Chapter 11: Higher-Order Equations: Extending First-Order Concepts

11.1 a. Setting $y' = \frac{dy}{dx} = v$ and $y'' = \frac{d^2y}{dx^2} = \frac{dv}{dx}$, we have

$$xy'' + 4y' = 18x^2 \quad \rightarrowtail \quad x\frac{dv}{dx} + 4v = 18x^2$$

$$\hookrightarrow \qquad \frac{dv}{dx} + \frac{4}{x}v = 18x \quad ,$$

which is a first-order linear equation with integrating factor

$$\mu = e^{\int (4/x)\,dx} = e^{4\ln|x|} = x^4 \quad .$$

Proceeding as usual,

$$x^4\left[\frac{dv}{dx} + \frac{4}{x}v = 18x\right] \quad \rightarrowtail \quad \frac{d}{dx}\left[x^4 v\right] = 18x^5$$

$$\hookrightarrow \quad x^4 v = \int 18x^5\,dx = 3x^6 + c_1 \quad \rightarrowtail \quad v = 3x^2 + C_1 x^{-4} \quad .$$

And since $v = {}^{dy}/_{dx}$,

$$y = \int \frac{dy}{dx}\,dx = \int v\,dx = \int \left[3x^2 + C_1 x^{-4}\right] dx$$

$$= x^3 - \frac{C_1}{3}x^{-3} + c_2 = x^3 - \frac{c_1}{x^3} + c_2 \quad .$$

11.1 c. Setting $y' = \frac{dy}{dx} = v$ and $y'' = \frac{d^2y}{dx^2} = \frac{dv}{dx}$, we have

$$y'' = y' \quad \rightarrowtail \quad \frac{dv}{dx} = v \quad ,$$

a simple separable equation with $v = 0$ as the only constant solution. Dividing through by v and proceeding as usual:

$$\frac{1}{v}\frac{dv}{dx} = 1 \quad \rightarrowtail \quad \int \frac{1}{v}\frac{dv}{dv}\,dx = \int 1\,dx$$

$$\hookrightarrow \qquad \ln|v| = x + c_1 \quad \rightarrowtail \quad v = \pm e^{x+c_1} = Ae^x \quad .$$

Since the last formula reduces to the constant solution $v = 0$ when $A = 0$, that last formula can be used as the general formula for $v(x)$. Integrating to get y :

$$y = \int \frac{dy}{dx}\,dx = \int v\,dx = \int Ae^x\,dx = Ae^x + c \quad .$$

11.1 e. Setting $y' = \frac{dy}{dx} = v$ and $y'' = \frac{d^2y}{dx^2} = \frac{dv}{dx}$, we have

$$xy'' = y' - 2x^2y' \quad \rightarrowtail \quad x\frac{dv}{dx} = v - 2x^2 v \quad \rightarrowtail \quad \frac{dv}{dx} = \left(\frac{1}{x} - 2x\right)v \quad ,$$

a simple separable equation with $v = 0$ as the only constant solution. Dividing through by v and proceeding as usual:

$$\frac{1}{v}\frac{dv}{dx} = \frac{1}{x} - 2x \quad \rightarrowtail \quad \int \frac{1}{v}\frac{dv}{dx}\,dx = \int \left[\frac{1}{x} - 2x\right] dx$$

$$\hookrightarrow \qquad \ln|v| = \ln|x| - x^2 + c_1$$

$$\hookrightarrow \qquad v = \pm e^{\ln|x| - x^2 + c_1} = \pm e^{\ln|x|} e^{-x^2} e^{c_1} = A_1 x e^{-x^2} \quad .$$

Since the last line reduces to the constant solution $v = 0$ when $A = 0$, that last formula can be used as the general formula for $v(x)$. Integrating to get y:

$$y = \int \frac{dy}{dx}\,dx = \int v\,dx$$

$$= \int A_1 x e^{-x^2}\,dx = -\tfrac{1}{2} A_1 e^{-x^2} + c = A e^{-x^2} + c \quad .$$

11.2 a. The equation does not contain a y, only y' and y''.
Setting $\quad y' = \frac{dy}{dx} = v \quad$ and $\quad y'' = \frac{d^2y}{dx^2} = \frac{dv}{dx} \quad$, we have

$$y'' = 4x\sqrt{y'} \quad \rightarrowtail \quad \frac{dv}{dx} = 4x\sqrt{v} \quad ,$$

a simple separable equation with $v = 0$ as the only constant solution. Dividing through by $\sqrt{v}$ and proceeding as usual:

$$v^{-1/2}\frac{dv}{dx} = 4x \quad \rightarrowtail \quad \int v^{-1/2}\frac{dv}{dx}\,dx = \int 4x\,dx$$

$$\hookrightarrow \qquad 2v^{1/2} = 2x^2 + C_1$$

$$\hookrightarrow \qquad v = \left(x^2 + c_1\right)^2 = x^4 + 2c_1x^2 + c_1{}^2 \quad .$$

This last formula for $v(x)$ does not reduce to the constant solution $v = 0$ for any choice of c_1. So, to describe all solutions to our original equation, we need both

$$y = \int \frac{dy}{dx}\,dx = \int v\,dx = \int 0\,dx = C \quad ,$$

and

$$y = \int \frac{dy}{dx}\,dx = \int \left[x^4 + 2c_1x^2 + c_1{}^2\right] dx = \tfrac{1}{5}x^5 + \tfrac{2}{3}c_1x^3 + c_1{}^2x + c_2 \quad .$$

11.2 c. This differential equation explicitly contains a y factor. Because of this, the substitution of $v = y'$ with $v' = y''$ is not appropriate.

11.2 e. The equation does not contain a y, only y', y'' and x.

Setting $y' = \frac{dy}{dx} = v$ and $y'' = \frac{d^2y}{dx^2} = \frac{dv}{dx}$, we have

$$xy'' - y' = 6x^5 \quad\rightarrowtail\quad x\frac{dv}{dx} - v = 6x^5 \quad\rightarrowtail\quad \frac{dv}{dx} - \frac{1}{x}v = 6x^4 \quad ,$$

a linear first-order equation with integrating factor

$$\mu = e^{\int \left(-1/x\right) dx} = e^{-\ln|x|} = \frac{1}{|x|} \quad .$$

As noted in the chapter on first-order linear equations, we can use $\mu = 1/x$ and proceed as usual with such equations:

$$\frac{1}{x}\left[\frac{dv}{dx} - \frac{1}{x}v = 6x^4\right] \quad\rightarrowtail\quad \frac{d}{dx}\left[\frac{1}{x}v\right] = 6x^3$$

$$\hookrightarrow \quad \frac{1}{x}v = \int 6x^3\,dx = \frac{6}{4}x^4 + C_1 \quad\rightarrowtail\quad v = \frac{6}{4}x^5 + C_1x$$

$$\hookrightarrow \quad y = \int v\,dx = \int \left[\frac{6}{4}x^5 + C_1x\right] dx = \frac{1}{4}x^6 + C_1\frac{1}{2}x^2 + c_2$$

$$\hookrightarrow \quad y = \frac{1}{4}x^6 + c_1x^2 + c_2 \quad .$$

11.2 g. The equation does not contain a y, only y' and y''.

Setting $y' = \frac{dy}{dx} = v$ and $y'' = \frac{d^2y}{dx^2} = \frac{dv}{dx}$, we have

$$y'' = 2y' - 6 \quad\rightarrowtail\quad \frac{dv}{dx} = 2v - 6 \quad\rightarrowtail\quad \frac{dv}{dx} - 2v = -6 \quad ,$$

a linear first-order equation with integrating factor

$$\mu = e^{\int(-2)\,dx} = e^{-2x} \quad .$$

Proceeding as usual with such equations:

$$e^{-2x}\left[\frac{dv}{dx} - 2v = -6\right] \quad\rightarrowtail\quad \frac{d}{dx}\left[e^{-2x}v\right] = -6e^{-2x}$$

$$\hookrightarrow \quad e^{-2x}v = -\int 6e^{-2x}\,dx = 3e^{-2x} + C$$

$$\hookrightarrow \quad v = 3 + Ce^{2x}$$

$$\hookrightarrow \quad y = \int v\,dx = \int \left[3 + Ce^{2x}\right] dx = 3x + \frac{C}{2}e^{2x} + c_2$$

$$\hookrightarrow \quad y = 3x + c_1e^{2x} + c_2 \quad .$$

11.2 i. The equation does not contain a y, only y', y'' and x.

Setting $y' = \frac{dy}{dx} = v$ and $y'' = \frac{d^2y}{dx^2} = \frac{dv}{dx}$, we have

$$y'' + 4y' = 9e^{-3x} \quad\rightarrowtail\quad \frac{dv}{dx} + 4v = 9e^{-3x} \quad ,$$

a linear first-order equation with integrating factor

$$\mu = e^{\int 4\,dx} = e^{4x} \quad .$$

Proceeding as usual with such equations:

$$e^{4x}\left[\frac{dv}{dx} + 4v = 9e^{-3x}\right] \quad\rightarrowtail\quad \frac{d}{dx}\left[e^{4x}v\right] = 9e^{x}$$

$$\hookrightarrow \qquad e^{4x}v = \int 9e^{x}\,dx = 9e^{x} + C$$

$$\hookrightarrow \qquad v = 9e^{-3x} + Ce^{-4x}$$

$$\hookrightarrow \qquad y = \int v\,dx = \int\left[9e^{-3x} + Ce^{-4x}\right]dx = -3e^{-3x} - \frac{1}{4}Ce^{-4x} + c_2$$

$$\hookrightarrow \qquad y = -3e^{-3x} + Ae^{-4x} + c_2 \quad .$$

11.3 a. Setting $y'' = \frac{d^2y}{dx^2} = v$ and $y''' = \frac{d^3y}{dx^3} = \frac{dv}{dx}$, we have

$$y''' = y'' \quad\rightarrowtail\quad \frac{dv}{dx} = v \quad ,$$

a simple separable differential equation with constant solution $v = 0$. Dividing through by v and proceeding as usual:

$$\frac{1}{v}\frac{dv}{dx} = 1 \quad\rightarrowtail\quad \int \frac{1}{v}\frac{dv}{dx}\,dx = \int 1\,dx$$

$$\hookrightarrow \qquad \ln|v| = x + c_1 \quad\rightarrowtail\quad v = \pm e^{x+c_1} = Ae^{x} \quad .$$

If $A = 0$ the last reduces to the constant solution $v = 0$. So the last equation describes all possible v's. Since $v = y''$,

$$y' = \int y''\,dx = \int v\,dx = \int Ae^{x}\,dx = Ae^{x} + B$$

and

$$y = \int y'\,dx = \int\left[Ae^{x} + B\right]dx = Ae^{x} + Bx + C \quad .$$

11.3 c. Setting $y'' = \frac{d^2y}{dx^2} = v$ and $y''' = \frac{d^3y}{dx^3} = \frac{dv}{dx}$, we have

$$y''' = 2\sqrt{y''} \quad\rightarrowtail\quad \frac{dv}{dx} = 2\sqrt{v} \quad ,$$

a simple separable differential equation with constant solution $v = 0$. Dividing through by $\sqrt{v}$ and proceeding as usual:

$$v^{-1/2}\frac{dv}{dx} = 2 \quad\rightarrowtail\quad \int v^{-1/2}\frac{dv}{dx}\,dx = \int 2\,dx$$

$$\hookrightarrow \qquad 2v^{1/2} = 2x + c_1 \quad\rightarrowtail\quad v = \pm(x + A)^2 \quad .$$

Since $v = y''$,

$$y' = \int y'' \, dx = \int v \, dx = \pm \int (x + A)^2 \, dx = \pm (x + A)^3 + B$$

and

$$y = \int y' \, dx = \int \left[\pm \tfrac{1}{3} (x + A)^3 + B \right] dx = \pm \tfrac{1}{12} (x + A)^4 + Bx + C \quad .$$

However, this formula for $y(x)$ came from $v = \pm(x + A)^2$, which does not reduce to the constant solution $v = 0$ for any value of A . So we also have to find the formulas of y corresponding to $v = 0$:

$$y' = \int y'' \, dx = \int v \, dx = \int 0 \, dx = B$$

and

$$y = \int y' \, dx = \int B \, dx = Bx + C \quad .$$

So, to describe all solutions to the original differential equation, we need both

$$y = \pm \tfrac{1}{12} (x + A)^4 + Bx + C \qquad \text{and} \qquad y = Bx + C \quad .$$

11.4 a. Setting $\quad y' = \frac{dy}{dx} = v \qquad \text{and} \qquad y'' = \frac{d^2 y}{dx^2} = \frac{dv}{dx} = \frac{dv}{dy} \frac{dy}{dx} = \frac{dv}{dy} v \quad ,$

we have

$$yy'' = (y')^2 \quad \rightarrowtail \quad y \frac{dv}{dx} v = v^2 \quad \rightarrowtail \quad \frac{dv}{dx} = \frac{v}{y} \quad ,$$

a simple separable first-order equation with constant solution $v = 0$. Dividing through by v and proceeding as usual

$$\frac{1}{v} \frac{dv}{dy} = \frac{1}{y} \quad \rightarrowtail \quad \int \frac{1}{v} \frac{dv}{dy} \, dy = \int \frac{1}{y} \, dy$$

$$\hookrightarrow \qquad \ln|v| = \ln|y| + c_1 \quad \rightarrowtail \quad v = \pm e^{\ln|y| + c_1} = Ay \quad .$$

Note that the constant solution $v = 0$ is obtained from the last equation when $A = 0$. So $v = Ay$ describes all possible formulas for v . Since $v = y'$ the last equation is

$$\frac{dy}{dx} = Ay \quad ,$$

another simple linear equation with constant solution $y = 0$. For the nonconstant solutions:

$$\frac{dy}{dx} = Ay \quad \rightarrowtail \quad \int \frac{1}{y} \frac{dy}{dx} \, dx = \int A \, dx$$

$$\hookrightarrow \qquad \ln|y| = Ax + c_2 \quad \rightarrowtail \quad y = \pm e^{Ax + c_2} = Be^{Ax} \quad .$$

Note that this last formula for y does reduce to the constant solution $y = 0$ when $B = 0$. So the general solution is $y(x) = Be^{Ax}$.

11.4 c. Setting $y' = \frac{dy}{dx} = v$ and $y'' = \frac{d^2y}{dx^2} = \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = \frac{dv}{dy}v$,
we have

$$\sin(y)y'' + \cos(y)(y')^2 = 0$$

$$\hookrightarrow \quad \sin(y)\frac{dv}{dy}v + \cos(y)(v)^2 = 0 \quad \rightarrowtail \quad \frac{dv}{dy} + \frac{\cos(y)}{\sin(y)}v = 0 \quad ,$$

which is both linear and separable. We'll use the method for linear first-order equations.
The integrating factor is given by

$$\mu = \exp\left(\int \frac{\cos(y)}{\sin(y)}\,dy\right) = e^{\ln|\sin(y)|} = |\sin(y)| \quad ,$$

As noted in the chapter on linear first-order equations, we can simply use $\mu = \sin(y)$. Doing so, and then remembering that $v = y'$:

$$\sin(y)\left[\frac{dv}{dy} + \frac{\cos(y)}{\sin(y)}v = 0\right] \quad \rightarrowtail \quad \frac{d}{dy}[\sin(y)v] = 0$$

$$\hookrightarrow \quad \sin(y)v = \int 0\,dy = c_1 \quad \rightarrowtail \quad \sin(y)\frac{dy}{dx} = c_1$$

$$\hookrightarrow \quad \int \sin(y)\frac{dy}{dx}\,dx = \int c_1\,dx \quad \rightarrowtail \quad -\cos(y) = c_1x + c_2$$

$$\hookrightarrow \quad y = \arccos(-c_1x - c_2) = \arccos(a - cx) \quad .$$

11.4 e. Setting $y' = \frac{dy}{dx} = v$ and $y'' = \frac{d^2y}{dx^2} = \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = \frac{dv}{dy}v$,
we have

$$(y')^2 + yy'' = 2yy' \quad \rightarrowtail \quad v^2 + y\frac{dv}{dy}v = 2yv$$

$$\hookrightarrow \quad \frac{dv}{dy} + \frac{1}{y}v = 2 \qquad \text{(a linear equation)}$$

$$\hookrightarrow \quad \mu = e^{\int \left({}^1\!/_y\right) dy} = \cdots = y \qquad \text{(integrating factor)}$$

$$\hookrightarrow \quad y\left[\frac{dv}{dy} + \frac{1}{y}v = 2\right] \quad \rightarrowtail \quad \frac{d}{dy}[yv] = 2y$$

$$\hookrightarrow \quad yv = \int 2y\,dy = y^2 + c_1$$

$$\hookrightarrow \quad \frac{dy}{dx} = v = y + \frac{c_1}{y} = \frac{y^2 + c_1}{y}$$

$$\hookrightarrow \quad \frac{y}{y^2 + c_1}\frac{dy}{dx} = 1 \quad \rightarrowtail \quad \int \frac{y}{y^2 + c_1}\frac{dy}{dx}\,dx = \int 1\,dx$$

$$\hookrightarrow \quad \frac{1}{2}\ln\left|y^2 + c_1\right| = x + c_2 \quad \rightarrowtail \quad y^2 + c_1 = \pm e^{2x + 2c_2} = Ae^{2x}$$

$$\hookrightarrow \quad y = \pm\sqrt{Ae^{2x} - c_2} \quad .$$

11.5 a. Because the variable, x, explicitly appears in the equation, the equation is not autonomous.

11.5 c. The variable, x, does not appear in the equation. So the equation is autonomous.

Setting $\quad y' = \frac{dy}{dx} = v \quad$ and $\quad y'' = \frac{d^2y}{dx^2} = \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = \frac{dv}{dy}v \quad ,$

we have

$$y'y'' = 1 \quad \rightarrowtail \quad v\frac{dv}{dy}v = 1 \quad \rightarrowtail \quad v^2\frac{dv}{dy} = 1$$

$$\hookrightarrow \qquad \int v^2\frac{dv}{dy}\,dy = \int 1\,dy \quad \rightarrowtail \quad \tfrac{1}{3}v^3 = y + c_1$$

$$\hookrightarrow \qquad \frac{dy}{dx} = v = (3y+3c_1)^{1/3} \quad \rightarrowtail \quad (3y+3c_1)^{-1/3}\frac{dy}{dx} = 1$$

$$\hookrightarrow \qquad \int (3y+3c_1)^{-1/3}\frac{dy}{dx}\,dx = \int 1\,dx \quad \rightarrowtail \quad \tfrac{1}{2}(3y+3c_1)^{2/3} = x + c_2$$

$$\hookrightarrow \qquad 3y + 3c_1 = \pm(2x+2c_2)^{3/2}$$

$$\hookrightarrow \qquad y = -c_1 \pm \frac{1}{3}(2x+2c_2)^{3/2} = A \pm \frac{1}{3}(2x+B)^{3/2} \quad .$$

11.5 e. Because the variable, x, explicitly appears in the equation, the equation is not autonomous.

11.5 g. The variable, x, does not appear in the equation. So the equation is autonomous.

Setting $\quad y' = \frac{dy}{dx} = v \quad$ and $\quad y'' = \frac{d^2y}{dx^2} = \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = \frac{dv}{dy}v \quad ,$

we have

$$yy'' = 2(y')^2 \quad \rightarrowtail \quad y\frac{dv}{dy}v = 2v^2$$

$$\hookrightarrow \qquad \frac{1}{v}\frac{dv}{dy} = \frac{2}{y} \quad \text{or} \quad v = 0 \quad . \qquad (\star)$$

If $v = 0$, then

$$y = \int \frac{dy}{dx}\,dx = \int v\,dx = \int 0\,dx = c \quad .$$

Otherwise, we integrate the other equation in line $(\star)$ and continue:

$$\int \frac{1}{v}\frac{dv}{dy}\,dy = \int \frac{2}{y}\,dy \quad \rightarrowtail \quad \ln|v| = 2\ln|y| + c_1$$

$$\hookrightarrow \qquad \frac{dy}{dx} = v = e^{2\ln|y| + c_1} = Ay^2 \quad \rightarrowtail \quad y^{-2}\frac{dy}{dx} = A$$

$$\hookrightarrow \qquad \int y^{-2}\frac{dy}{dx}\,dx = \int A\,dx \quad \rightarrowtail \quad -y^{-1} = Ax + c_2$$

$$\hookrightarrow \qquad y = -(Ax+c_2)^{-1} = \frac{1}{Bx+C} \quad .$$

If $B = 0$, the last line reduces to the constant solution $y = c$ (with $c = {}^{1}/_{C}$). So this last line describes all solutions to the given differential equation.

11.5 i. Because the variable, x , explicitly appears in the equation, the equation is not autonomous.

11.6 a. From solving exercise 11.1 a, we know the general solution is

$$y = x^3 + \frac{c_1}{x^3} + c_2 \quad . \qquad (\star)$$

Hence,

$$y' = \frac{d}{dx}\left[x^3 + \frac{c_1}{x^3} + c_2\right] = 3x^2 - 3\frac{c_1}{x^4} \quad .$$

Applying the initial conditions:

$$8 = y(1) = 1^3 + \frac{c_1}{1^3} + c_2 \quad \rightarrowtail \quad c_1 + c_2 = 7 \quad ,$$

and

$$-3 = y'(1) = 3 \cdot 1^2 - 3\frac{c_1}{1^4} \quad \rightarrowtail \quad c_1 = 2 \quad .$$

So $c_1 = 2$ and $c_2 = 7 - c_1 = 7 - 2 = 5$. Using these with equation ($\star$) gives

$$y = x^3 + \frac{2}{x^3} + 5 \quad .$$

11.6 c. From solving exercise 11.1 c, we know the general solution is

$$y = Ae^x + c \quad . \qquad (\star)$$

Hence,

$$y' = \frac{d}{dx}\left[Ae^x + c\right] = Ae^x \quad .$$

Applying the initial conditions:

$$8 = y(0) = Ae^0 + c \quad \rightarrowtail \quad A + c = 8$$

and

$$5 = y'(0) = Ae^0 \quad \rightarrowtail \quad A = 5 \quad .$$

So $A = 5$ and $c = 8 - A = 8 - 5 = 3$. Using these with equation ($\star$) gives

$$y = 5e^x + 3 \quad .$$

11.6 e. From solving exercise 11.3 a, we know the general solution is

$$y = Ae^x + Bx + C \quad . \qquad (\star)$$

Hence,

$$y' = \frac{d}{dx}\left[Ae^x + Bx + C\right] = Ae^x + B$$

and

$$y'' = \frac{d}{dx}\left[Ae^x + B\right] = Ae^x \quad .$$

Applying the initial conditions:

$$10 = y(0) = Ae^0 + B \cdot 0 + C \quad \rightarrowtail \quad A + C = 10 \quad ,$$

$$5 = y'(0) = Ae^0 + B \quad \rightarrowtail \quad A + B = 5$$

and

$$2 = y''(0) = Ae^0 \quad \rightarrowtail \quad A = 2 \quad .$$

So $A = 2$, $B = 5 - A = 5 - 2 = 3$ and $C = 10 - A = 10 - 2 = 8$. Using these with equation ($\star$) gives

$$y = 2e^x + 3x + 8 \quad .$$

11.6 g. First, we must find the general solution to the differential equation. Since the equation does not explicitly contain y, we use the substitution $v = y' = \frac{dy}{dx}$ with $\frac{dv}{dx} = \frac{d^2y}{dx^2}$:

$$xy'' + 2y' = 6 \quad \rightarrowtail \quad x\frac{dv}{dx} + 2v = 6$$

$$\hookrightarrow \qquad \frac{dv}{dx} + \frac{2}{x}v = \frac{6}{x} \qquad \text{(a linear equation)}$$

$$\hookrightarrow \qquad \mu = e^{\int (2/x)\,dx} = e^{2\ln|x|} = x^2 \qquad \text{(integrating factor)}$$

$$\hookrightarrow \qquad x^2\left[\frac{dv}{dx} + \frac{2}{x}v = \frac{6}{x}\right] \quad \rightarrowtail \quad \frac{d}{dx}\left[x^2 v\right] = 6x$$

$$\hookrightarrow \qquad x^2 v = 3x^2 + c_1$$

$$\hookrightarrow \qquad y' = \frac{dy}{dx} = v = 3 + c_1 x^{-2}$$

$$\hookrightarrow \qquad y = \int \left[3 + c_1 x^{-2}\right] = 3x - c_1 x^{-1} + c_2 \quad . \qquad (\star)$$

Applying the initial conditions:

$$4 = y(1) = 3 \cdot 1 - c_1 \cdot 1^{-1} + c_2 \quad \rightarrowtail \quad c_2 - c_1 = 1$$

and

$$5 = y'(1) = 3 + c_1 \cdot 1^{-2} \quad \rightarrowtail \quad c_1 = 2 \quad .$$

So $c_1 = 2$ and $c_2 = 1 + c_1 = 1 + 2 = 3$. Using these with equation ($\star$) gives

$$y = 3x - 2x^{-1} + 3 \quad .$$

11.7 a. From solving exercise 11.4 a, we know the general solution is

$$y = Be^{ax} \quad . \qquad (\star)$$

Hence,

$$y' = \frac{d}{dx}\left[Be^{ax}\right] = Bae^{ax} \quad .$$

Applying the initial conditions:

$$5 = y(0) = Be^0 \quad \rightarrowtail \quad B = 5$$

and

$$15 \;=\; y'(0) \;=\; Bae^0 \quad\rightarrowtail\quad Ba \;=\; 15 \quad .$$

So $B = 5$ and $a \;=\; \frac{15}{B} = \frac{15}{5} = 3$. Using these with equation ($\star$) gives

$$y \;=\; 5e^{3x} \quad .$$

11.7 c. From solving exercise 11.4 b, we know the general solution is

$$y \;=\; (Ax + c)^3 \quad . \qquad (\star)$$

Hence,

$$y' \;=\; \frac{d}{dx}\left[(Ax + c)^3\right] \;=\; 3(Ax + c)^2 A \quad .$$

Applying the initial conditions:

$$1 \;=\; y(1) \;=\; (A \cdot 1 + c)^3 \quad\rightarrowtail\quad A \;+\; c \;=\; 1$$

and

$$9 \;=\; y'(1) \;=\; 3(A \cdot 1 + c)^2 A \quad\rightarrowtail\quad A(A + c)^2 \;=\; 3 \quad .$$

Together, these two conditions yield

$$3 \;=\; A(A + c)^2 \;=\; A \cdot 1^2 \;=\; A \qquad \text{and} \qquad c \;=\; 1 \;-\; A \;=\; 1 \;-\; 3 \;=\; -2 \quad .$$

Using these with equation ($\star$) yields $\quad y \;=\; (3x - 2)^3 \quad .$

11.7 e. First, we must find the general solution to the differential equation. Since the equation does not explicitly contain x but does contain y, we use the substitution $v \;=\; y' \;=\; \frac{dy}{dx}$ with $y'' = \frac{d^2y}{dx^2} = \frac{dv}{dx} = \frac{dv}{dy}v$:

$$y'' \;=\; -y'e^{-y} \quad\rightarrowtail\quad \frac{dv}{dy}v \;=\; -ve^{-y}$$

$$\hookrightarrow \qquad \frac{dv}{dy} \;=\; -e^{-y} \qquad \text{or} \qquad v \;=\; 0$$

$$\hookrightarrow \quad y' \;=\; v \;=\; -\int e^{-y}\,dy \;=\; e^{-y} \;+\; c_1 \qquad \text{or} \qquad y' \;=\; v \;=\; 0 \quad .$$

Clearly, $y' = 0$ will not be able to satisfy the initial condition $y'(0) = 2$. So we are only interested in the cases with

$$y' \;=\; e^{-y} \;+\; c_1 \quad .$$

Using the initial conditions with the last equation, we have

$$y'(0) \;=\;=\; e^{-y(0)} \;+\; c_1 \quad\rightarrowtail\quad 2 \;=\; e^0 \;+\; c_1 \quad\rightarrowtail\quad c_1 \;=\; 1 \quad .$$

Letting $c_1 = 1$ in our last formula for y and continuing yields

$$\frac{dy}{dx} \;=\; e^{-y} \;+\; 1 \quad\rightarrowtail\quad \frac{1}{e^{-y} + 1}\frac{dy}{dx} \;=\; 1$$

$$\hookrightarrow \qquad \frac{e^y}{e^y\left[e^{-y} + 1\right]}\frac{dy}{dx} \;=\; 1 \quad\rightarrowtail\quad \int \frac{e^y}{1 + e^y}\frac{dy}{dx}\,dx \;=\; \int 1\,dx$$

$$\hookrightarrow \qquad \ln\left|1+e^y\right| = x + c_2 \quad \rightarrowtail \quad 1+e^y = \pm e^{x+c_2} = Ae^x$$

$$\hookrightarrow \qquad y = \ln\big(Ae^x - 1\big) \quad .$$

Applying an initial condition:

$$0 = y(0) = \ln\Big(Ae^0 - 1\Big) = \ln(A-1) \quad \rightarrowtail \quad A - 1 = 1 \quad \rightarrowtail \quad A = 2 \quad .$$

So the solution is $\quad y = \ln\big(2e^x - 1\big) \quad .$

11.8 a. In every case, we start the same way, by setting $y' = \frac{dy}{dx} = v$ and $y'' = \frac{d^2y}{dx^2} = \frac{dv}{dx}$. Doing so, we get

$$y'' = -2x(y')^2 \quad \rightarrowtail \quad \frac{dv}{dx} = -2xv^2 \quad ,$$

which is a separable equation with constant solution $v = 0$. Corresponding to this constant v solution, we then have

$$\frac{dy}{dx} = v = 0 \quad \rightarrowtail \quad y = c_1 \quad .$$

For the nonzero formulas of v:

$$\frac{dv}{dx} = -2xv^2 \quad \rightarrowtail \quad v^{-2}\frac{dv}{dx} = -2x \quad \rightarrowtail \quad \int v^{-2}\frac{dv}{dx}\,dx = -\int 2x\,dx$$

$$\hookrightarrow \qquad -v^{-1} = -x^2 + c_1$$

$$\hookrightarrow \qquad \frac{dy}{dx} = v = \frac{1}{x^2 - c_1} \quad .$$

So, the derivative of any solution y is given by either

$$y' = \frac{dy}{dx} = 0 \qquad \text{or} \qquad y' = \frac{dy}{dx} = \frac{1}{x^2 - c_1} \tag{$\star$}$$

Now consider the initial values.

11.8 a i. Since $y'(0) = 4 \neq 0$ we use

$$\frac{dy}{dx} = \frac{1}{x^2 - c_1} \qquad \text{with} \quad y'(0) = 4$$

$$\hookrightarrow \qquad 4 = y'(0) = \frac{1}{0^2 - c_1} \quad \rightarrowtail \quad c_1 = -\frac{1}{4}$$

$$\hookrightarrow \qquad \frac{dy}{dx} = \frac{1}{x^2 + (1/4)} = \frac{4}{(2x)^2 + 1}$$

$$\hookrightarrow \qquad y = \int \frac{4}{(2x)^2 + 1}\,dx = 2\arctan(2x) + c_2 \quad .$$

Applying the other initial condition:

$$3 = y(0) = 2\arctan(2\cdot 0) + c_2 = 0 + c_2 \quad .$$

So $c_2 = 3$ and the solution is $\quad y = 2\arctan(2x) + 3 \quad .$

11.8 a iii. Since $y'(1) = 1 \neq 0$ we use

$$\frac{dy}{dx} = \frac{1}{x^2 - c_1} \qquad \text{with} \quad y'(1) = 1$$

$$\hookrightarrow \qquad 1 = y'(1) = \frac{1}{1^2 - c_1} \quad \rightarrowtail \quad c_1 = 0$$

$$\hookrightarrow \qquad \frac{dy}{dx} = \frac{1}{x^2 + 0} = x^{-2}$$

$$\hookrightarrow \qquad y = \int x^{-2}\,dx = -x^{-1} + c_2 = c_2 - \frac{1}{x} \quad .$$

Applying the other initial condition:

$$0 = y(1) = c_2 - \frac{1}{1} \quad \rightarrowtail \quad c_2 = 1 \quad .$$

So the solution is $\quad y = 1 - \dfrac{1}{x} \quad .$

11.10. Setting $\quad y' = \frac{dy}{dx} = v \qquad$ and $\qquad y'' = \frac{d^2y}{dx^2} = \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = \frac{dv}{dy}v \quad ,$
we have

$$y'' = 2y' - 6 \quad \rightarrowtail \quad \frac{dv}{dy}v = 2v - 6 \quad \rightarrowtail \quad \frac{dv}{dy} = 2\frac{v-3}{v} \quad ,$$

which is a separable differential equation with constant solution $v = 3$. To find the nonconstant v solutions:

$$\frac{dv}{dy} = 2\frac{v-3}{v} \quad \rightarrowtail \quad \frac{v}{v-3}\frac{dv}{dy} = 2 \quad \rightarrowtail \quad \frac{v-3+3}{v-3}\frac{dv}{dy} = 2$$

$$\hookrightarrow \qquad \int \left[1 + \frac{3}{v-3}\right]\frac{dv}{dy}\,dy = \int 2\,dy$$

$$\hookrightarrow \qquad v + 3\ln|v-3| = 2y + c_1$$

$$\hookrightarrow \qquad \frac{dy}{dx} + 3\ln\left|\frac{dy}{dx} - 3\right| = 2y + c_1 \quad .$$

This is not an “easily solved” differential equation by any stretch of the imagination.
Moral: Sometimes, some approaches are better than others.