Chapter 14: Verifying the Big Theorems and an Introduction to Differential Operators

14.2 a. By inspection: $L = \frac{d^2}{dx^2} + 5\frac{d}{dx} + 6$.

14.2 b i.

$$\begin{aligned} L[\sin(x)] &= \frac{d^2}{dx^2}\big[\sin(x)\big] + 5\frac{d}{dx}\big[\sin(x)\big] + 6\big[\sin(x)\big] \\ &= -\sin(x) + 5\cos(x) + 6\sin(x) = 5\sin(x) + 5\cos(x) \quad . \end{aligned}$$

14.2 b iii.

$$\begin{aligned} L\left[e^{-3x}\right] &= \frac{d^2}{dx^2}\big[e^{-3x}\big] + 5\frac{d}{dx}\big[e^{-3x}\big] + 6\big[e^{-3x}\big] \\ &= 9e^{-3x} + 5\left[-3e^{-3x}\right] + 6e^{-3x} = [9 - 15 + 6]\,e^{-3x} = 0 \quad . \end{aligned}$$

14.2 c. Since the differential equation can be written as

$$L[y] = 0 \quad ,$$

and we have

$$L\left[e^{-3x}\right] = 0 \quad ,$$

it follows that $y(x) = e^{-3x}$ is a solution to the differential equation.

14.3 a. By inspection: $L = \frac{d^2}{dx^2} - 5\frac{d}{dx} + 9$.

14.3 b i.

$$\begin{aligned} L[\sin(x)] &= \frac{d^2}{dx^2}\big[\sin(x)\big] - 5\frac{d}{dx}\big[\sin(x)\big] + 9\big[\sin(x)\big] \\ &= -\sin(x) - 5\cos(x) + 9\sin(x) = 8\sin(x) - 5\cos(x) \quad . \end{aligned}$$

14.3 b iii.

$$\begin{aligned} L\left[e^{2x}\right] &= \frac{d^2}{dx^2}\big[e^{2x}\big] - 5\frac{d}{dx}\big[e^{2x}\big] + 9\big[e^{2x}\big] \\ &= 4e^{2x} - 5\cdot 2e^{2x} + 9e^{2x} = 3e^{2x} \quad . \end{aligned}$$

14.4 a. By inspection: $L = x^2\frac{d^2}{dx^2} + 5x\frac{d}{dx} + 6$.

14.4 b i.

$$\begin{aligned} L[\sin(x)] &= x^2\frac{d^2}{dx^2}\big[\sin(x)\big] + 5x\frac{d}{dx}\big[\sin(x)\big] + 6\big[\sin(x)\big] \\ &= -x^2\sin(x) + 5x\cos(x) + 6\sin(x) \quad . \end{aligned}$$

14.4 b iii.

$$\begin{aligned} L\left[x^3\right] &= x^2\frac{d^2}{dx^2}\big[x^3\big] + 5x\frac{d}{dx}\big[x^3\big] + 6\big[x^3\big] \\ &= x^2\,[3\cdot 2x] + 5x\left[3x^2\right] + 6x^3 = 6x^3 + 15x^3 + 6x^3 = 27x^3 \quad . \end{aligned}$$

14.5 a. By inspection: $L = \frac{d^3}{dx^3} - \sin(x)\frac{d}{dx} + \cos(x)$.

14.5 b i.

$$L[\sin(x)] = \frac{d^3}{dx^3}\big[\sin(x)\big] - \sin(x)\frac{d}{dx}\big[\sin(x)\big] + \cos(x)\big[\sin(x)\big]$$
$$= -\cos(x) - \sin(x)\cos(x) + \cos(x)\sin(x) = -\cos(x) \quad .$$

14.5 b iii.

$$L\left[x^2\right] = \frac{d^3}{dx^3}\left[x^2\right] - \sin(x)\frac{d}{dx}\left[x^2\right] + \cos(x)\left[x^2\right]$$
$$= 0 - \sin(x)\cdot 2x + \cos(x)\cdot x^2 = x^2\cos(x) - 2x\sin(x) \quad .$$

14.6 a. Let ϕ be any sufficiently differentiable function. Then

$$\begin{aligned} L_2L_1[\phi] = L_2\left[L_1[\phi]\right] &= L_2\left[\frac{d\phi}{dx} + x\phi\right] \\ &= \frac{d}{dx}\left[\frac{d\phi}{dx} + x\phi\right] - x\left[\frac{d\phi}{dx} + x\phi\right] \\ &= \frac{d^2\phi}{dx^2} + \left[\phi + x\frac{d\phi}{dx}\right] - x\frac{d\phi}{dx} - x^2\phi \\ &= \frac{d^2\phi}{dx^2} + \left(1 - x^2\right)\phi \quad . \end{aligned}$$

So, $L_2L_1 = \dfrac{d^2}{dx^2} + \left(1 - x^2\right) \quad .$

On the other hand,

$$\begin{aligned} L_1L_2[\phi] = L_1\left[L_2[\phi]\right] &= L_1\left[\frac{d\phi}{dx} - x\phi\right] \\ &= \frac{d}{dx}\left[\frac{d\phi}{dx} - x\phi\right] + x\left[\frac{d\phi}{dx} - x\phi\right] \\ &= \frac{d^2\phi}{dx^2} - \left[\phi + x\frac{d\phi}{dx}\right] + x\frac{d\phi}{dx} - x^2\phi \\ &= \frac{d^2\phi}{dx^2} - \left(1 + x^2\right)\phi \quad . \end{aligned}$$

So, $L_1L_2 = \dfrac{d^2}{dx^2} - \left(1 + x^2\right) \quad .$

14.6 c. Let ϕ be any sufficiently differentiable function. Then

$$\begin{aligned} L_2L_1[\phi] = L_2\left[L_1[\phi]\right] &= L_2\left[x\frac{d\phi}{dx} + 3\phi\right] \\ &= \frac{d}{dx}\left[x\frac{d\phi}{dx} + 3\phi\right] + 2x\left[x\frac{d\phi}{dx} + 3\phi\right] \\ &= \left[\frac{d\phi}{dx} + x\frac{d^2\phi}{dx^2}\right] + 3\frac{d\phi}{dx} + 2x^2\frac{d\phi}{dx} + 6x\phi \\ &= x\frac{d^2\phi}{dx^2} + \left(4 + 2x^2\right)\frac{d\phi}{dx} + 6x\phi \quad . \end{aligned}$$

So, $L_2L_1 = x\dfrac{d^2}{dx^2} + \left(4 + 2x^2\right)\dfrac{d}{dx} + 6x \quad .$

On the other hand,

$$L_1L_2[\phi] \;=\; L_1\left[L_2[\phi]\right] \;=\; L_1\left[\frac{d\phi}{dx} + 2x\phi\right]$$
$$= x\frac{d}{dx}\left[\frac{d\phi}{dx} + 2x\phi\right] \;+\; 3\left[\frac{d\phi}{dx} + 2x\phi\right]$$
$$= x\frac{d^2\phi}{dx^2} \;+\; x\left[2\phi + 2x\frac{d\phi}{dx}\right] \;+\; 3\frac{d\phi}{dx} \;+\; 6x\phi$$
$$= x\frac{d^2\phi}{dx^2} \;+\; \left(3 + 2x^2\right)\frac{d\phi}{dx} \;+\; 8x\phi \quad .$$

So, $\quad L_1L_2 = x\dfrac{d^2}{dx^2} \;+\; \left(3 + 2x^2\right)\dfrac{d}{dx} \;+\; 8x \quad .$

14.6 e. Let ϕ be any sufficiently differentiable function. Then

$$L_2L_1[\phi] \;=\; L_2\left[L_1[\phi]\right] \;=\; L_2\left[\frac{d^2\phi}{dx^2}\right] \;=\; x^3\frac{d^2\phi}{dx^2} \quad .$$

So, $\quad L_2L_1 = x^3\dfrac{d^2}{dx^2} \quad .$

On the other hand,

$$L_1L_2[\phi] \;=\; L_1\left[L_2[\phi]\right] \;=\; L_2\left[x^3\phi\right]$$
$$= \frac{d^2}{dx^2}\left[x^3\phi\right]$$
$$= \frac{d}{dx}\left[\frac{d}{dx}\left[x^3\phi\right]\right]$$
$$= \frac{d}{dx}\left[3x^2\phi \;+\; x^3\frac{d\phi}{dx}\right]$$
$$= \left[6x\phi + 3x^2\frac{d\phi}{dx}\right] \;+\; \left[3x^2\frac{d\phi}{dx} + x^3\frac{d^2\phi}{dx^2}\right]$$
$$= x^3\frac{d^2\phi}{dx^2} \;+\; 6x^2\frac{d\phi}{dx} \;+\; 6x\phi \quad .$$

So, $\quad L_1L_2 = x^3\dfrac{d^2}{dx^2} \;+\; 6x^2\dfrac{d}{dx} \;+\; 6x \quad .$

14.7 a. For any sufficiently differentiable function ϕ,

$$\left(\frac{d}{dx} + 2\right)\left(\frac{d}{dx} + 3\right)[\phi] \;=\; \left(\frac{d}{dx} + 2\right)\left[\frac{d\phi}{dx} + 3\phi\right]$$
$$= \frac{d}{dx}\left[\frac{d\phi}{dx} + 3\phi\right] \;+\; 2\left[\frac{d\phi}{dx} + 3\phi\right]$$
$$= \frac{d^2\phi}{dx^2} \;+\; 3\frac{d\phi}{dx} \;+\; 2\frac{d\phi}{dx} \;+\; 6\phi$$
$$= \frac{d^2\phi}{dx^2} \;+\; 5\frac{d\phi}{dx} \;+\; 6\phi \quad .$$

So,

$$\left(\frac{d}{dx}+2\right)\left(\frac{d}{dx}+3\right) = \frac{d^2}{dx^2} + 5\frac{d}{dx} + 6 \quad .$$

14.7 c. For any sufficiently differentiable function ϕ,

$$\begin{aligned}\left(x\frac{d}{dx}+4\right)\left(\frac{d}{dx}+\frac{1}{x}\right)[\phi] &= \left(x\frac{d}{dx}+4\right)\left[\frac{d\phi}{dx}+\frac{1}{x}\phi\right] \\ &= x\frac{d}{dx}\left[\frac{d\phi}{dx}+\frac{1}{x}\phi\right] + 4\left[\frac{d\phi}{dx}+\frac{1}{x}\phi\right] \\ &= x\frac{d^2\phi}{dx^2} + x\left[\frac{-1}{x^2}\phi+\frac{1}{x}\frac{d\phi}{dx}\right] + 4\frac{d\phi}{dx} + \frac{4}{x}\phi \\ &= x\frac{d^2\phi}{dx^2} + 5\frac{d\phi}{dx} + \frac{3}{x}\phi \quad .\end{aligned}$$

So,

$$\left(x\frac{d}{dx}+4\right)\left(\frac{d}{dx}+\frac{1}{x}\right) = x\frac{d^2}{dx^2} + 5\frac{d}{dx} + \frac{3}{x} \quad .$$

14.7 e. For any sufficiently differentiable function ϕ,

$$\begin{aligned}\left(\frac{d}{dx}+\frac{1}{x}\right)\left(\frac{d}{dx}+4x\right)[\phi] &= \left(\frac{d}{dx}+\frac{1}{x}\right)\left[\frac{d\phi}{dx}+4x\phi\right] \\ &= \frac{d}{dx}\left[\frac{d\phi}{dx}+4x\phi\right] + \frac{1}{x}\left[\frac{d\phi}{dx}+4x\phi\right] \\ &= \frac{d^2\phi}{dx^2} + \left[4\phi+4x\frac{d\phi}{dx}\right] + \frac{1}{x}\frac{d\phi}{dx} + 4\phi \\ &= \frac{d^2\phi}{dx^2} + \left(4x+\frac{1}{x}\right)\frac{d\phi}{dx} + 8\phi \quad .\end{aligned}$$

So,

$$\left(\frac{d}{dx}+\frac{1}{x}\right)\left(\frac{d}{dx}+4x\right) = \frac{d^2}{dx^2} + \left(4x+\frac{1}{x}\right)\frac{d}{dx} + 4 \quad .$$

14.7 g. For any sufficiently differentiable function ϕ,

$$\begin{aligned}\left(\frac{d}{dx}+x^2\right)\left(\frac{d^2}{dx^2}+\frac{d}{dx}\right)[\phi] &= \left(\frac{d}{dx}+x^2\right)\left[\frac{d^2\phi}{dx^2}+\frac{d\phi}{dx}\right] \\ &= \frac{d}{dx}\left[\frac{d^2\phi}{dx^2}+\frac{d\phi}{dx}\right] + x^2\left[\frac{d^2\phi}{dx^2}+\frac{d\phi}{dx}\right] \\ &= \frac{d^3\phi}{dx^3} + \frac{d^2\phi}{dx^2} + x^2\frac{d^2\phi}{dx^2} + x^2\frac{d\phi}{dx} \\ &= \frac{d^3\phi}{dx^3} + \left(1+x^2\right)\frac{d^2\phi}{dx^2} + x^2\frac{d\phi}{dx} \quad .\end{aligned}$$

So,

$$\left(\frac{d}{dx}+x^2\right)\left(\frac{d^2}{dx^2}+\frac{d}{dx}\right) = \frac{d^3}{dx^3} + \left(1+x^2\right)\frac{d^2}{dx^2} + x^2\frac{d}{dx} \quad .$$

14.9. *Verifying the factorization:* For any sufficiently differentiable function ϕ ,

$$\begin{aligned}\left(\frac{d}{dx} - x\right)\left(\frac{d}{dx} + 2x\right)[\phi] &= \left(\frac{d}{dx} - x\right)\left[\frac{d\phi}{dx} + 2x\phi\right] \\ &= \frac{d}{dx}\left[\frac{d\phi}{dx} + 2x\phi\right] - x\left[\frac{d\phi}{dx} + 2x\phi\right] \\ &= \frac{d^2\phi}{dx^2} + \left[2\phi + 2x\frac{d\phi}{dx}\right] - x\frac{d\phi}{dx} - 2x^2\phi \\ &= \frac{d^2\phi}{dx^2} + x\frac{d\phi}{dx} + \left(2 - 2x^2\right)\phi \quad .\end{aligned}$$

So,

$$\frac{d^2}{dx^2} + x\frac{d}{dx} + \left(2 - 2x^2\right) = \left(\frac{d}{dx} - x\right)\left(\frac{d}{dx} + 2x\right) \quad .$$

Finding the solution: From the factoring above, we know the differential equation can be written as

$$\left(\frac{d}{dx} - x\right)\left(\frac{d}{dx} + 2x\right)[y] = 0 \quad ,$$

and from theorem 14.6 on page 294 of the text, we know that any solution y to

$$\left(\frac{d}{dx} + 2x\right)[y] = 0$$

also satisfies the previous equation. Rewriting the last equation in the more traditional form

$$\frac{dy}{dx} + 2xy = 0$$

we see that we have a simple first-order linear and separable differential equation. Using the integrating factor

$$\mu = \mu(x) = e^{\int 2x\,dx} = e^{x^2}$$

we have

$$e^{x^2}\left[\frac{dy}{dx} + 2xy\right] = e^{x^2} \cdot 0$$

$$\hookrightarrow \qquad \frac{d}{dx}\left[e^{x^2}y\right] = 0$$

$$\hookrightarrow \qquad e^{x^2}y = c$$

$$\hookrightarrow \qquad y = y(x) = c^{-x^2} \quad .$$