Chapter 15: Second-Order Homogeneous Linear Equations with Constant Coefficients

15.1 a. The characteristic equation is

$$r^2 - 7r + 10 = 0 \quad ,$$

which factors easily,

$$(r-2)(r-5) = 0 \quad .$$

Thus, the solutions to the characteristic equation are $r = 2$ and $r = 5$. The corresponding solutions to the differential equation are then e^{2x} and e^{5x}. These form a fundamental set of solutions, and, so, our general solution to the differential equation is

$$y(x) = c_1 e^{2x} + c_2 e^{5x} \quad .$$

15.1 c. Writing out and solving the characteristic equation:

$$r^2 - 25 = 0 \quad \rightarrowtail \quad r^2 = 25 \quad \rightarrowtail \quad r = \pm\sqrt{25} = \pm 5 \quad .$$

So the solutions to the characteristic equation are $r = 5$ and $r = -5$. The corresponding solutions to the differential equation are then e^{5x} and e^{-5x}, and our general solution to the differential equation is

$$y(x) = c_1 e^{5x} + c_2 e^{-5x} \quad .$$

15.1 e.

$$4r^2 - 1 = 0 \quad \rightarrowtail \quad r^2 = \frac{1}{4} \quad \rightarrowtail \quad r = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}$$

$$\hookrightarrow \qquad r = \frac{1}{2} \quad \text{and} \quad r = -\frac{1}{2}$$

$$\hookrightarrow \qquad y(x) = c_1 e^{x/2} + c_2 e^{-x/2} \quad .$$

15.2 a. Finding the general solution and its derivative:

$$0 = r^2 - 8r + 15 = (r-3)(r-5)$$

$$\hookrightarrow \qquad r = 3 \quad \text{and} \quad r = 5$$

$$\hookrightarrow \qquad y(x) = c_1 e^{3x} + c_2 e^{5x} \quad \text{and} \quad y'(x) = 3c_1 e^{3x} + 5c_2 e^{5x} \quad . \qquad (\star)$$

Applying the initial conditions:

$$1 = y(0) = c_1 e^{3\cdot 0} + c_2 e^{5\cdot 0} = c_1 + c_2$$

and

$$0 = y'(0) = 3c_1 e^{3\cdot 0} + 5c_2 e^{5\cdot 0} = 3c_1 + 5c_2 \quad .$$

Solving for the constants:

$$1 = c_1 + c_2 \quad \text{and} \quad 0 = 3c_1 + 5c_2$$

$$\hookrightarrow \qquad 1 = c_1 + c_2 \quad \text{and} \quad c_2 = -\frac{5}{3}c_1$$

$$\hookrightarrow \qquad 1 = c_1 - \tfrac{5}{3}c_1 = -\tfrac{2}{3}c_1 \qquad \text{and} \qquad c_2 = -\tfrac{5}{3}c_1$$

$$\hookrightarrow \qquad c_1 = -\tfrac{3}{2} \qquad \text{and} \qquad c_2 = -\tfrac{5}{3} \times \left(-\tfrac{3}{2}\right) = \tfrac{5}{2} \quad .$$

Plugging these values into the formula for y in line ($\star$) then gives the solution,

$$y(x) = -\tfrac{5}{2}e^{3x} + \tfrac{4}{2}e^{5x} \quad .$$

15.2 c. From solving exercise 15.2 a, we know the differential equation's general solution and its derivative are

$$y(x) = c_1e^{3x} + c_2e^{5x} \qquad \text{and} \qquad y'(x) = 3c_1e^{3x} + 5c_2e^{5x} \quad . \qquad (\star)$$

Applying the initial conditions:

$$5 = y(0) = c_1e^{3\cdot 0} + c_2e^{5\cdot 0} = c_1 + c_2$$

and

$$19 = y'(0) = 3c_1e^{3\cdot 0} + 5c_2e^{5\cdot 0} = 3c_1 + 5c_2 \quad .$$

Solving for the constants:

$$5 = c_1 + c_2 \qquad \text{and} \qquad 19 = 3c_1 + 5c_2$$

$$\hookrightarrow \quad c_1 = 5 - c_2 \qquad \text{and} \qquad 19 = 3(5 - c_2) + 5c_2 = 15 + 2c_2$$

$$\hookrightarrow \qquad c_1 = 5 - c_2 \qquad \text{and} \qquad c_2 = \frac{19 - 15}{2} = 2$$

$$\hookrightarrow \qquad c_1 = 5 - 2 = 3 \qquad \text{and} \qquad c_2 = 2 \quad .$$

Plugging these values into the formula for y in line ($\star$) then gives the solution,

$$y(x) = 3e^{3x} + 2e^{5x} \quad .$$

15.2 e. Finding the general solution and its derivative:

$$0 = r^2 - 9 = (r-3)(r+3) = (r-3)(r-[-3])$$

$$\hookrightarrow \qquad r = 3 \qquad \text{and} \qquad r = -3$$

$$\hookrightarrow \quad y(x) = c_1e^{3x} + c_2e^{-3x} \qquad \text{and} \qquad y'(x) = 3c_1e^{3x} - 3c_2e^{-3x} \quad . \qquad (\star)$$

Applying the initial conditions:

$$0 = y(0) = c_1e^{3\cdot 0} + c_2e^{-3\cdot 0} = c_1 + c_2$$

and

$$1 = y'(0) = 3c_1e^{3\cdot 0} - 3c_2e^{-3\cdot 0} = 3c_1 - 3c_2 \quad .$$

Solving for the constants:

$$0 = c_1 + c_2 \quad \text{and} \quad 1 = 3c_1 - 3c_2$$

$$\hookrightarrow \quad c_1 = -c_2 \quad \text{and} \quad 1 = 3(-c_2) - 3c_2 = -6c_2$$

$$\hookrightarrow \quad c_1 = -c_2 \quad \text{and} \quad c_2 = -\frac{1}{6}$$

$$\hookrightarrow \quad c_1 = -\left(-\frac{1}{6}\right) = \frac{1}{6} \quad \text{and} \quad c_2 = -\frac{1}{6} \quad .$$

Plugging these values into the formula for y in line $(\star)$ then gives the solution,

$$y(x) = \frac{1}{6}e^{3x} - \frac{1}{6}e^{5x} \quad .$$

15.3 a. $0 = r^2 - 10r + 25 = (r-5)^2 \quad \rightarrowtail \quad r = 5 \quad .$

So one solution to the differential equation is $y_1(x) = e^{5x}$. As derived in section 15.4, the second solution to the differential equation when the characteristic polynomial only has one root is

$$y_2(x) = xy_1(x) = xe^{5x} \quad .$$

Hence, the general solution to the differential equation is

$$y(x) = c_1 e^{5x} + c_2 x e^{5x} \quad .$$

15.3 c.

$$0 = 4r^2 - 4r + 1 = (2r-1)^2 \quad \rightarrowtail \quad r = \frac{1}{2}$$

$$\hookrightarrow \quad y_1(x) = e^{x/2} \quad \text{and} \quad y_2(x) = xy_1(x) = xe^{x/2}$$

$$\hookrightarrow \quad y(x) = c_1 e^{x/2} + c_2 x e^{x/2} \quad .$$

15.4 a. Finding the general solution and its derivative:

$$0 = r^2 - 8r + 16 = (r-4)^2 \quad \rightarrowtail \quad r = 4$$

$$\hookrightarrow \quad y_1(x) = e^{4x} \quad \text{and} \quad y_2(x) = xy_1(x) = xe^{4x}$$

$$\hookrightarrow \quad y(x) = c_1 e^{4x} + c_2 x e^{4x} \qquad (\star)$$

$$\hookrightarrow \quad y'(x) = 4c_1 e^{4x} + c_2\left[e^{4x} + 4xe^{4x}\right] \quad .$$

Applying the initial conditions:

$$1 = y(0) = c_1 e^{4\cdot 0} + c_2 \cdot 0 e^{4\cdot 0} = c_1$$

and

$$0 = y'(0) = 4c_1 e^{4\cdot 0} + c_2\left[e^{4\cdot 0} + 4\cdot 0 e^{4\cdot 0}\right] = 4c_1 + c_2 \quad .$$

Solving for the constants and plugging them back into formula ($\star$) for y :

$$1 = c_1 \qquad \text{and} \qquad 0 = 4c_1 + c_2 = 4 \cdot 1 + c_2$$

$$\hookrightarrow \qquad c_1 = 1 \qquad \text{and} \qquad c_2 = -4$$

$$\hookrightarrow \qquad y(x) = e^{4x} - 4xe^{4x} \quad .$$

15.4 c. From solving exercise 15.4 a, we know the differential equation's general solution and its derivative are

$$y(x) = c_1 e^{4x} + c_2 x e^{4x} \qquad \text{and} \qquad y'(x) = 4c_1 e^{4x} + c_2 \left[e^{4x} + 4xe^{4x} \right] \quad . \qquad (\star)$$

Applying the initial conditions:

$$3 = y(0) = c_1 e^{4 \cdot 0} + c_2 \cdot 0 e^{4 \cdot 0} = c_1$$

and

$$14 = y'(0) = 4c_1 e^{4 \cdot 0} + c_2 \left[e^{4 \cdot 0} + 4 \cdot 0 e^{4 \cdot 0} \right] = 4c_1 + c_2 \quad .$$

Solving for the constants and plugging them back into formula ($\star$) for y :

$$3 = c_1 \qquad \text{and} \qquad 14 = 4c_1 + c_2 = 4 \cdot 3 + c_2$$

$$\hookrightarrow \qquad c_1 = 3 \qquad \text{and} \qquad c_2 = 2$$

$$\hookrightarrow \qquad y(x) = 3e^{4x} + 2xe^{4x} \quad .$$

15.4 e. Finding the general solution and its derivative:

$$0 = 4r^2 + 4r + 1 = (2r + 1)^2$$

$$\hookrightarrow \qquad 2r + 1 = 0 \quad \rightarrowtail \quad r = -\frac{1}{2}$$

$$\hookrightarrow \qquad y_1(x) = e^{-x/2} \qquad \text{and} \qquad y_2(x) = xy_1(x) = xe^{-x/2}$$

$$\hookrightarrow \qquad y(x) = c_1 e^{-x/2} + c_2 x e^{-x/2} \qquad (\star)$$

$$\hookrightarrow \qquad y'(x) = -\frac{1}{2} c_1 e^{-x/2} + c_2 \left[e^{-x/2} - \frac{1}{2} x e^{-x/2} \right] \quad .$$

Applying the initial conditions:

$$0 = y(0) = c_1 e^{-0/2} + c_2 \cdot 0 e^{-0/2} = c_1$$

and

$$1 = y'(0) = -\frac{1}{2} c_1 e^{-0/2} + c_2 \left[e^{-0/2} - \frac{1}{2} \cdot 0 e^{-0/2} \right] = -\frac{1}{2} c_1 + c_2 \quad .$$

Solving for the constants and plugging them back into formula ($\star$) for y :

$$0 = c_1 \qquad \text{and} \qquad 1 = -\frac{1}{2}c_1 + c_2 = -\frac{1}{2}\cdot 0 + c_2$$

$$\hookrightarrow \qquad c_1 = 0 \qquad \text{and} \qquad c_2 = 1$$

$$\hookrightarrow \qquad y(x) = 0\cdot e^{-x/2} + 1\cdot xe^{-x/2} = xe^{-x/2} \quad .$$

15.5 a. The characteristic equation and its solutions:

$$r^2 + 25 = 0 \quad \rightarrowtail \quad r^2 = -25 \quad \rightarrowtail \quad r = \pm\sqrt{-25} = \pm 5i \quad .$$

So, two solutions to the differential equation are

$$y_+(x) = e^{i5x} = \cos(5x) + i\sin(5x)$$

and

$$y_-(x) = e^{i5x} = \cos(5x) + i\sin(5x) \quad .$$

However, because of the i , these are not real valued. To get corresponding real-valued solutions y_1 and y_2 , we can either set

$$y_1 = y_+ + y_- \qquad \text{and} \qquad y_2 = y_+ - y_- \quad ,$$

or, equivalently, simply take the real and the imaginary parts of

$$y_+(x) = \cos(5x) + i\sin(5x)$$

for y_1 and y_2 , respectively. Either way, we get

$$y_1(x) = \cos(5x) \qquad \text{and} \qquad y_2(x) = \sin(5x) \quad .$$

The general solution is then

$$y(x) = Ay_1(x) + By_2(x) = A\cos(5x) + B\sin(5x) \quad .$$

15.5 c. The characteristic equation and its solutions:

$$r^2 - 2r + 5 = 0$$

$$\hookrightarrow \qquad r = \frac{-(-2)\pm\sqrt{(-2)^2 - 4(1)(5)}}{2\cdot 1} = \frac{2\pm\sqrt{-16}}{2} = 1 \pm 2i \quad .$$

So the complex exponential solutions to the differential equation are given by

$$\begin{aligned} y_\pm(x) = e^{rx} = e^{(1\pm 2i)x} = e^{x\pm i2x} &= e^x e^{i2s} \\ &= e^x\left[\cos(2x) \pm i\sin(2x)\right] \\ &= e^x\cos(2x) \pm ie^x\sin(2x) \quad . \end{aligned}$$

Taking the real and imaginary parts of y_+ ,

$$y_+(x) = \underbrace{e^x\cos(2x)}_{y_1(x)} + i\,\underbrace{e^x\sin(2x)}_{y_2(x)} \quad ,$$

gives us the linearly independent pair of real-valued solutions

$$y_1(x) = e^x \cos(2x) \quad \text{and} \quad y_2(x) = e^x \sin(2x) \quad .$$

So the general solution in terms of real-valued functions is

$$y(x) = Ae^x \cos(2x) + Be^x \sin(2x)$$

15.6 a. $r^2 + 16 = 0 \quad \rightarrowtail \quad r^2 = -16 \quad \rightarrowtail \quad r = \pm\sqrt{-16} = \pm 4i \quad .$
The corresponding solutions to the differential equation are then given by

$$y_\pm(x) = e^{r_\pm x} = e^{\pm i4x} = \underbrace{\cos(4x)}_{y_1(x)} \pm i \underbrace{\sin(4x)}_{y_2(x)} \quad .$$

So, in terms of real-valued functions, the general solution to the differential equation is

$$y(x) = A\cos(4x) + B\sin(4x) \quad . \qquad (\star)$$

Its derivative is

$$y'(x) = -4A\sin(4x) + 4B\cos(4x) \quad .$$

Applying the initial conditions, we have

$$\begin{aligned} 1 = y(0) &= A\cos(4\cdot 0) + B\sin(4\cdot 0) \\ &= A\cdot 1 + B\cdot 0 = A \end{aligned}$$

and

$$\begin{aligned} 0 = y'(0) &= -4A\sin(4\cdot 0) + 4B\cos(4\cdot 0) \\ &= -4A\cdot 0 + 4B\cdot 1 = 4B \quad . \end{aligned}$$

Thus, $A = 1$, $B = 0$, and the solution to the initial-value problem is

$$y(x) = A\cos(4x) + B\sin(4x) = \cos(4x) \quad .$$

15.6 c. From solving exercise 15.6 a, we know the general solution to the differential equation is

$$y(x) = A\cos(4x) + B\sin(4x) \quad , \qquad (\star)$$

and its derivative is

$$y'(x) = -4A\sin(4x) + 4B\cos(4x) \quad .$$

Applying the initial conditions, we have

$$\begin{aligned} 4 = y(0) &= A\cos(4\cdot 0) + B\sin(4\cdot 0) \\ &= A\cdot 1 + B\cdot 0 = A \end{aligned}$$

and

$$\begin{aligned} 12 = y'(0) &= -4A\sin(4\cdot 0) + 4B\cos(4\cdot 0) \\ &= -4A\cdot 0 + 4B\cdot 1 = 4B \quad . \end{aligned}$$

Thus, $A = 4$, $B = \frac{12}{4} = 3$, and the solution to the initial-value problem is

$$y(x) = 4\cos(4x) + 3\sin(4x) \quad .$$

15.6 e.
$$r^2 - 4r + 13 = 0$$

$$\hookrightarrow \quad r = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2 \cdot 1} = \frac{4 \pm 2\sqrt{-9}}{2} = 2 \pm 3i \quad .$$

The corresponding solutions to the differential equation are then given by

$$\begin{aligned} y_\pm(x) = e^{r_\pm x} = e^{(2\pm 3i)x} = e^{2x \pm i3x} &= e^{2x} e^{\pm i3x} \\ &= e^{2x}\left[\cos(3x) \pm i \sin(3x)\right] \\ &= \underbrace{e^{2x}\cos(3x)}_{y_1(x)} \pm i \underbrace{e^{2x}\sin(ex)}_{y_2(x)} \quad . \end{aligned}$$

So, in terms of real-valued functions, the general solution to the differential equation is

$$\begin{aligned} y(x) &= Ae^{2x}\cos(3x) + Be^{2x}\sin(3x) \qquad (\star) \\ &= \big(A\cos(3x) + B\sin(3x)\big)e^{2x} \quad , \end{aligned}$$

and its derivative is

$$\begin{aligned} y'(x) &= A\left[2e^{2x}\cos(3x) - 3e^{2x}\sin(3x)\right] + B\left[2e^{2x}\sin(3x) + 3e^{2x}\cos(3x)\right] \\ &= \big(A\left[2\cos(3x) - 3\sin(3x)\right] + B\left[2\sin(3x) + 3\cos(3x)\right]\big)e^{2x} \quad . \end{aligned}$$

Applying the initial conditions, we have

$$0 = y(0) = \big(A\cos(3\cdot 0) + B\sin(3\cdot 0)\big)e^{2\cdot 0} = A$$

and

$$\begin{aligned} 1 = y'(0) &= \big(A\left[2\cos(3\cdot 0) - 3\sin(3\cdot 0)\right] + B\left[2\sin(3\cdot 0) + 3\cos(3\cdot 0)\right]\big)e^{2\cdot 0} \\ &= 2A + 3B \quad . \end{aligned}$$

Thus, $A = 0$, $B = \frac{1}{3}[1 - 2A] = \frac{1}{3}$ and the solution to the initial-value problem is

$$y(x) = Ae^{2x}\cos(3x) + Be^{2x}\sin(3x) = \frac{1}{3}e^{2x}\sin(3x) \quad .$$

15.7 a.
$$r^2 - r + \left(\frac{1}{4} + 4\pi^2\right) = 0$$

$$\hookrightarrow \quad r = \frac{-(-1) \pm \sqrt{(-1)^2 - 4\left(\frac{1}{4} + 4\pi^2\right)}}{2} = \frac{1 \pm 2\sqrt{-4\pi^2}}{2} = \frac{1}{2} \pm i2\pi \quad .$$

The corresponding solutions to the differential equation are then given by

$$\begin{aligned} y_\pm(x) = e^{r_\pm x} = e^{\left(1/2 \pm i2\pi\right)x} &= e^{x/2} e^{\pm i2\pi} \\ &= e^{x/2}\left[\cos(2\pi x) \pm i \sin(2\pi x)\right] \\ &= \underbrace{e^{x/2}\cos(2\pi x)}_{y_1(x)} \pm i \underbrace{e^{x/2}\sin(2\pi x)}_{y_2(x)} \quad . \end{aligned}$$

So, in terms of real-valued functions, the general solution to the differential equation is

$$\begin{aligned} y(x) &= Ae^{x/2}\cos(2\pi x) + Be^{x/2}\sin(2\pi x) \\ &= \big(A\cos(2\pi x) + B\sin(2\pi x)\big)e^{x/2} \quad , \qquad (\star) \end{aligned}$$

and its derivative is

$$\begin{aligned} y'(x) &= 2\pi\big(-A\sin(2\pi x) + B\cos(2\pi x)\big)e^{x/2} \\ &\quad + \frac{1}{2}\big(A\cos(2\pi x) + B\sin(2\pi x)\big)e^{x/2} \\ &= \left(A\left[\frac{1}{2}\cos(2\pi x) - 2\pi\sin(2\pi x)\right] + B\left[2\pi\cos(2\pi x) + \frac{1}{2}\sin(2\pi x)\right]\right)e^{x/2} \quad . \end{aligned}$$

Applying the initial conditions, we have

$$1 = y(0) = \big(A\cos(0) + B\sin(0)\big)e^0 = A$$

and

$$\begin{aligned} \frac{1}{2} = y'(0) &= \left(A\left[\frac{1}{2}\cos(0) - 2\pi\sin(0)\right] + B\left[2\pi\cos(0) + \frac{1}{2}\sin(0)\right]\right)e^0 \\ &= \frac{1}{2}A + 2\pi B \quad . \end{aligned}$$

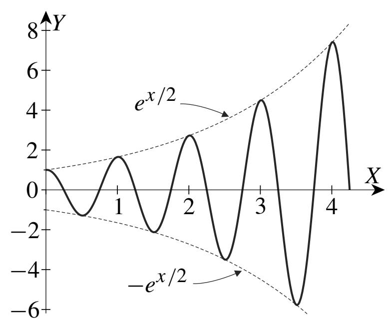
Thus, $A = 1$,

$$B = \frac{1}{2\pi}\left[\frac{1}{2} - \frac{1}{2}A\right] = \frac{1}{2\pi}\left[\frac{1}{2} - \frac{1}{2}\cdot 1\right] = 0$$

and the solution to the initial-value problem is

$$y(x) = Ae^{x/2}\cos(2\pi x) + Be^{x/2}\sin(2\pi x) = e^{x/2}\cos(2\pi x) \quad .$$

The graph of this is basically the graph of a cosine "with an exponentially increasing amplitude", as indicated by the heavy line in the figure below:



15.8 a. $$r^2 - 9 = 0 \quad \rightarrowtail \quad r^2 = 9 \quad \rightarrowtail \quad r = \pm\sqrt{9} = \pm 3$$

$$\hookrightarrow \qquad y(x) = c_1e^{3x} + c_2e^{-3x} \quad .$$

15.8 c. $$0 = r^2 + 6r + 9 = (r+3)^2 \quad \rightarrowtail \quad r = -3$$

$$\hookrightarrow \qquad y(x) = c_1e^{-3x} + c_2xe^{-3x} \quad .$$

15.8 e. $$0 = 9r^2 - 6r + 1 = (3r-1)^2 \quad \rightarrowtail \quad r = \frac{1}{3}$$

$$\hookrightarrow \qquad y(x) = c_1e^{x/3} + c_2xe^{x/3} \quad .$$

15.8 g.

$$r^2 - 4r + 40 = 0$$

$$\hookrightarrow \quad r = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(40)}}{2} = \frac{4 \pm \sqrt{-64}}{2} = 2 \pm 4i \quad .$$

Finding the general solution to the differential equation in terms of real-valued functions:

$$\begin{aligned} y_{\pm}(x) = e^{(2\pm 4i)x} = e^{2x \pm i4x} &= e^{2x} e^{\pm i4x} \\ &= e^{2x}\left[\cos(4x) \pm i \sin(4x)\right] \\ &= \underbrace{e^{2x}\cos(4x)}_{y_1(x)} \pm i\, \underbrace{e^{2x}\sin(4x)}_{y_2(x)} \quad . \end{aligned}$$

$$\hookrightarrow \quad y(x) = c_1 e^{2x}\cos(4x) + c_2 e^{2x}\sin(4x) \quad .$$

15.8 i.

$$0 = r^2 + 10r + 25 = (r+5)^2 \quad \rightarrowtail \quad r = -5$$

$$\hookrightarrow \quad y(x) = c_1 e^{-5x} + c_2 x e^{-5x} \quad .$$

15.8 k.

$$9r^2 + 1 = 0 \quad \rightarrowtail \quad r^2 = -\tfrac{1}{9} \quad \rightarrowtail \quad r = \pm\sqrt{-\tfrac{1}{9}} = \pm\tfrac{1}{3}i$$

$$\hookrightarrow \quad y_{\pm}(x) = e^{\pm ix/3} = \underbrace{\cos\left(\frac{x}{3}\right)}_{y_1(x)} \pm i\, \underbrace{\sin\left(\frac{x}{3}\right)}_{y_2(x)}$$

$$\hookrightarrow \quad y(x) = c_1 \cos\left(\frac{x}{3}\right) + c_2 \sin\left(\frac{x}{3}\right) \quad .$$

15.8 m.

$$r^2 + 4r + 7 = 0$$

$$\hookrightarrow \quad r = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(7)}}{2} = \frac{-4 \pm \sqrt{-12}}{2} = -2 \pm \sqrt{3}i \quad .$$

Finding the general solution to the differential equation in terms of real-valued functions:

$$\begin{aligned} y_{\pm}(x) = e^{\left(2\pm\sqrt{3}i\right)x} = e^{-2x \pm i\sqrt{3}x} &= e^{-2x} e^{\pm i\sqrt{3}x} \\ &= e^{-2x}\left[\cos\left(\sqrt{3}x\right) \pm i \sin\left(\sqrt{3}x\right)\right] \\ &= \underbrace{e^{-2x}\cos\left(\sqrt{3}x\right)}_{y_1(x)} \pm i\, \underbrace{e^{-2x}\sin\left(\sqrt{3}x\right)}_{y_2(x)} \quad . \end{aligned}$$

$$\hookrightarrow \quad y(x) = c_1 e^{-2x}\cos\left(\sqrt{3}x\right) + c_2 e^{-2x}\sin\left(\sqrt{3}x\right) \quad .$$

15.8 o.

$$0 = r^2 + 4r + 4 = (r+2)^2 \quad \rightarrowtail \quad r = -2$$

$$\hookrightarrow \quad y(x) = c_1 e^{-2x} + c_2 x e^{-2x} \quad .$$

15.8 q.

$$0 = r^2 - 4r = r(r-4)$$

$$\hookrightarrow \quad r = 0 \quad \text{and} \quad r = 4$$

$$\hookrightarrow \quad y(x) = c_1 e^{0x} + c_2 e^{4x} = c_1 + c_2 e^{4x} \quad .$$

15.8 s.

$$4r^2 + 3 = 0 \quad \rightarrowtail \quad r^2 = -\frac{3}{4} \quad \rightarrowtail \quad r = \pm\sqrt{-\frac{3}{4}} = \pm\frac{\sqrt{3}}{2}i$$

$$\hookrightarrow \quad y_{\pm}(x) = e^{\pm i\sqrt{3}x/2} = \underbrace{\cos\left(\frac{\sqrt{3}}{2}x\right)}_{y_1(x)} \pm i \underbrace{\sin\left(\frac{\sqrt{3}}{2}x\right)}_{y_2(x)}$$

$$\hookrightarrow \quad y(x) = c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right) \quad .$$

15.9 a.

$$\begin{aligned}
\sin^2(t) + \cos^2(t) &= \left(\frac{e^t - e^{-t}}{2i}\right)^2 + \left(\frac{e^t + e^{-t}}{2}\right)^2 \\
&= \frac{\left(e^t\right)^2 - 2e^t e^{-t} + \left(e^{-t}\right)^2}{4i^2} + \frac{\left(e^t\right)^2 + 2e^t e^{-t} + \left(e^{-t}\right)^2}{4} \\
&= \frac{e^2 t - 2 + e^{-2t}}{-4} + \frac{e^2 t + 2 + e^{-2t}}{4} \\
&= -\frac{e^{2t} - 2 + e^{-2t}}{4} + \frac{e^{2t} + 2 + e^{-2t}}{4} \\
&= \frac{\left[-e^{2t} + 2 - e^{-2t}\right] + \left[e^{2t} + 2 + e^{-2t}\right]}{4} = \frac{4}{4} = 1 \quad .
\end{aligned}$$

15.9 c. $\cos(A)\cos(B) - \sin(A)\sin(B)$

$$\begin{aligned}
&= \left(\frac{e^A + e^{-A}}{2}\right)\left(\frac{e^B + e^{-B}}{2}\right) - \left(\frac{e^A - e^{-A}}{2i}\right)\left(\frac{e^B - e^{-B}}{2i}\right) \\
&= \frac{e^A e^B + e^A e^{-B} + e^{-A} e^B + e^{-A} e^{-B}}{4} - \frac{e^A e^B - e^A e^{-B} - e^{-A} e^B + e^{-A} e^{-B}}{-4} \\
&= \frac{e^{A+B} + e^{A-B} + e^{-A+B} + e^{-A-B}}{4} + \frac{e^{A+B} - e^{A-B} - e^{-A+B} + e^{-A-B}}{4} \\
&= \frac{2e^{A+B} + 2e^{-A-B}}{4} \\
&= \frac{e^{(A+B)} + e^{-(A+B)}}{2} = \cos(A+B) \quad .
\end{aligned}$$