr^2 + \gamma r + \kappa = 0 \quad \implies \quad 4r^2 + 4r + 37 = 0
\]
\[
c \implies \quad r = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 4 \cdot 37}}{2 \cdot 4} = -\frac{1}{2} \pm 3i
\]
\[
c \implies \quad \alpha = -\frac{1}{2} \quad \text{and} \quad \omega = 3
\]
\[
c \implies \quad p = \frac{2\pi}{\omega} = \frac{2\pi}{3} \quad \text{and} \quad v = \frac{\omega}{2\pi} = \frac{3}{2\pi}.
\]

16.6 c. For each of the following, because this is an underdamped system with \( \alpha = \frac{1}{2} \) and \( \omega = 3 \),
\[
y(t) = e^{-\alpha t} \left[ c_1 \cos(\omega t) + c_2 \sin(\omega t) \right] = e^{-t/2} \left[ c_1 \cos(3t) + c_2 \sin(3t) \right].
\]

and
\[
y'(t) = -\frac{1}{2} e^{-t/2} \left[ c_1 \cos(3t) + c_2 \sin(3t) \right]
\]
\[
+ e^{-t/2} \left[ -3c_1 \sin(3t) + 3c_2 \cos(3t) \right]
\]
\[
= e^{-t/2} \left[ c_1 \left(-\frac{1}{2} \cos(3t) - \sin(3t) \right) + c_2 \left[3 \cos(3t) - \frac{1}{2} \sin(3t) \right] \right].
\]
Since we will be solving several initial-value problems with this \( y \) at \( t = 0 \), let’s go ahead and observe that
\[
y(0) = e^{-0/2} \left[ c_1 \cos(0) + c_2 \sin(0) \right] = c_1 \quad \text{(\( \star.1 \))}
\]

and
\[
y'(0) = e^{-0/2} \left[ c_1 \left(-\frac{1}{2} \cos(0) - \sin(0) \right) + c_2 \left[3 \cos(0) - \frac{1}{2} \sin(0) \right] \right]
\]
\[
= -\frac{1}{2} c_1 + 3c_2. \quad \text{(\( \star.2 \))}
\]

16.6 c i. Applying the initial conditions to equation set (\( \star \)), above, yields
\[
1 = y(0) = c_1 \quad \text{and} \quad 0 = y'(0) = -\frac{1}{2} c_1 + 3c_2.
\]
So \( c_1 = 1 \) and \( c_2 = \frac{1}{2} \cdot 3 \cdot c_1 = \frac{1}{2} \cdot 1 = \frac{1}{2} \). Hence,
\[
y = e^{-t/2} \left[ c_1 \cos(3t) + c_2 \sin(3t) \right] = e^{-t/2} \left[ \cos(3t) + \frac{1}{6} \sin(3t) \right].
\]
and

\[ A = \sqrt{(c_1)^2 + (c_2)^2} = \sqrt{1^2 + \left(\frac{1}{6}\right)^2} = \frac{1}{6}\sqrt{37}. \]

**16.6 c ii.** Applying the initial conditions to equation set (●), above, yields

\[ 4 = y(0) = c_1 \quad \text{and} \quad -2 = y'(0) = -\frac{1}{2}c_1 + 3c_2. \]

So \( c_1 = 4 \) and \( c_2 = \frac{1}{3} \left[ -2 + \frac{1}{2}c_1 \right] = \frac{1}{3} \left[ -2 + \frac{1}{2} \cdot 4 \right] = 0. \) Hence,

\[ y = e^{-t/2} \left[ c_1 \cos(3t) + c_2 \sin(3t) \right] = 4e^{-t/2} \cos(3t) \]

and

\[ A = \sqrt{(c_1)^2 + (c_2)^2} = \sqrt{4^2 + 0^2} = 4. \]

**16.6 c iii.** Applying the initial conditions to equation set (●), above, yields

\[ 0 = y(0) = c_1 \quad \text{and} \quad 1 = y'(0) = -\frac{1}{2}c_1 + 3c_2. \]

So \( c_1 = 0 \) and \( c_2 = \frac{1}{3} \left[ 1 + \frac{1}{2}c_1 \right] = \frac{1}{3} \left[ 1 + \frac{1}{2} \cdot 0 \right] = \frac{1}{3}. \) Hence,

\[ y = e^{-t/2} \left[ c_1 \cos(3t) + c_2 \sin(3t) \right] = \frac{1}{3}e^{-t/2} \sin(3t) \]

and

\[ A = \sqrt{(c_1)^2 + (c_2)^2} = \sqrt{0^2 + \left(\frac{1}{3}\right)^2} = \frac{1}{3}. \]

**16.6 c iv.** Applying the initial conditions to equation set (●), above, yields

\[ 2 = y(0) = c_1 \quad \text{and} \quad 2 = y'(0) = -\frac{1}{2}c_1 + 3c_2. \]

So \( c_1 = 2 \) and \( c_2 = \frac{1}{3} \left[ 2 + \frac{1}{2}c_1 \right] = \frac{1}{3} \left[ 2 + \frac{1}{2} \cdot 2 \right] = \frac{2}{3}. \) Hence,

\[ y(t) = e^{-t/2} \left[ c_1 \cos(3t) + c_2 \sin(3t) \right] = e^{-t/2} \left[ 2 \cos(3t) + \frac{2}{3} \sin(3t) \right] \]

and

\[ A = \sqrt{(c_1)^2 + (c_2)^2} = \sqrt{2^2 + \left(\frac{2}{3}\right)^2} = \frac{2}{3}\sqrt{10}. \]

**16.8 a.** We have

\[ \gamma = 4 \quad \text{and} \quad 2\sqrt{\kappa m} = 2\sqrt{4 \cdot 1} = 4 = \gamma. \]

So \( 0 < \gamma = 2\sqrt{\kappa m}, \) which means we classify this system as critically damped.
16.8 b. Since this is a critically damped system, 
\[ y(t) = c_1 e^{-\alpha t} + c_2 t e^{-\alpha t} \]
where \( \alpha = \sqrt{\frac{k}{m}} = \sqrt{\frac{4}{1}} = 2 \). 
So, 
\[ y(t) = c_1 e^{-2t} + c_2 t e^{-2t} \]
and 
\[ y'(t) = -2c_1 e^{-2t} + c_2 [1 - 2t] e^{-2t} . \]
Applying the initial conditions, we have 
\[ 2 = y(0) = c_1 e^{-2 \cdot 0} + c_2 \cdot 0 e^{-2 \cdot 0} = c_1 \]
and 
\[ 0 = y'(0) = -2c_1 e^{-2 \cdot 0} + c_2 [1 - 2 \cdot 0] e^{-2 \cdot 0} = -2c_1 + c_2 \]
So \( c_1 = 2 \), \( c_2 = 2c_2 = 2 \cdot 2 = 4 \), and 
\[ y(t) = c_1 e^{-2t} + c_2 t e^{-2t} = 2e^{-2t} + 2te^{-2t} . \]
The graph of this is

16.8 c. From the exercise above, we know 
\[ y(t) = c_1 e^{-2t} + c_2 t e^{-2t} \]
and 
\[ y'(t) = -2c_1 e^{-2t} + c_2 [1 - 2t] e^{-2t} . \]
Applying the initial conditions, we have 
\[ 0 = y(0) = c_1 e^{-2 \cdot 0} + c_2 \cdot 0 e^{-2 \cdot 0} = c_1 \]
and 
\[ 2 = y'(0) = -2c_1 e^{-2 \cdot 0} + c_2 [1 - 2 \cdot 0] e^{-2 \cdot 0} = -2c_1 + c_2 \]
So \( c_1 = 0 \), \( c_2 = 2 + 2c_2 = 2 + 2 \cdot 0 = 2 \), and 
\[ y(t) = c_1 e^{-2t} + c_2 t e^{-2t} = 2te^{-2t} . \]
The graph of this is