

### Chapter 15: Second-Order Homogeneous Linear Equations with Constant Coefficients

**15.1 a.** The characteristic equation is

$$r^2 - 7r + 10 = 0 \quad ,$$

which factors easily,

$$(r - 2)(r - 5) = 0 \quad .$$

Thus, the solutions to the characteristic equation are  $r = 2$  and  $r = 5$ . The corresponding solutions to the differential equation are then  $e^{2x}$  and  $e^{5x}$ . These form a fundamental set of solutions, and, so, our general solution to the differential equation is

$$y(x) = c_1 e^{2x} + c_2 e^{5x} \quad .$$

**15.1 c.** Writing out and solving the characteristic equation:

$$r^2 - 25 = 0 \quad \rightsquigarrow \quad r^2 = 25 \quad \rightsquigarrow \quad r = \pm\sqrt{25} = \pm 5 \quad .$$

So the solutions to the characteristic equation are  $r = 5$  and  $r = -5$ . The corresponding solutions to the differential equation are then  $e^{5x}$  and  $e^{-5x}$ , and our general solution to the differential equation is

$$y(x) = c_1 e^{5x} + c_2 e^{-5x} \quad .$$

**15.1 e.**

$$4r^2 - 1 = 0 \quad \rightsquigarrow \quad r^2 = \frac{1}{4} \quad \rightsquigarrow \quad r = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}$$

$$\hookrightarrow \quad r = \frac{1}{2} \quad \text{and} \quad r = -\frac{1}{2}$$

$$\hookrightarrow \quad y(x) = c_1 e^{x/2} + c_2 e^{-x/2} \quad .$$

**15.2 a.** Finding the general solution and its derivative:

$$0 = r^2 - 8r + 15 = (r - 3)(r - 5)$$

$$\hookrightarrow \quad r = 3 \quad \text{and} \quad r = 5$$

$$\hookrightarrow \quad y(x) = c_1 e^{3x} + c_2 e^{5x} \quad \text{and} \quad y'(x) = 3c_1 e^{3x} + 5c_2 e^{5x} \quad . \quad (\star)$$

Applying the initial conditions:

$$1 = y(0) = c_1 e^{3 \cdot 0} + c_2 e^{5 \cdot 0} = c_1 + c_2$$

and

$$0 = y'(0) = 3c_1 e^{3 \cdot 0} + 5c_2 e^{5 \cdot 0} = 3c_1 + 5c_2 \quad .$$

Solving for the constants:

$$1 = c_1 + c_2 \quad \text{and} \quad 0 = 3c_1 + 5c_2$$

$$\hookrightarrow \quad 1 = c_1 + c_2 \quad \text{and} \quad c_2 = -\frac{5}{3}c_1$$

$$\hookrightarrow 1 = c_1 - \frac{5}{3}c_1 = -\frac{2}{3}c_1 \quad \text{and} \quad c_2 = -\frac{5}{3}c_1$$

$$\hookrightarrow c_1 = -\frac{3}{2} \quad \text{and} \quad c_2 = -\frac{5}{3} \times \left(-\frac{3}{2}\right) = \frac{5}{2} .$$

Plugging these values into the formula for  $y$  in line (★) then gives the solution,

$$y(x) = -\frac{5}{2}e^{3x} + \frac{4}{2}e^{5x} .$$

**15.2 c.** From solving exercise 15.2 a, we know the differential equation's general solution and its derivative are

$$y(x) = c_1e^{3x} + c_2e^{5x} \quad \text{and} \quad y'(x) = 3c_1e^{3x} + 5c_2e^{5x} . \quad (\star)$$

Applying the initial conditions:

$$5 = y(0) = c_1e^{3 \cdot 0} + c_2e^{5 \cdot 0} = c_1 + c_2$$

and

$$19 = y'(0) = 3c_1e^{3 \cdot 0} + 5c_2e^{5 \cdot 0} = 3c_1 + 5c_2 .$$

Solving for the constants:

$$5 = c_1 + c_2 \quad \text{and} \quad 19 = 3c_1 + 5c_2$$

$$\hookrightarrow c_1 = 5 - c_2 \quad \text{and} \quad 19 = 3(5 - c_2) + 5c_2 = 15 + 2c_2$$

$$\hookrightarrow c_1 = 5 - c_2 \quad \text{and} \quad c_2 = \frac{19 - 15}{2} = 2$$

$$\hookrightarrow c_1 = 5 - 2 = 3 \quad \text{and} \quad c_2 = 2 .$$

Plugging these values into the formula for  $y$  in line (★) then gives the solution,

$$y(x) = 3e^{3x} + 2e^{5x} .$$

**15.2 e.** Finding the general solution and its derivative:

$$0 = r^2 - 9 = (r - 3)(r + 3) = (r - 3)(r - [-3])$$

$$\hookrightarrow r = 3 \quad \text{and} \quad r = -3$$

$$\hookrightarrow y(x) = c_1e^{3x} + c_2e^{-3x} \quad \text{and} \quad y'(x) = 3c_1e^{3x} - 3c_2e^{-3x} . \quad (\star)$$

Applying the initial conditions:

$$0 = y(0) = c_1e^{3 \cdot 0} + c_2e^{-3 \cdot 0} = c_1 + c_2$$

and

$$1 = y'(0) = 3c_1e^{3 \cdot 0} - 3c_2e^{-3 \cdot 0} = 3c_1 - 3c_2 .$$

Solving for the constants:

$$0 = c_1 + c_2 \quad \text{and} \quad 1 = 3c_1 - 3c_2$$

$$\hookrightarrow c_1 = -c_2 \quad \text{and} \quad 1 = 3(-c_2) - 3c_2 = -6c_2$$

$$\hookrightarrow c_1 = -c_2 \quad \text{and} \quad c_2 = -\frac{1}{6}$$

$$\hookrightarrow c_1 = -\left(-\frac{1}{6}\right) = \frac{1}{6} \quad \text{and} \quad c_2 = -\frac{1}{6} .$$

Plugging these values into the formula for  $y$  in line (★) then gives the solution,

$$y(x) = \frac{1}{6}e^{3x} - \frac{1}{6}e^{5x} .$$

**15.3 a.**  $0 = r^2 - 10r + 25 = (r - 5)^2 \rightsquigarrow r = 5 .$

So one solution to the differential equation is  $y_1(x) = e^{5x}$ . As derived in section 15.4, the second solution to the differential equation when the characteristic polynomial only has one root is

$$y_2(x) = xy_1(x) = xe^{5x} .$$

Hence, the general solution to the differential equation is

$$y(x) = c_1e^{5x} + c_2xe^{5x} .$$

**15.3 c.**  $0 = 4r^2 - 4r + 1 = (2r - 1)^2 \rightsquigarrow r = \frac{1}{2}$

$$\hookrightarrow y_1(x) = e^{x/2} \quad \text{and} \quad y_2(x) = xy_1(x) = xe^{x/2}$$

$$\hookrightarrow y(x) = c_1e^{x/2} + c_2xe^{x/2} .$$

**15.4 a.** Finding the general solution and its derivative:

$$0 = r^2 - 8r + 16 = (r - 4)^2 \rightsquigarrow r = 4$$

$$\hookrightarrow y_1(x) = e^{4x} \quad \text{and} \quad y_2(x) = xy_1(x) = xe^{4x}$$

$$\hookrightarrow y(x) = c_1e^{4x} + c_2xe^{4x} \quad (\star)$$

$$\hookrightarrow y'(x) = 4c_1e^{4x} + c_2[e^{4x} + 4xe^{4x}] .$$

Applying the initial conditions:

$$1 = y(0) = c_1e^{4 \cdot 0} + c_2 \cdot 0e^{4 \cdot 0} = c_1$$

and

$$0 = y'(0) = 4c_1e^{4 \cdot 0} + c_2[e^{4 \cdot 0} + 4 \cdot 0e^{4 \cdot 0}] = 4c_1 + c_2 .$$

Solving for the constants and plugging them back into formula (★) for  $y$ :

$$1 = c_1 \quad \text{and} \quad 0 = 4c_1 + c_2 = 4 \cdot 1 + c_2$$

$$\hookrightarrow \quad c_1 = 1 \quad \text{and} \quad c_2 = -4$$

$$\hookrightarrow \quad y(x) = e^{4x} - 4xe^{4x} \quad .$$

**15.4 c.** From solving exercise 15.4 a, we know the differential equation's general solution and its derivative are

$$y(x) = c_1e^{4x} + c_2xe^{4x} \quad \text{and} \quad y'(x) = 4c_1e^{4x} + c_2[e^{4x} + 4xe^{4x}] \quad . \quad (\star)$$

Applying the initial conditions:

$$3 = y(0) = c_1e^{4 \cdot 0} + c_2 \cdot 0e^{4 \cdot 0} = c_1$$

and

$$14 = y'(0) = 4c_1e^{4 \cdot 0} + c_2[e^{4 \cdot 0} + 4 \cdot 0e^{4 \cdot 0}] = 4c_1 + c_2 \quad .$$

Solving for the constants and plugging them back into formula (★) for  $y$ :

$$3 = c_1 \quad \text{and} \quad 14 = 4c_1 + c_2 = 4 \cdot 3 + c_2$$

$$\hookrightarrow \quad c_1 = 3 \quad \text{and} \quad c_2 = 2$$

$$\hookrightarrow \quad y(x) = 3e^{4x} + 2xe^{4x} \quad .$$

**15.4 e.** Finding the general solution and its derivative:

$$0 = 4r^2 + 4r + 1 = (2r + 1)^2$$

$$\hookrightarrow \quad 2r + 1 = 0 \quad \rightsquigarrow \quad r = -\frac{1}{2}$$

$$\hookrightarrow \quad y_1(x) = e^{-x/2} \quad \text{and} \quad y_2(x) = xy_1(x) = xe^{-x/2}$$

$$\hookrightarrow \quad y(x) = c_1e^{-x/2} + c_2xe^{-x/2} \quad (\star)$$

$$\hookrightarrow \quad y'(x) = -\frac{1}{2}c_1e^{-x/2} + c_2\left[e^{-x/2} - \frac{1}{2}xe^{-x/2}\right] \quad .$$

Applying the initial conditions:

$$0 = y(0) = c_1e^{-0/2} + c_2 \cdot 0e^{-0/2} = c_1$$

and

$$1 = y'(0) = -\frac{1}{2}c_1e^{-0/2} + c_2\left[e^{-0/2} - \frac{1}{2} \cdot 0e^{-0/2}\right] = -\frac{1}{2}c_1 + c_2 \quad .$$

Solving for the constants and plugging them back into formula (\*) for  $y$ :

$$0 = c_1 \quad \text{and} \quad 1 = -\frac{1}{2}c_1 + c_2 = -\frac{1}{2} \cdot 0 + c_2$$

$$\hookrightarrow \quad c_1 = 0 \quad \text{and} \quad c_2 = 1$$

$$\hookrightarrow \quad y(x) = 0 \cdot e^{-x/2} + 1 \cdot x e^{-x/2} = x e^{-x/2} .$$

**15.5 a.** The characteristic equation and its solutions:

$$r^2 + 25 = 0 \quad \rightsquigarrow \quad r^2 = -25 \quad \rightsquigarrow \quad r = \pm\sqrt{-25} = \pm 5i .$$

So, two solutions to the differential equation are

$$y_+(x) = e^{i5x} = \cos(5x) + i \sin(5x)$$

and

$$y_-(x) = e^{-i5x} = \cos(5x) - i \sin(5x) .$$

However, because of the  $i$ , these are not real valued. To get corresponding real-valued solutions  $y_1$  and  $y_2$ , we can either set

$$y_1 = y_+ + y_- \quad \text{and} \quad y_2 = y_+ - y_- ,$$

or, equivalently, simply take the real and the imaginary parts of

$$y_+(x) = \cos(5x) + i \sin(5x)$$

for  $y_1$  and  $y_2$ , respectively. Either way, we get

$$y_1(x) = \cos(5x) \quad \text{and} \quad y_2(x) = \sin(5x) .$$

The general solution is then

$$y(x) = Ay_1(x) + By_2(x) = A \cos(5x) + B \sin(5x) .$$

**15.5 c.** The characteristic equation and its solutions:

$$r^2 - 2r + 5 = 0$$

$$\hookrightarrow \quad r = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2 \cdot 1} = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i .$$

So the complex exponential solutions to the differential equation are given by

$$\begin{aligned} y_{\pm}(x) &= e^{rx} = e^{(1 \pm 2i)x} = e^{x \pm i2x} = e^x e^{i2x} \\ &= e^x [\cos(2x) \pm i \sin(2x)] \\ &= e^x \cos(2x) \pm i e^x \sin(2x) . \end{aligned}$$

Taking the real and imaginary parts of  $y_+$ ,

$$y_+(x) = \underbrace{e^x \cos(2x)}_{y_1(x)} + i \underbrace{e^x \sin(2x)}_{y_2(x)} ,$$

gives us the linearly independent pair of real-valued solutions

$$y_1(x) = e^x \cos(2x) \quad \text{and} \quad y_2(x) = e^x \sin(2x) .$$

So the general solution in terms of real-valued functions is

$$y(x) = Ae^x \cos(2x) + Be^x \sin(2x)$$

**15.6 a.**  $r^2 + 16 = 0 \rightsquigarrow r^2 = -16 \rightsquigarrow r = \pm\sqrt{-16} = \pm 4i$  .

The corresponding solutions to the differential equation are then given by

$$y_{\pm}(x) = e^{r_{\pm}x} = e^{\pm i4x} = \underbrace{\cos(4x)}_{y_1(x)} \pm i \underbrace{\sin(4x)}_{y_2(x)} .$$

So, in terms of real-valued functions, the general solution to the differential equation is

$$y(x) = A \cos(4x) + B \sin(4x) . \quad (\star)$$

Its derivative is

$$y'(x) = -4A \sin(4x) + 4B \cos(4x) .$$

Applying the initial conditions, we have

$$\begin{aligned} 1 &= y(0) = A \cos(4 \cdot 0) + B \sin(4 \cdot 0) \\ &= A \cdot 1 + B \cdot 0 = A \end{aligned}$$

and

$$\begin{aligned} 0 &= y'(0) = -4A \sin(4 \cdot 0) + 4B \cos(4 \cdot 0) \\ &= -4A \cdot 0 + 4B \cdot 1 = 4B . \end{aligned}$$

Thus,  $A = 1$ ,  $B = 0$ , and the solution to the initial-value problem is

$$y(x) = A \cos(4x) + B \sin(4x) = \cos(4x) .$$

**15.6 c.** From solving exercise 15.6 a, we know the general solution to the differential equation is

$$y(x) = A \cos(4x) + B \sin(4x) , \quad (\star)$$

and its derivative is

$$y'(x) = -4A \sin(4x) + 4B \cos(4x) .$$

Applying the initial conditions, we have

$$\begin{aligned} 4 &= y(0) = A \cos(4 \cdot 0) + B \sin(4 \cdot 0) \\ &= A \cdot 1 + B \cdot 0 = A \end{aligned}$$

and

$$\begin{aligned} 12 &= y'(0) = -4A \sin(4 \cdot 0) + 4B \cos(4 \cdot 0) \\ &= -4A \cdot 0 + 4B \cdot 1 = 4B . \end{aligned}$$

Thus,  $A = 4$ ,  $B = \frac{12}{4} = 3$ , and the solution to the initial-value problem is

$$y(x) = 4 \cos(4x) + 3 \sin(4x) .$$

15.6 e.

$$r^2 - 4r + 13 = 0$$

$$\Leftrightarrow r = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2 \cdot 1} = \frac{4 \pm 2\sqrt{-9}}{2} = 2 \pm 3i .$$

The corresponding solutions to the differential equation are then given by

$$\begin{aligned} y_{\pm}(x) &= e^{r_{\pm}x} = e^{(2 \pm 3i)x} = e^{2x \pm i3x} = e^{2x} e^{\pm i3x} \\ &= e^{2x} [\cos(3x) \pm i \sin(3x)] \\ &= \underbrace{e^{2x} \cos(3x)}_{y_1(x)} \pm i \underbrace{e^{2x} \sin(3x)}_{y_2(x)} . \end{aligned}$$

So, in terms of real-valued functions, the general solution to the differential equation is

$$\begin{aligned} y(x) &= Ae^{2x} \cos(3x) + Be^{2x} \sin(3x) \quad (\star) \\ &= (A \cos(3x) + B \sin(3x))e^{2x} , \end{aligned}$$

and its derivative is

$$\begin{aligned} y'(x) &= A [2e^{2x} \cos(3x) - 3e^{2x} \sin(3x)] + B [2e^{2x} \sin(3x) + 3e^{2x} \cos(3x)] \\ &= (A [2 \cos(3x) - 3 \sin(3x)] + B [2 \sin(3x) + 3 \cos(3x)])e^{2x} . \end{aligned}$$

Applying the initial conditions, we have

$$0 = y(0) = (A \cos(3 \cdot 0) + B \sin(3 \cdot 0))e^{2 \cdot 0} = A$$

and

$$\begin{aligned} 1 &= y'(0) = (A [2 \cos(3 \cdot 0) - 3 \sin(3 \cdot 0)] + B [2 \sin(3 \cdot 0) + 3 \cos(3 \cdot 0)])e^{2 \cdot 0} \\ &= 2A + 3B . \end{aligned}$$

Thus,  $A = 0$ ,  $B = \frac{1}{3}[1 - 2A] = \frac{1}{3}$  and the solution to the initial-value problem is

$$y(x) = Ae^{2x} \cos(3x) + Be^{2x} \sin(3x) = \frac{1}{3}e^{2x} \sin(3x) .$$

15.7 a.

$$r^2 - r + \left(\frac{1}{4} + 4\pi^2\right) = 0$$

$$\Leftrightarrow r = \frac{-(-1) \pm \sqrt{(-1)^2 - 4\left(\frac{1}{4} + 4\pi^2\right)}}{2} = \frac{1 \pm 2\sqrt{-4\pi^2}}{2} = \frac{1}{2} \pm i2\pi .$$

The corresponding solutions to the differential equation are then given by

$$\begin{aligned} y_{\pm}(x) &= e^{r_{\pm}x} = e^{\left(\frac{1}{2} \pm i2\pi\right)x} = e^{x/2} e^{\pm i2\pi x} \\ &= e^{x/2} [\cos(2\pi x) \pm i \sin(2\pi x)] \\ &= \underbrace{e^{x/2} \cos(2\pi x)}_{y_1(x)} \pm i \underbrace{e^{x/2} \sin(2\pi x)}_{y_2(x)} . \end{aligned}$$

So, in terms of real-valued functions, the general solution to the differential equation is

$$\begin{aligned} y(x) &= Ae^{x/2} \cos(2\pi x) + Be^{x/2} \sin(2\pi x) \\ &= (A \cos(2\pi x) + B \sin(2\pi x))e^{x/2} , \quad (\star) \end{aligned}$$

and its derivative is

$$\begin{aligned} y'(x) &= 2\pi(-A \sin(2\pi x) + B \cos(2\pi x))e^{x/2} \\ &\quad + \frac{1}{2}(A \cos(2\pi x) + B \sin(2\pi x))e^{x/2} \\ &= \left( A \left[ \frac{1}{2} \cos(2\pi x) - 2\pi \sin(2\pi x) \right] + B \left[ 2\pi \cos(2\pi x) + \frac{1}{2} \sin(2\pi x) \right] \right) e^{x/2} . \end{aligned}$$

Applying the initial conditions, we have

$$1 = y(0) = (A \cos(0) + B \sin(0))e^0 = A$$

and

$$\begin{aligned} \frac{1}{2} = y'(0) &= \left( A \left[ \frac{1}{2} \cos(0) - 2\pi \sin(0) \right] + B \left[ 2\pi \cos(0) + \frac{1}{2} \sin(0) \right] \right) e^0 \\ &= \frac{1}{2}A + 2\pi B . \end{aligned}$$

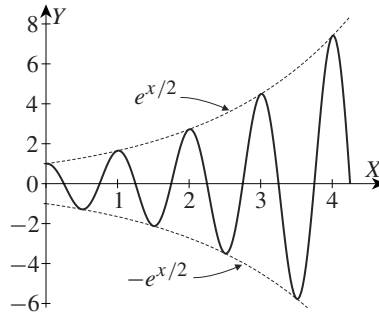
Thus,  $A = 1$ ,

$$B = \frac{1}{2\pi} \left[ \frac{1}{2} - \frac{1}{2}A \right] = \frac{1}{2\pi} \left[ \frac{1}{2} - \frac{1}{2} \cdot 1 \right] = 0$$

and the solution to the initial-value problem is

$$y(x) = Ae^{x/2} \cos(2\pi x) + Be^{x/2} \sin(2\pi x) = e^{x/2} \cos(2\pi x) .$$

The graph of this is basically the graph of a cosine “with an exponentially increasing amplitude”; as indicated by the heavy line in the figure below:



**15.8 a.**  $r^2 - 9 = 0 \rightsquigarrow r^2 = 9 \rightsquigarrow r = \pm\sqrt{9} = \pm 3$

$\hookrightarrow y(x) = c_1 e^{3x} + c_2 e^{-3x} .$

**15.8 c.**  $0 = r^2 + 6r + 9 = (r+3)^2 \rightsquigarrow r = -3$

$\hookrightarrow y(x) = c_1 e^{-3x} + c_2 x e^{-3x} .$

**15.8 e.**  $0 = 9r^2 - 6r + 1 = (3r-1)^2 \rightsquigarrow r = \frac{1}{3}$

$\hookrightarrow y(x) = c_1 e^{x/3} + c_2 x e^{x/3} .$



15.8 g.

$$r^2 - 4r + 40 = 0$$

$$\hookrightarrow r = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(40)}}{2} = \frac{4 \pm \sqrt{-144}}{2} = 2 \pm 6i \quad .$$

Finding the general solution to the differential equation in terms of real-valued functions:

$$\begin{aligned} y_{\pm}(x) &= e^{(2 \pm 6i)x} = e^{2x \pm i6x} = e^{2x} e^{\pm i6x} \\ &= e^{2x} [\cos(6x) \pm i \sin(6x)] \\ &= \underbrace{e^{2x} \cos(6x)}_{y_1(x)} \pm i \underbrace{e^{2x} \sin(6x)}_{y_2(x)} \quad . \end{aligned}$$

$$\hookrightarrow y(x) = c_1 e^{2x} \cos(6x) + c_2 e^{2x} \sin(6x) \quad .$$

15.8 i.

$$0 = r^2 + 10r + 25 = (r + 5)^2 \quad \rightsquigarrow \quad r = -5$$

$$\hookrightarrow y(x) = c_1 e^{-5x} + c_2 x e^{-5x} \quad .$$

15.8 k.

$$9r^2 + 1 = 0 \quad \rightsquigarrow \quad r^2 = -\frac{1}{9} \quad \rightsquigarrow \quad r = \pm \sqrt{-\frac{1}{9}} = \pm \frac{1}{3}i$$

$$\hookrightarrow y_{\pm}(x) = e^{\pm i x/3} = \underbrace{\cos\left(\frac{x}{3}\right)}_{y_1(x)} \pm i \underbrace{\sin\left(\frac{x}{3}\right)}_{y_2(x)}$$

$$\hookrightarrow y(x) = c_1 \cos\left(\frac{x}{3}\right) + c_2 \sin\left(\frac{x}{3}\right) \quad .$$

15.8 m.

$$r^2 + 4r + 7 = 0$$

$$\hookrightarrow r = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(7)}}{2} = \frac{-4 \pm \sqrt{-12}}{2} = -2 \pm \sqrt{3}i \quad .$$

Finding the general solution to the differential equation in terms of real-valued functions:

$$\begin{aligned} y_{\pm}(x) &= e^{(2 \pm \sqrt{3}i)x} = e^{-2x \pm i\sqrt{3}x} = e^{-2x} e^{\pm i\sqrt{3}x} \\ &= e^{-2x} [\cos(\sqrt{3}x) \pm i \sin(\sqrt{3}x)] \\ &= \underbrace{e^{-2x} \cos(\sqrt{3}x)}_{y_1(x)} \pm i \underbrace{e^{-2x} \sin(\sqrt{3}x)}_{y_2(x)} \quad . \end{aligned}$$

$$\hookrightarrow y(x) = c_1 e^{-2x} \cos(\sqrt{3}x) + c_2 e^{-2x} \sin(\sqrt{3}x) \quad .$$

15.8 o.

$$0 = r^2 + 4r + 4 = (r + 2)^2 \quad \rightsquigarrow \quad r = -2$$

$$\hookrightarrow y(x) = c_1 e^{-2x} + c_2 x e^{-2x} \quad .$$

15.8 q.

$$0 = r^2 - 4r = r(r - 4)$$

$$\hookrightarrow r = 0 \quad \text{and} \quad r = 4$$

$$\hookrightarrow y(x) = c_1 e^{0x} + c_2 e^{4x} = c_1 + c_2 e^{4x} .$$

15.8 s.

$$4r^2 + 3 = 0 \quad \rightsquigarrow \quad r^2 = -\frac{3}{4} \quad \rightsquigarrow \quad r = \pm \sqrt{-\frac{3}{4}} = \pm \frac{\sqrt{3}}{2}i$$

$$\hookrightarrow y_{\pm}(x) = e^{\pm i\sqrt{3}x/2} = \underbrace{\cos\left(\frac{\sqrt{3}}{2}x\right)}_{y_1(x)} \pm i \underbrace{\sin\left(\frac{\sqrt{3}}{2}x\right)}_{y_2(x)}$$

$$\hookrightarrow y(x) = c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right) .$$

15.9 a.

$$\begin{aligned} \sin^2(t) + \cos^2(t) &= \left(\frac{e^t - e^{-t}}{2i}\right)^2 + \left(\frac{e^t + e^{-t}}{2}\right)^2 \\ &= \frac{(e^t)^2 - 2e^t e^{-t} + (e^{-t})^2}{4i^2} + \frac{(e^t)^2 + 2e^t e^{-t} + (e^{-t})^2}{4} \\ &= \frac{e^{2t} - 2 + e^{-2t}}{-4} + \frac{e^{2t} + 2 + e^{-2t}}{4} \\ &= -\frac{e^{2t} - 2 + e^{-2t}}{4} + \frac{e^{2t} + 2 + e^{-2t}}{4} \\ &= \frac{[-e^{2t} + 2 - e^{-2t}] + [e^{2t} + 2 + e^{-2t}]}{4} = \frac{4}{4} = 1 . \end{aligned}$$

15.9 c.  $\cos(A)\cos(B) - \sin(A)\sin(B)$ 

$$\begin{aligned} &= \left(\frac{e^A + e^{-A}}{2}\right)\left(\frac{e^B + e^{-B}}{2}\right) - \left(\frac{e^A - e^{-A}}{2i}\right)\left(\frac{e^B - e^{-B}}{2i}\right) \\ &= \frac{e^A e^B + e^A e^{-B} + e^{-A} e^B + e^{-A} e^{-B}}{4} - \frac{e^A e^B - e^A e^{-B} - e^{-A} e^B + e^{-A} e^{-B}}{-4} \\ &= \frac{e^{A+B} + e^{A-B} + e^{-A+B} + e^{-A-B}}{4} + \frac{e^{A+B} - e^{A-B} - e^{-A+B} + e^{-A-B}}{4} \\ &= \frac{2e^{A+B} + 2e^{-A-B}}{4} \\ &= \frac{e^{(A+B)} + e^{-(A+B)}}{2} = \cos(A+B) . \end{aligned}$$