

**Chapter 12: Higher-Order Linear Equation and the Reduction of Order Method**

**12.1 a.** The equation is in the form

$$ay'' + by' + cy = g$$

where  $a$ ,  $b$ ,  $c$  and  $g$  are the functions of  $x$

$$a = 1 \quad , \quad b = x^2 \quad , \quad c = -4 \quad \text{and} \quad g = x^3 \quad .$$

which is the standard form for a second-order linear differential equation. So the given equation is second order and linear. Since the forcing function  $g$  is  $x^3$  and not 0, the equation is not homogeneous.

**12.1 c.** The equation can be rewritten in the form

$$ay'' + by' + cy = g$$

where  $a$ ,  $b$ ,  $c$  and  $g$  are the functions of  $x$

$$a = 1 \quad , \quad b = x^2 \quad , \quad c = -4 \quad \text{and} \quad g = 0 \quad .$$

which is the standard form for a second-order linear differential equation. So the given equation is second order and linear. Since the forcing function  $g$  is  $x^3$  and not 0, the equation is homogeneous.

**12.1 e.** The equation is in the form

$$ay' + by = g$$

where  $a$ ,  $b$ ,  $c$  and  $g$  are the functions of  $x$

$$a = x \quad , \quad b = 3 \quad \text{and} \quad g = e^{2x} \quad .$$

which is the standard form for a first-order linear differential equation. So the given equation is first order and linear. Since the forcing function  $g$  is  $e^{2x}$  and not 0, the equation is nonhomogeneous.

**12.1 g.** The highest order derivative of  $y$  in the equation is  $y''$ ; so the equation is second order. Because of the  $yy''$  and  $(y'')^3$  terms, the equation cannot be put in the form

$$ay'' + by' + cy = g$$

where  $a$ ,  $b$ ,  $c$  and  $g$  are functions of  $x$  only. So the equation is not linear.

**12.1 i.** The highest order derivative of  $y$  in the equation is  $y^{(iv)}$ ; so the equation is fourth order. By adding 25 to both sides of the equation we get

$$y^{(iv)} + 0y''' + 6y'' + 3y' - 83y = 25 \quad .$$

Since each term on the left is  $y$  or a derivative of  $y$  multiplied by a function of  $x$  (constants in this case), and the right side is a nonzero function of  $x$  only (again, just a constant), the equation is linear and nonhomogeneous.

**12.1 k.** The highest order derivative of  $y$  in the equation is  $y'''$ ; so the equation is third order. Rewriting the equation as

$$y''' + 0y'' + 3y' + x^2y = 0 \quad ,$$

we see that the equation is linear and homogeneous.

**12.2 a.** Letting  $y = y_1 = e^{2x}$ , we have

$$y = e^{2x} \quad \rightsquigarrow \quad y' = 2e^{2x} \quad \rightsquigarrow \quad y'' = 4e^{2x} \quad .$$

Thus,

$$\begin{aligned} y'' - 5y' + 6y &= 4e^{2x} - 5 \cdot 2e^{2x} + 6e^{2x} \\ &= [4 - 10 + 6]e^{2x} = 0 \cdot e^{2x} = 0 \quad , \end{aligned}$$

showing that  $y_1 = e^{2x}$  is one solution to the differential equation.

To find the general solution, now let  $y = y_1u = e^{2x}u$  where  $u = u(x)$  is a yet unknown function of  $x$ . Computing the derivatives, we have

$$y' = (e^{2x}u)' = 2e^{2x}u + e^{2x}u'$$

and

$$\begin{aligned} y'' &= (y')' = (2e^{2x}u + e^{2x}u')' \\ &= (2e^{2x}u)' + (e^{2x}u')' \\ &= 4e^{2x}u + 2e^{2x}u' + 2e^{2x}u' + e^{2x}u'' \\ &= 4e^{2x}u + 4e^{2x}u' + e^{2x}u'' \quad . \end{aligned}$$

So,

$$\begin{aligned} 0 &= y'' - 5y' + 6y \\ &= [4e^{2x}u + 4e^{2x}u' + e^{2x}u''] - 5[2e^{2x}u + e^{2x}u'] + 6[e^{2x}u] \\ &= e^{2x}u'' + [4 - 5 \cdot 1]e^{2x}u' + [4 - 5 \cdot 2 + 6]e^{2x}u \\ &= e^{2x}[u'' - u' + 0u] \quad , \end{aligned}$$

which reduces to  $u'' - u' = 0$ . Setting  $v = u'$ , we have

$$v' - v = 0 \quad \rightsquigarrow \quad v' = v$$

a simple separable equation with constant solution  $v = 0$ . For the other solutions, we divide by  $v$  and continue:

$$\frac{1}{v} \frac{dv}{dx} = 1 \quad \rightsquigarrow \quad \int \frac{1}{v} \frac{dv}{dx} dx = \int 1 dx$$

$$\iff \ln |v| = x + c_1 \quad \rightsquigarrow \quad v = Ae^x \quad .$$

Since this last equation reduces to the constant solution  $v = 0$  if  $A = 0$ , it describes all possible formulas for  $v$ . And since  $v = u'$ ,

$$u = \int u' dx = \int v dx = \int Ae^x dx = Ae^x + B \quad .$$

Hence, the general solution to the original differential equation is

$$y = y_1 u = e^{2x} [Ae^x + B] = Ae^{3x} + Be^x .$$

**12.2 c.** Letting  $y(x) = y_1 = x^3$ , we have

$$y = x^3 \rightsquigarrow y' = 3x^2 \rightsquigarrow y'' = 6x .$$

Thus,

$$\begin{aligned} x^2 y'' - 6xy' + 12y &= x^2[6x] - 6x[3x^2] + 12[x^3] \\ &= 6x^3 - 18x^3 + 12x^3 = 0x^3 = 0 , \end{aligned}$$

showing that  $y_1 = x^3$  is one solution to the differential equation.

To find the general solution, now let  $y = y_1 u = x^3 u$  where  $u = u(x)$  is a yet unknown function of  $x$ . Computing the derivatives, we have

$$y' = (x^3 u)' = 3x^2 u + x^3 u'$$

and

$$\begin{aligned} y'' &= (y')' = (3x^2 u + x^3 u')' \\ &= (3x^2 u)' + (x^3 u')' \\ &= 6xu + 3x^2 u' + 3x^2 u' + x^3 u'' \\ &= 6xu + 6x^2 u' + x^3 u'' . \end{aligned}$$

So,

$$\begin{aligned} 0 &= x^2 y'' - 6xy' + 12y \\ &= x^2 [6xu + 6x^2 u' + x^3 u''] - 6x [3x^2 u + x^3 u'] + 12 [x^3 u] \\ &= 6x^3 u + 6x^4 u' + x^5 u'' - 18x^3 u - 6x^4 u' + 12x^3 u \\ &= x^5 u'' + [6x^4 - 6x^4] u' + [6x^3 - 18x^3 + 12x^3] u \\ &= x^5 u'' + 0u' + 0u , \end{aligned}$$

which reduces to  $u'' = 0$ . Setting  $v = u'$ , we have

$$v' = 0 \rightsquigarrow u' = v = c_1$$

$$\hookrightarrow u = \int u' dx = \int c_1 dx = c_1 x + c_2 .$$

Hence, the general solution to the original differential equation is

$$y = y_1 u = x^3 [c_1 x + c_2] = c_1 x^4 + c_2 x^3 .$$

**12.2 e.** Letting  $y = y_1 = \sqrt{x}$ , we have

$$y = x^{1/2} \rightsquigarrow y' = \frac{1}{2} x^{-1/2} \rightsquigarrow y'' = -\frac{1}{4} x^{-3/2} .$$

Thus,

$$4x^2 y'' + y = 4x^2 \left[ -\frac{1}{4}x^{-3/2} \right] + x^{1/2} = -x^{1/2} + x^{1/2} = 0 \quad ,$$

showing that  $y_1 = \sqrt{x}$  is one solution to the differential equation.

Now, letting the general solution be  $y = y_1 u = x^{1/2} u$ , we have

$$y' = \left( x^{1/2} u \right)' = \frac{1}{2}x^{-1/2} u + x^{1/2} u'$$

and

$$\begin{aligned} y'' &= \left( \frac{1}{2}x^{-1/2} u + x^{1/2} u' \right)' \\ &= -\frac{1}{4}x^{-3/2} u + \frac{1}{2}x^{-1/2} u' + \frac{1}{2}x^{-1/2} u' + x^{1/2} u'' \\ &= -\frac{1}{4}x^{-3/2} u + x^{-1/2} u' + x^{1/2} u'' \quad . \end{aligned}$$

So,

$$\begin{aligned} 0 &= 4x^2 y'' + y \\ &= 4x^2 \left[ -\frac{1}{4}x^{-3/2} u + x^{-1/2} u' + x^{1/2} u'' \right] + \left[ x^{1/2} u \right] \\ &= -x^{1/2} u + 4x^{3/2} u' + 4x^{5/2} u'' + x^{1/2} u \\ &= 4x^{5/2} u'' + 4x^{3/2} u' + 0u \\ &= 4x^{3/2} [xu'' + u'] \quad , \end{aligned}$$

which reduces to  $xu'' + u' = 0$ . Setting  $v = u'$ , we have

$$xv' + v = 0 \quad .$$

A simple equation both linear and separable. For variety, let's solve this as a linear equation:

$$xv' + v = 0 \quad \rightsquigarrow \quad \frac{dv}{dx} + \frac{1}{x}v = 0$$

$$\hookrightarrow \quad \mu = e^{\int (1/x) dx} = e^{\ln|x|} = x \quad (\text{integrating factor on } x > 0)$$

$$\hookrightarrow \quad x \left[ v' + \frac{1}{x}v = 0 \right] \rightsquigarrow \frac{d}{dx} [xv] = 0 \rightsquigarrow xv = c_1$$

$$\hookrightarrow \quad u' = v = \frac{c_1}{x}$$

$$\hookrightarrow \quad u = \int u' dx = \int \frac{c_1}{x} dx = c_1 \ln|x| + c_2 \quad .$$

Hence, the general solution to the original differential equation is

$$y = y_1 u = \sqrt{x} [c_1 \ln|x| + c_2] = c_1 \sqrt{x} \ln|x| + c_2 \sqrt{x} \quad .$$

**12.2 g.** Letting  $y = y_1 = e^{-x}$ , we have

$$y = e^{-x} \rightsquigarrow y' = -e^{-x} \rightsquigarrow y'' = e^{-x} \quad .$$

Thus,

$$\begin{aligned}(x+1)y'' + xy' - y &= (x+1)e^{-x} - xe^{-x} - e^{-x} \\ &= [x+1-x-1]e^{-x} = 0e^{-x} = 0 \quad ,\end{aligned}$$

showing that  $y_1 = e^{-x}$  is one solution to the differential equation.

For the general solution, let  $y = y_1u = e^{-x}u$ . Computing the derivatives, we have

$$y' = (e^{-x}u)' = -e^{-x}u + e^{-x}u'$$

and

$$\begin{aligned}y'' &= (-e^{-x}u + e^{-x}u')' \\ &= e^{-x}u - e^{-x}u' - e^{-x}u' + e^{-x}u'' \\ &= e^{-x}u - 2e^{-x}u' + e^{-x}u'' \quad .\end{aligned}$$

So,

$$\begin{aligned}0 &= (x+1)y'' + xy' - y \\ &= (x+1)[e^{-x}u - 2e^{-x}u' + e^{-x}u''] + x[-e^{-x}u + e^{-x}u'] - [e^{-x}u] \\ &= ([x+1]u'' + [-2x-2+x]u' + [x+1-x-1]u)e^{-x} \\ &= ([x+1]u'' - [x+2]u' + 0u)e^{-x} \quad .\end{aligned}$$

Setting  $v = u'$ , this reduces to

$$[x+1]v' - [x+2]v = 0 \quad \rightsquigarrow \quad \frac{dv}{dx} = \frac{x+2}{x+1}v \quad ,$$

a separable equation with constant solution  $v = 0$ . For the other solutions:

$$\begin{aligned}\frac{dv}{dx} &= \frac{x+2}{x+1}v \quad \rightsquigarrow \quad \frac{1}{v} \frac{dv}{dx} = \frac{x+2}{x+1} = 1 + \frac{1}{x+1} \\ \hookrightarrow & \int \frac{1}{v} \frac{dv}{dx} dx = \int \left[1 + \frac{1}{x+1}\right] dx \\ \hookrightarrow & \ln|v| = x + \ln|x+1| + c_1 \\ \hookrightarrow & v = \pm e^{x+\ln|x+1|+c_1} = A(x+1)e^x \quad .\end{aligned}$$

Since this last equation reduces to the constant solution  $v = 0$  if  $A = 0$ , it describes all possible formulas for  $v = u'$ . Then, using integration by parts

$$\begin{aligned}u &= \int v dx = \int A(x+1)e^x dx \\ &= A \left[ (x+1)e^x - \int e^x dx \right] \\ &= A [xe^x + e^x - e^x + c_2] = Axe^x + B \quad .\end{aligned}$$

Hence, the general solution to the original differential equation is

$$y = y_1u = e^{-x} [Axe^x + B] = Ax + Be^{-x} \quad .$$

12.2 i. Letting  $y = y_1 = \sin(x)$ , we have

$$y = \sin(x) \quad \rightsquigarrow \quad y' = \cos(x) \quad \rightsquigarrow \quad y'' = -\sin(x) \quad .$$

Thus,

$$y'' + y = -\sin(x) + \sin(x) = 0 \quad ,$$

showing that  $y_1 = \sin(x)$  is one solution to the differential equation.

For the general solution, let  $y = y_1 u = \sin(x)u$ . Computing the derivatives,

$$y' = (\sin(x)u)' = \cos(x)u + \sin(x)u'$$

and

$$\begin{aligned} y'' &= (\cos(x)u + \sin(x)u')' \\ &= -\sin(x)u + \cos(x)u' + \cos(x)u' + \sin(x)u'' \\ &= -\sin(x)u + 2\cos(x)u' + \sin(x)u'' \quad . \end{aligned}$$

So,

$$\begin{aligned} 0 &= y'' + y \\ &= -\sin(x)u + 2\cos(x)u' + \sin(x)u'' + \sin(x)u \\ &= \sin(x)u'' + 2\cos(x)u' + 0u \quad . \end{aligned}$$

Setting  $v = u'$ , this reduces to

$$\sin(x)v' + 2\cos(x)v = 0 \quad \rightsquigarrow \quad \frac{dv}{dx} = -2\frac{\cos(x)}{\sin(x)}v$$

a separable equation with constant solution  $v = 0$ . For the other solutions, we divide by  $v$  and continue:

$$\frac{1}{v} \frac{dv}{dx} = -2\frac{\cos(x)}{\sin(x)} \quad \rightsquigarrow \quad \int \frac{1}{v} \frac{dv}{dx} dx = -2 \int \frac{\cos(x)}{\sin(x)} dx$$

$$\hookrightarrow \quad \ln|v| = -2\ln|\sin(x)| + c_1$$

$$\hookrightarrow \quad u' = v = \pm e^{-2\ln|\sin(x)|+c_1} = C \sin^{-2}(x) \quad .$$

Since this last equation reduces to the constant solution  $v = 0$  if  $A = 0$ , it describes all possible formulas for  $v = u'$ . Then, after recalling that  $\sin^{-2}(x)$  is the derivative of  $-\cot(x)$ , we have

$$\begin{aligned} u &= \int C \sin^{-2}(x) dx \\ &= -C \cot(x) + B = A \frac{\cos(x)}{\sin(x)} + B \quad . \end{aligned}$$

Hence, the general solution to the original differential equation is

$$y = y_1 u = \sin(x) \left[ A \frac{\cos(x)}{\sin(x)} + c_2 \right] = A \cos(x) + B \sin(x) \quad .$$

**12.2 k.** Letting  $y = y_1 = \sin(x)$ , we have

$$y = \sin(x) \rightsquigarrow y' = \cos(x) \rightsquigarrow y'' = -\sin(x) \quad .$$

Thus,

$$\begin{aligned} \sin^2(x)y'' - 2\cos(x)\sin(x)y' + (1 + \cos^2(x))y &= \\ &= [-\sin^2(x) - 2\cos^2(x) + 1 + \cos^2(x)]\sin(x) \\ &= [1 - (\sin^2(x) + \cos^2(x))]\sin(x) = [1 - 1]\sin(x) = 0 \quad , \end{aligned}$$

showing that  $y_1 = \sin(x)$  is one solution to the differential equation.

For the general solution, let  $y = y_1u = \sin(x)u$ . Computing the derivatives, we have

$$y' = (\sin(x)u)' = \cos(x)u + \sin(x)u'$$

and

$$\begin{aligned} y'' &= (\cos(x)u + \sin(x)u')' \\ &= -\sin(x)u + \cos(x)u' + \cos(x)u' + \sin(x)u'' \\ &= -\sin(x)u + 2\cos(x)u' + \sin(x)u'' \quad . \end{aligned}$$

So,

$$\begin{aligned} 0 &= \sin^2(x)y'' - 2\cos(x)\sin(x)y' + (1 + \cos^2(x))y \\ &= \sin^2(x)[- \sin(x)u + 2\cos(x)u' + \sin(x)u''] \\ &\quad - 2\cos(x)\sin(x)[\cos(x)u + \sin(x)u'] + (1 + \cos^2(x))[\sin(x)u] \\ &= \sin^3(x)u'' + [2\cos(x)\sin^2(x) - 2\cos(x)\sin^2(x)]u' \\ &\quad + [-\sin^3(x) - 2\cos^2(x)\sin(x) + (1 + \cos^2(x))\sin(x)]u \\ &= \sin(x)(\sin^2(x)u'' + 0u' + [1 - \sin^2(x) + \cos^2(x)]u) \\ &= \sin(x)(\sin^2(x)u'' + 0u' + 0u) \quad , \end{aligned}$$

which reduces to  $u'' = 0$ . Setting  $v = u'$  would be silly; just integrate:

$$\begin{aligned} u' &= \int u'' dx = \int 0 dx = c_1 \\ \Leftrightarrow u &= \int u' dx = \int c_1 dx = c_1x + c_2 \quad . \end{aligned}$$

Thus, the general solution to the original differential equation is

$$y = y_1u = \sin(x)[c_1x + c_2] = c_1x\sin(x) + c_2\sin(x) \quad .$$

**12.2 m.** Letting  $y = y_1 = \sin(\ln|x|)$ , we have

$$y = \sin(\ln|x|) \rightsquigarrow y' = \cos(\ln|x|)x^{-1}$$

$$\Leftrightarrow y'' = -\sin(\ln|x|)x^{-2} - \cos(\ln|x|)x^{-2} \quad .$$

Thus,

$$\begin{aligned} x^2 y'' + xy' + y &= x^2 \left[ -\sin(\ln|x|)x^{-2} - \cos(\ln|x|)x^{-2} \right] \\ &\quad + x [\cos(\ln|x|)] + \sin(\ln|x|) \\ &= -\sin(\ln|x|) - \cos(\ln|x|) + \cos(\ln|x|) + \sin(\ln|x|) \\ &= 0 \quad , \end{aligned}$$

showing that  $y_1 = \sin(\ln|x|)$  is one solution to the differential equation.

For the general solution, let  $y = y_1 u = \sin(\ln|x|)u$ . Computing the derivatives, we have

$$y' = (\sin(\ln|x|)u)' = \cos(\ln|x|)x^{-1}u + \sin(\ln|x|)u'$$

and

$$\begin{aligned} y'' &= (y')' = \left( \cos(\ln|x|)x^{-1}u + \sin(\ln|x|)u' \right)' \\ &= \left[ -\sin(\ln|x|)x^{-2} - \cos(\ln|x|)x^{-2} \right] u \\ &\quad + \cos(\ln|x|)x^{-1}u' + \cos(\ln|x|)x^{-1}u' + \sin(\ln|x|)u'' \\ &= -x^{-2}[\sin(\ln|x|) + \cos(\ln|x|)]u \\ &\quad + 2\cos(\ln|x|)x^{-1}u' + \sin(\ln|x|)u'' \quad . \end{aligned}$$

So,

$$\begin{aligned} 0 &= x^2 y'' + xy' + y \\ &= x^2 \left[ x^{-2}[-\sin(\ln|x|) - \cos(\ln|x|)]u + 2\cos(\ln|x|)x^{-1}u' + \sin(\ln|x|)u'' \right] \\ &\quad + x \left[ \cos(\ln|x|)x^{-1}u + \sin(\ln|x|)u' \right] + \sin(\ln|x|)u \\ &= x^2 \sin(\ln|x|)u'' + x [2\cos(\ln|x|) + \sin(\ln|x|)]u' \\ &\quad + [-\sin(\ln|x|) - \cos(\ln|x|) + \cos(\ln|x|) + \sin(\ln|x|)]u \\ &= x (x \sin(\ln|x|)u'' + [2\cos(\ln|x|) + \sin(\ln|x|)]u') + 0u \quad . \end{aligned}$$

Setting  $v = u'$ , this reduces to

$$x \sin(\ln|x|)v' + [2\cos(\ln|x|) + \sin(\ln|x|)]v = 0$$

$$\hookrightarrow \frac{dv}{dx} = \left[ -2\frac{\cos(\ln|x|)}{x \sin(\ln|x|)} - \frac{1}{x} \right] v$$

a separable equation with constant solution  $v = 0$ . For the other solutions, we divide by  $v$ , integrate, and continue:

$$\int \frac{1}{v} \frac{dv}{dx} dx = \int \left[ -2\frac{\cos(\ln|x|)}{x \sin(\ln|x|)} - \frac{1}{x} \right] dx$$

$$\hookrightarrow \ln|v| = -2 \ln|\sin(\ln|x|)| - \ln|x| + c_1$$

$$\hookrightarrow v = Ax^{-1} \sin^{-2}(\ln|x|) \quad .$$



Since this last equation reduces to the constant solution  $v = 0$  if  $A = 0$ , it describes all possible formulas for  $v = u'$ . Integrating (use a substitution  $s = \ln|x|$  and then recall that  $\sin^{-2}(s)$  is the derivative of  $-\cot(s)$ ), we get

$$\begin{aligned} u &= \int v \, dx = \int Ax^{-1} \sin^{-2}(\ln|x|) \, dx \\ &= -A \cot(\ln|x|) + c_2 = C \frac{\cos(\ln|x|)}{\sin(\ln|x|)} + B . \end{aligned}$$

Hence, the general solution to the original differential equation is

$$y = y_1 u = \sin(\ln|x|) \left[ C \frac{\cos(\ln|x|)}{\sin(\ln|x|)} + B \right] = C \cos(\ln|x|) + B \sin(\ln|x|) .$$

**12.3 a.** Letting  $y = y_1 = e^{3x}$ , we have

$$y = e^{3x} \quad \rightsquigarrow \quad y' = 3e^{3x} \quad \rightsquigarrow \quad y'' = 9e^{3x} .$$

Thus,

$$\begin{aligned} y'' - 4y' + 3y &= 9y'' - 4 \cdot 3e^{2x} + 3e^{2x} \\ &= [9 - 12 + 3]e^{2x} = 0e^{2x} = 0 , \end{aligned}$$

showing that  $y_1 = e^{3x}$  is one solution to the corresponding homogeneous differential equation.

For the general solution, let  $y = y_1 u = e^{3x} u$ . Computing the derivatives, we have

$$y' = (e^{3x} u)' = 3e^{3x} u + e^{3x} u' = [3u + u'] e^{3x}$$

and

$$\begin{aligned} y'' &= (y')' = (3e^{3x} u + e^{3x} u')' \\ &= 9e^{3x} u + 3e^{3x} u' + 3e^{3x} u' + e^{3x} u'' \\ &= [9u + 6u' + u''] e^{3x} . \end{aligned}$$

So,

$$\begin{aligned} 9e^{2x} &= y'' - 4y' + 3y \\ &= [9u + 6u' + u''] e^{3x} - 4[3u + u'] e^{3x} + 3e^{3x} u \\ &= (u'' + [6 - 4]u' + [9 - 12 + 3]u) e^{3x} \\ &= (u'' + 2u') e^{3x} . \end{aligned}$$

Setting  $v = u'$  and continuing:

$$9e^{2x} = (v' + 2v) e^{3x} \quad \rightsquigarrow \quad \frac{dv}{dx} + 2v = 9e^{-x}$$

$$\hookrightarrow \quad \mu = e^{\int 2 \, dx} = e^{2x} \quad (\text{integrating factor for above equation})$$

$$\hookrightarrow \quad e^{2x} \left[ \frac{dv}{dx} + 2v = 9e^{-x} \right] \quad \rightsquigarrow \quad \frac{d}{dx} [e^{2x} v] = 9e^x$$

$$\hookrightarrow e^{2x}v = \int 9e^x dx = 9e^x + c_1$$

$$\hookrightarrow u' = v = 9e^{-x} + c_1e^{-2x}$$

$$\hookrightarrow u = \int [9e^{-x} + c_1e^{-2x}] = -9e^{-x} - \frac{c_1}{2}e^{-2x} + c_3$$

$$\hookrightarrow u = A + Be^{-2x} - 9e^{-x} .$$

Hence, the general solution to the original differential equation is

$$y = y_1u = e^{3x} [A + Be^{-2x} - 9e^{-x}] = Ae^{3x} + Be^x - 9e^{2x} .$$

**12.3 c.** Letting  $y = y_1 = x$ , we have

$$y = x \rightsquigarrow y' = 1 \rightsquigarrow y'' = 0 .$$

Thus,

$$x^2y'' + xy' - y = x^2 \cdot 0 + x \cdot 1 - x = x - x = 0 ,$$

showing that  $y_1 = x$  is one solution to the corresponding homogeneous differential equation.

Now let  $y = y_1u = xu$ . Computing the derivatives, we have

$$y' = (xu)' = u + xu'$$

and

$$y'' = (u + xu')' = u' + u' + xu'' = 2u' + xu'' .$$

So,

$$\begin{aligned} \sqrt{x} &= x^2y'' + xy' - y \\ &= x^2[2u' + xu''] + x[u + xu'] - xu \\ &= x^3u'' + [2x^2 + x^2]u' + [x - x]u \\ &= x^2[xu'' + 3u'] . \end{aligned}$$

Setting  $v = u'$  and continuing:

$$\sqrt{x} = x^2[xv' + 3v] \rightsquigarrow \frac{dv}{dx} + \frac{3}{x}v = x^{-5/2}$$

$$\hookrightarrow \mu = e^{\int (3/x) dx} = x^3 \quad (\text{integrating factor for above equation})$$

$$\hookrightarrow x^3 \left[ \frac{dv}{dx} + \frac{3}{x}v = x^{-5/2} \right] \rightsquigarrow \frac{d}{dx} [x^3v] = x^{1/2}$$

$$\hookrightarrow x^3v = \int x^{1/2} dx = \frac{2}{3}x^{3/2} + c_1$$

$$\hookrightarrow u' = v = \frac{2}{3}x^{-3/2} + c_1x^{-3}$$

$$\hookrightarrow u = \int \left[ \frac{2}{3}x^{-3/2} + c_1x^{-3} \right] dx = -\frac{3}{4}x^{-1/2} - \frac{c_1}{2}x^{-2} + c_2$$

$$\hookrightarrow u = A + Bx^{-2} - \frac{3}{4}x^{-1/2} .$$

Hence, the general solution to the original differential equation is

$$y = y_1u = x \left[ A + Bx^{-2} - \frac{3}{4}x^{-1/2} \right] = Ax + Bx^{-1} - \frac{3}{4}x^{1/2} .$$

**12.3 e.** Letting  $y = y_1 = x^{-1}$ , we have

$$y = x^{-1} \quad \rightsquigarrow \quad y' = -x^{-2} \quad \rightsquigarrow \quad y'' = 2x^{-3} .$$

Thus,

$$\begin{aligned} xy'' + (2+2x)y' + 2y &= x \left[ 2x^{-3} \right] + (2+2x) \left[ -x^{-2} \right] + 2 \left[ x^{-1} \right] \\ &= 2x^{-2} - 2x^{-2} - 2x^{-1} + 2x^{-1} = 0 , \end{aligned}$$

showing that  $y_1 = x^{-1}$  is one solution to the corresponding homogeneous differential equation.

Now let  $y = y_1u = x^{-1}u$ . Computing the derivatives, we have

$$y' = \left( x^{-1}u \right)' = -x^{-2}u + x^{-1}u'$$

and

$$\begin{aligned} y'' &= \left( -x^{-2}u + x^{-1}u' \right)' \\ &= 2x^{-3}u - x^{-2}u' - x^{-2}u' + x^{-1}u'' \\ &= 2x^{-3}u - 2x^{-2}u' + x^{-1}u'' . \end{aligned}$$

So,

$$\begin{aligned} 8e^{2x} &= xy'' + (2+2x)y' + 2y \\ &= x \left[ 2x^{-3}u - 2x^{-2}u' + x^{-1}u'' \right] \\ &\quad + (2+2x) \left[ -x^{-2}u + x^{-1}u' \right] + 2 \left[ x^{-1}u \right] \\ &= u'' + \left[ -2x^{-1} + 2x^{-1} + 2 \right] u' + \left[ 2x^{-2} - 2x^{-2} - 2x^{-1} + 2x^{-1} \right] u \\ &= u'' + 2u' . \end{aligned}$$

Setting  $v = u'$  and continuing:

$$8e^{2x} = v' + 2v \quad \rightsquigarrow \quad \frac{dv}{dx} + 2v = 8e^{2x}$$

$$\hookrightarrow \mu = e^{\int 2dx} = e^{2x} \quad (\text{integrating factor for above equation})$$

$$\hookrightarrow e^{2x} \left[ \frac{dv}{dx} + 2v = 8e^{2x} \right] \quad \rightsquigarrow \quad \frac{d}{dx} \left[ e^{2x}v \right] = 8e^{4x}$$

$$\hookrightarrow e^{2x}v = \int 8e^{4x} dx = 2e^{4x} + c_1 \quad \rightsquigarrow v = 2e^{2x} + c_1e^{-2x}$$

$$\hookrightarrow u = \int v dx = \int [2e^{2x} + c_1e^{-2x}] dx = e^{2x} - \frac{c_1}{2}e^{-2x} + c_2$$

$$\hookrightarrow u = e^{2x} + Be^{-2x} + A .$$

Hence, the general solution to the original differential equation is

$$y = y_1u = x^{-1} [e^{2x} + A + Be^{-2x}] .$$

**12.5 a.** Letting  $y = y_1 = e^{3x}$ , we have

$$y = e^{3x} \quad \rightsquigarrow \quad y' = 3e^{3x} \quad \rightsquigarrow \quad y'' = 9e^{3x} \quad \rightsquigarrow \quad y''' = 27e^{3x} .$$

Thus,

$$\begin{aligned} y''' - 9y'' + 27y' - 27y &= 27e^{3x} - 9 \cdot 9e^{3x} + 27 \cdot 3e^{3x} - 27e^{3x} \\ &= [27 - 81 + 81 - 27]e^{3x} = 0 , \end{aligned}$$

showing that  $y_1 = e^{3x}$  is one solution to the homogeneous differential equation.

Now let  $y = y_1u = e^{3x}u$ . Computing the derivatives, we have

$$y' = (e^{3x}u)' = 3e^{3x}u + e^{3x}u' = e^{3x} [3u + u'] ,$$

$$\begin{aligned} y'' &= (3e^{3x}u + e^{3x}u')' \\ &= 9e^{3x}u + 3e^{3x}u' + 3e^{3x}u' + e^{3x}u'' = e^{3x} [9u + 6u' + u''] , \end{aligned}$$

and

$$\begin{aligned} y''' &= (e^{3x} [9u + 6u' + u''])' \\ &= 3e^{3x} [9u + 6u' + u''] + e^{3x} [9u' + 6u'' + u'''] \\ &= e^{3x} [27u + 27u' + 9u'' + u'''] . \end{aligned}$$

So,

$$\begin{aligned} 0 &= y''' - 9y'' + 27y' - 27y \\ &= e^{3x} ([27u + 27u' + 9u'' + u'''] \\ &\quad - 9[9u + 6u' + u''] + 27[3u + u'] - 27[u]) \\ &= e^{3x} (u''' + [9 - 9]u'' + [27 - 54 + 27]u' + [27 - 81 + 81 - 27]u) \\ &= e^{3x} (u''' + 0u'' + 0u' + 0u) . \end{aligned}$$

So  $u''' = 0$ , and we can find  $u$  simply by integrating:

$$u'' = \int u''' dx = \int 0 dx = c_1$$

$$\hookrightarrow u' = \int u'' dx = \int c_1 dx = c_1x + c_2$$

$$\hookrightarrow u = \int u' dx = \int [c_1x + c_2] dx = \frac{c_1}{2}x^2 + c_2x + c_3$$

$$\hookrightarrow u = Cx^2 + Bx + A .$$

Hence, the general solution to the original differential equation is

$$y = y_1u = e^{3x} [A + Bx + Cx^2] .$$

**12.5 c.** Letting  $y = y_1 = e^{3x}$ , we have

$$y = e^{2x} \rightsquigarrow y' = 2e^{2x} \rightsquigarrow y'' = 4e^{3x}$$

$$\hookrightarrow y''' = 8e^{2x} \rightsquigarrow y^{(4)} = 16e^{2x} .$$

Thus,

$$\begin{aligned} y^{(4)} - 8y''' + 24y'' - 32y' + 16y &= 16e^{2x} - 8 \cdot 8e^{2x} + 24 \cdot 4e^{2x} - 32 \cdot 2e^{2x} + 16e^{2x} \\ &= [16 - 64 + 96 - 64 + 16]e^{2x} = [0]e^{2x} = 0 , \end{aligned}$$

showing that  $y_1 = e^{2x}$  is one solution to the homogeneous differential equation.

Now let  $y = y_1u = e^{2x}u$ . Computing the derivatives, we have

$$y' = (e^{2x}u)' = 2e^{2x}u + e^{2x}u' = e^{2x} [2u + u'] ,$$

$$\begin{aligned} y'' &= (2e^{2x}u + e^{2x}u')' \\ &= 4e^{2x}u + 2e^{2x}u' + 2e^{2x}u' + e^{2x}u'' = e^{2x} [4u + 4u' + u''] , \end{aligned}$$

$$\begin{aligned} y''' &= (e^{2x} [4u + 4u' + u''])' \\ &= 2e^{2x} [4u + 4u' + u''] + e^{2x} [4u' + 4u'' + u'''] \\ &= e^{2x} [8u + 12u' + 6u'' + u'''] , \end{aligned}$$

and

$$\begin{aligned} y^{(4)} &= (e^{2x} [8u + 12u' + 6u'' + u'''])' \\ &= 2e^{2x} [8u + 12u' + 6u'' + u'''] + e^{2x} [8u' + 12u'' + 6u''' + u^{(4)}] \\ &= e^{2x} [16u + 32u' + 24u'' + 8u''' + u^{(4)}] . \end{aligned}$$

So,

$$\begin{aligned} 0 &= y^{(4)} - 8y''' + 24y'' - 32y' + 16y \\ &= e^{2x} \left( [16u + 32u' + 24u'' + 8u''' + u^{(4)}] - 8[8u + 12u' + 6u'' + u'''] \right. \\ &\quad \left. + 24[4u + 4u' + u''] - 32[2u + u'] + 16[u] \right) \end{aligned}$$

$$\begin{aligned}
&= e^{2x} \left( u^{(4)} + [8 - 8]u''' + [24 - 48 + 24]u'' \right. \\
&\quad \left. + [32 - 96 + 96 - 32]u' + [16 - 64 + 96 - 64 + 16]u \right) \\
&= e^{2x} \left( u^{(4)} + 0u''' + 0u'' + 0u' + 0u \right) .
\end{aligned}$$

So  $u^{(4)} = 0$ , and we can find  $u$  simply by integrating:

$$u''' = \int u^{(4)} dx = \int 0 dx = c_1$$

$$\hookrightarrow u'' = \int u''' dx = \int c_1 dx = c_1x + c_2$$

$$\hookrightarrow u' = \int u'' dx = \int [c_1x + c_2] dx = \frac{c_1}{2}x^2 + c_2x + c_3$$

$$\hookrightarrow u = \int u' dx = \int \left[ \frac{c_1}{2}x^2 + c_2x + c_3 \right] dx = \frac{c_1}{6}x^3 + \frac{c_2}{2}x^2 + c_3x + c_4$$

$$\hookrightarrow u = Dx^3 + Cx^2 + Bx + A .$$

Hence, the general solution to the original differential equation is

$$y = y_1u = e^{2x} \left[ A + Bx + Cx^2 + Dx^3 \right] .$$