Chapter 10: The Art and Science of Modeling with First-Order Equations

10.2 a. Since the formula gives the number of rabbits after t months and there are 12 months in a year, the number of rabbits after one year is

$$R(12) = 2e^{\beta t} \quad \text{with} \quad \beta = \frac{5}{4} \quad \text{and} \quad t = 12$$
$$= 2\exp\left(\frac{5}{4}\cdot 12\right) = 2e^{15} = 2\cdot 3{,}269{,}017.372 \approx 6{,}538{,}035 \quad .$$

10.2 b. Let t_2, t_4, t_8 and t_{16} be the number of months that pass from $t = 0$ before the rabbit population reaches 2, 4, 8 and 16, respectively. Since we started with two rabbits at $t = 0$, we automatically have $t_2 = 0$. And using formula (10.7),

$$4 = 2e^{\beta\cdot t_4} \quad , \quad 8 = 2e^{\beta\cdot t_8} \quad \text{and} \quad 16 = 2e^{\beta\cdot t_16} \quad .$$

Dividing by 2 and taking the natural logarithm of each side yields

$$\beta\cdot t_4 = \ln 2 \quad , \quad \beta\cdot t_8 = \ln 4 \quad \text{and} \quad \beta\cdot t_{16} = \ln 8 \quad .$$

Since $\beta = {}^5\!/_4$, dividing through by β gives

$$t_4 = \frac{4}{5}\ln 2 \quad , \quad t_8 = \frac{4}{5}\ln 4 \quad \text{and} \quad t_{16} = \frac{4}{5}\ln 16 \quad .$$

Hence, the time it takes for the number of rabbits to increase

1. from 2 to 4 is

$$t_4 - t_2 = \frac{4}{5}\ln 2 - 0 = \frac{4}{5}(0.69314718\ldots) \approx 0.55 \quad .$$

2. from 4 to 8 is

$$t_8 - t_4 = \frac{4}{5}\ln 8 - \frac{4}{5}\ln 4$$
$$= \frac{4}{5}[\ln 8 - \ln 4] = \frac{4}{5}\ln\frac{8}{4} = \frac{4}{5}\ln 2 \approx 0.55 \quad .$$

3. from 8 to 16 is

$$t_{16} - t_8 = \frac{4}{5}\ln 16 - \frac{4}{5}\ln 8 = \frac{4}{5}\ln\frac{16}{8} = \frac{4}{5}\ln 2 \approx 0.55 \quad .$$

10.2 d. Let t_E be the time at which the mass of the rabbits equals the mass of the Earth. Then, using the information in the section,

$$\text{mass of rabbits at time } t_E = \text{mass of Earth}$$

$$\hookrightarrow \quad R(t_E) \times \text{ave. mass of one rabbit} = \text{mass of Earth}$$

$$\hookrightarrow \quad 2e^{\beta t_E}\cdot 3 = 6\times 10^{24} \quad \rightarrowtail \quad \beta t_E = \ln\left(\frac{6}{2\cdot 3}\times 10^{24}\right)$$

$$\hookrightarrow \quad t_E = \frac{1}{\beta}\ln 10^{24} = \frac{4}{5}\cdot 24\ln 10 \approx 44.21 \quad .$$

10.4 a. Start by remembering that $A_0 = A_0 e^{-\delta\cdot 0} = A(0)$. By that and the definition of $\tau_{1/2}$,

$$A_0 e^{-\delta\tau_{1/2}} = A(\tau_{1/2}) = \frac{1}{2}A(0) = \frac{1}{2}A_0 \quad .$$

Dividing out the A_0 then gives $e^{-\delta\tau_{1/2}} = {}^1\!/_2$. Hence, for every t ,

$$A(t+\tau_{1/2}) = A_0 e^{-\delta[t+\tau_{1/2}]} = A_0 e^{-\delta t - \delta\tau_{1/2}}$$
$$= A_0 e^{-\delta t} e^{-\delta\tau_{1/2}} = e^{-\delta\tau_{1/2}}\left[A_0 e^{-\delta t}\right] = \frac{1}{2}A(t) \quad .$$

10.4 b. From the answer to the last exercise, we know $e^{-\delta\tau_{1/2}} = {}^1\!/_2$. Using this, the formula for $A(t)$, and some algebra:

$$A(t) = A_0 e^{-\delta t} = A_0 e^{-\delta(\tau_{1/2})\left({}^t\!/_{\tau_{1/2}}\right)} = A_0\left(e^{-\delta\tau_{1/2}}\right)^{{}^t\!/_{\tau_{1/2}}} = A_0\left(\frac{1}{2}\right)^{{}^t\!/_{\tau_{1/2}}} \quad .$$

10.6 a. Note: Since these exercises concern radioactive decay, we know that the amount of carbon-14 remaining after t years is given by $A(t) = A_0 e^{-\delta t}$ where $A_0 = A(0)$.

Using equation (10.10) on page 203 relating the decay constant and half-life, we have

$$\delta = \frac{\ln 2}{\tau_{1/2}} = \frac{\ln 2}{5,730} = 0.000120968\ldots \approx 0.000121 \quad (\text{/year}) \quad .$$

So, in all the following: $A(t) = A_0 e^{-\delta t} = A_0 e^{-0.000121t}$.

10.6 b. The fraction of the original amount of carbon-14 remaining after t years is

$$\frac{A(t)}{A(0)} = \frac{A_0 e^{-\delta t}}{A_0} = e^{-\delta t} \approx e^{-0.000121t}$$

Multiplying this by 100 then gives the percentage remaining. In particular, the percentage remaining:

after 10 years is $100e^{-0.000121\cdot 10}\,\% \approx 99.88\,\%$.
after 1,000 years is $100e^{-0.000121\cdot 1000}\,\% \approx 88.60\,\%$.
after 10,000 years is $100e^{-0.000121\cdot 10000}\,\% \approx 29.82\,\%$.

10.6 c. From the answer to the last exercise we know $e^{-\delta t}$ is the fraction of the original carbon-14 remaining after t years. So, if 30 percent (i.e., ${}^3\!/_{10}$) still remains:

$$e^{-\delta t} = \frac{3}{10} \quad\rightarrowtail\quad -\delta t = \ln\left(\frac{3}{10}\right) \quad\rightarrowtail\quad t = -\frac{1}{\delta}\ln\left(\frac{3}{10}\right)$$

$$\hookrightarrow \qquad t = \frac{1}{0.000121}\ln\left(\frac{10}{3}\right) = 9950.18846\ldots \approx 9950 \quad .$$

10.6 d. Repeating the above with ${}^6\!/_{10}$ replacing ${}^3\!/_{10}$,

$$e^{-\delta t} = \frac{6}{10} \quad\rightarrowtail\quad -\delta t = \ln\left(\frac{6}{10}\right) \quad\rightarrowtail\quad t = -\frac{1}{\delta}\ln\left(\frac{6}{10}\right)$$

$$\hookrightarrow \qquad t = \frac{1}{0.000121}\ln\left(\frac{10}{6}\right) = 4221.69937\ldots \approx 4222 \quad .$$

10.6 e. The ratio ${}^{A}/_{A_0}$ is the fraction of the original carbon-14 remaining after t years, which we also know (from above) to be given by $e^{-\delta t}$. So

$$e^{-\delta t} = \frac{A}{A_0} \quad \rightarrowtail \quad t = -\frac{1}{\delta}\ln\left(\frac{A}{A_0}\right) \quad \rightarrowtail \quad t \approx -\frac{1}{0.000121}\ln\left(\frac{A}{A_0}\right) \quad .$$

Note: Because of the relation between the half-life and decay rate, we can rewrite the above as

$$t = -\frac{\tau_{1/2}}{\ln 2}\ln\left(\frac{A}{A_0}\right) \approx -\frac{5730}{\ln 2}\ln\left(\frac{A}{A_0}\right)$$

10.8 a. Since rate of change in the population depends only on births and harvesting, and we know that the monthly monthly birth rate per rabbit is $\beta = {}^{5}/_{4}$ and that 500 rabbits are harvested per month, we have

$$\begin{aligned}\frac{dR}{dt} &= \text{change in the number of rabbits per month}\\ &= \text{number of births per month} - \text{number of rabbits harvested per month}\\ &= \frac{5}{4}R - 500 \quad .\end{aligned}$$

So the differential equation is: $\quad \dfrac{dR}{dt} = \dfrac{5}{4}R - 500 \quad .$

10.8 b. For the equilibrium solution:

$$0 = \frac{dR}{dt} = \frac{5}{4}R - 500 \quad \rightarrowtail \quad R = \frac{4}{5}\cdot 500 = 400 \quad .$$

So $R = 400$ is the only equilibrium solution.
If $R < 400$, then $\dfrac{dR}{dt} = \dfrac{5}{4}R - 500 = \dfrac{5}{4}(R-400) < 0$.
If $R > 400$, then $\dfrac{dR}{dt} = \dfrac{5}{4}R - 500 = \dfrac{5}{4}(R-400) > 0$.
So the population increases over time if we start with more than 400 rabbits, and decreases over time if we start with fewer than 400 rabbits. (If fact, since the derivative is -500 when $R = 0$, this model has the number of rabbit becoming negative if we start with fewer than 400 rabbits!)

10.8 c. The differential equation is both linear and separable. Since we have already found $R = 400$ to be the only equilibrium solution, we will proceed as follows for the other solutions:

$$\frac{dR}{dt} = \frac{5}{4}R - 500 \quad \rightarrowtail \quad \frac{dR}{dt} = \frac{5}{4}(R-400)$$

$$\hookrightarrow \quad \frac{1}{R-400}\frac{dR}{dt} = \frac{5}{4} \quad \rightarrowtail \quad \int \frac{1}{R-400}\frac{dR}{dt}\,dt = \int \frac{5}{4}\,dt$$

$$\hookrightarrow \quad \ln|R-400| = \frac{5}{4}t + c \quad \rightarrowtail \quad R - 400 = Ae^{5t/4}$$

$$\hookrightarrow \quad R = 400 + Ae^{5t/4} \quad .$$

Note that the last equation reduces to the equilibrium solution $R = 400$ when $A = 0$. Applying the initial condition $R(0) = R_0$, we have

$$R_0 = R(0) = 400 + Ae^0 = 400 + A \quad \rightarrowtail \quad A = R_0 - 400 \quad .$$

So the solution is $\quad R = 400 + (R_0 - 400)e^{5t/4} \quad .$

10.10. The answer to both parts of this exercise come from the same basic equation,

$$\begin{aligned}\frac{dR}{dt} &= \text{change in the number of rabbits per month} \\ &= \text{number of births per month} - \text{number of deaths per month} \\ &\qquad - \text{number of rabbits harvested per month} \quad .\end{aligned}$$

where, by the assumptions in the problem,

$$\text{number of births per month} - \text{number of deaths per month} = \beta R - \gamma R^2 \quad .$$

So, in general, the differential equation is

$$\frac{dR}{dt} = \beta R - \gamma R^2 - h$$

where h is the number of rabbits harvested per month.

In particular, if we harvest a constant number h_0 of rabbits each month, then

$$h = \text{number of rabbits harvested per month} = h_0$$

and

$$\frac{dR}{dt} = \beta R - \gamma R^2 - h = \beta R - \gamma R^2 - h_0 \quad .$$

That is, the differential equation is $\quad \frac{dR}{dt} = \beta R - \gamma R^2 - h_0 \quad .$

On the other hand, if we harvest one fourth of all the rabbits on the ranch each month, then

$$h = \text{number of rabbits harvested per month} = \frac{1}{4}R$$

and

$$\begin{aligned}\frac{dR}{dt} &= \beta R - \gamma R^2 - h \\ &= \beta R - \gamma R^2 - \frac{1}{4}R = \left(\beta - \frac{1}{4}\right) R - \gamma R^2 \quad .\end{aligned}$$

Hence, in this case, the differential equation is $\quad \frac{dR}{dt} = \left(\beta - \frac{1}{4}\right) R - \gamma R^2 \quad .$

10.12 a. The differential equation is

$$\begin{aligned}\frac{dy}{dt} &= \text{change in the amount of alcohol in the tank per minute} \\ &= \text{rate alcohol is added to the tank} - \text{rate alcohol is drained from the tank}\end{aligned}$$

where

$$\text{rate alcohol is added to the tank} = 3\left(\frac{\text{gal. of input mix}}{\text{minute}}\right) \times \frac{75}{100}\left(\frac{\text{gal. of alcohol}}{\text{gal. of input mix}}\right)$$
$$= \frac{9}{4}\left(\frac{\text{gal. of alcohol}}{\text{minute}}\right) \quad ,$$

and

$$\text{rate alcohol is drained from the tank} = 3\left(\frac{\text{gal. of tank mix}}{\text{minute}}\right)$$
$$\times \text{ amount of alcohol per gal. of tank mix}$$
$$= 3\left(\frac{\text{gal. of tank mix}}{\text{minute}}\right) \times \frac{y(t)\,(\text{gal. of alcohol})}{1000\,(\text{gal. of tank mix})}$$
$$= \frac{3}{1000}y(t)\left(\frac{\text{gal. of alcohol}}{\text{minute}}\right) \quad .$$

Combining the above gives

$$\frac{dy}{dt} = \frac{9}{4} - \frac{3}{1000}y \quad .$$

10.12 b. $\frac{dy}{dt} = \frac{9}{4} - \frac{3}{1000}y = \frac{3}{1000}(750 - y) \quad .$

Since ${}^{dy}\!/_{dt} = 0$ if and only if $y = 750$, the only equilibrium solution is $y = 750$.

If $y < 750$, then $\frac{dy}{dt} = \frac{3}{1000}(750 - y) > 0$.

If $y > 750$, then $\frac{dy}{dt} = \frac{3}{1000}(750 - y) < 0$.

10.12 c. The differential equation is both linear and separable. This time we will use the method for linear equations. First we find the integrating factor:

$$\frac{dy}{dt} = \frac{9}{4} - \frac{3}{1000}y \quad \rightarrowtail \quad \frac{dy}{dt} + \frac{3}{1000}y = \frac{9}{4}$$

$$\hookrightarrow \qquad \mu = e^{\int 3/1000\,dt} = e^{3t/1000} \quad .$$

Multiplying the differential equation by μ and continuing:

$$e^{3t/1000}\left[\frac{dy}{dt} + \frac{3}{1000}y = \frac{9}{4}\right] \quad \rightarrowtail \quad \frac{d}{dt}\left[e^{3t/1000}y\right] = \frac{9}{4}e^{3t/1000}$$

$$\hookrightarrow \qquad e^{3t/1000}y = \int \frac{9}{4}e^{3t/1000}\,dt = 750e^{3t/1000} + c$$

$$\hookrightarrow \qquad y = 750 + ce^{-3t/1000} \quad .$$

Since the tank initially contains no alcohol,

$$0 = y(0) = 750 + ce^0 = 750 + c \quad \rightarrowtail \quad c = -750 \quad .$$

So, $\quad y(t) = 750 - 750e^{-3t/1000} \quad .$

10.12 d i.
$$y(10) = 750 - 750e^{-3(10)/1000} = 750 - 750e^{-3/100} = 22.165849\ldots \approx 22.2 \quad .$$

10.12 d ii.
$$y(60) = 750 - 750e^{-3(60)/1000} = 750 - 750e^{-18/100} = 123.547341\ldots \approx 123.6 \quad .$$

10.12 d iii.
$$y(1000) = 750 - 750e^{-3(1000)/1000} = 750 - 750e^{-3} = 712.659698\ldots \approx 712.7 \quad .$$

10.12 e. Since the tank contains 1000 gallons, the mixture will be half alcohol at the time t when $y(t) = 500$. Solving for this t:

$$500 = y(t) = 750 - 750e^{-3t/1000}$$

$$\hookrightarrow \quad e^{-3t/1000} = \frac{750 - 500}{750} = \frac{1}{3} \quad \rightarrowtail \quad -3t/1000 = \ln\left(\frac{1}{3}\right) = -\ln 3$$

$$\hookrightarrow \quad t = \frac{1000}{3}\ln 3 = 366.204096\ldots \approx 3.66.2 \quad .$$

10.14 a. The differential equation is

$$\frac{dy}{dt} = \text{change in the amount of salt in the tank per minute} = \text{rate salt is added to the tank} - \text{rate salt is drained from the tank}$$

where

$$\text{rate salt is added to the tank} = \frac{1}{2}\left(\frac{\text{gal. of input mix}}{\text{minute}}\right) \times 3 \left(\frac{\text{oz. of salt}}{\text{gal. of input mix}}\right) = \frac{3}{2}\left(\frac{\text{oz. of salt}}{\text{minute}}\right) \quad ,$$

and

$$\text{rate salt is drained from the tank} = \frac{1}{2}\left(\frac{\text{gal. of tank mix}}{\text{minute}}\right) \times \text{amount of salt per gal. of tank mix}$$
$$= \frac{1}{2}\left(\frac{\text{gal. of tank mix}}{\text{minute}}\right) \times \frac{y(t)\ (\text{oz. of salt})}{200\ (\text{gal. of tank mix})}$$
$$= \frac{1}{400} y(t) \left(\frac{\text{oz. of salt}}{\text{minute}}\right) \quad .$$

Combining the above gives $\displaystyle \frac{dy}{dt} = \frac{3}{2} - \frac{1}{400}y \quad .$

10.14 b. $\frac{dy}{dt} = \frac{3}{2} - \frac{1}{400}y = \frac{1}{400}(600 - y)$.

Since $dy/dt = 0$ if and only if $y = 600$, the only equilibrium solution is $y = 600$.

If $y < 600$, then $\frac{dy}{dt} = \frac{1}{400}(600 - y)) > 0$.

If $y > 600$, then $\frac{dy}{dt} = \frac{1}{400}(600 - y)) < 0$.

10.14 c.

$$\frac{dy}{dt} = \frac{3}{2} - \frac{1}{400}y = \frac{1}{400}(600 - y)$$

$$\hookrightarrow \quad \frac{1}{600 - y}\frac{dy}{dt} = \frac{1}{400} \quad \rightarrowtail \quad \int \frac{1}{600 - y}\frac{dy}{dt}\,dt = \int \frac{1}{400}\,dt$$

$$\hookrightarrow \quad -\ln|y - 600| = \frac{1}{400}t + c \quad \rightarrowtail \quad y = 600 + Ae^{-t/400} \quad .$$

Since there is no salt in the tank at $t = 0$,

$$0 = y(0) = 600 + Ae^0 = 600 + A \quad \rightarrowtail \quad A = -600 \quad .$$

So, $y(t) = 600 - 600e^{-t/400}$.

10.14 d i.

$$\begin{aligned} y(10) &= 600 - 600e^{-10/400} \\ &= 600 - 600e^{-1/40} = 14.814052\ldots \approx 14.8 \quad . \end{aligned}$$

10.14 d ii.

$$\begin{aligned} y(10) &= 600 - 600e^{-60/400} \\ &= 600 - 600e^{-3/20} = 83.575214\ldots \approx 83.6 \quad . \end{aligned}$$

10.14 d iii.

$$\begin{aligned} y(100) &= 600 - 600e^{-100/400} \\ &= 600 - 600e^{-1/4} = 132.719530\ldots \approx 132.7 \quad . \end{aligned}$$

10.14 e. In terms of "ounces of salt per gallon of mix", the concentration as $t \to \infty$ is

$$\begin{aligned} \lim_{t\to\infty} \frac{\text{oz. of salt in the tank at time } t}{\text{gal. of mix in the tank}} &= \lim_{t\to\infty} \frac{y(t)}{200} \\ &= \lim_{t\to\infty} \frac{600 - 600e^{-t/400}}{200} = \frac{600 - 600 \cdot 0}{200} = 3 \quad . \end{aligned}$$

10.14 f. Since the tank contains 200 gallons, the total amount of salt in the tank at the time t when the concentration is 2 ounces of salt per gallon will be $y(t) = 2 \cdot 200 = 400$. Replacing $y(t)$ with its formula and then solving for t:

$$400 = 600 - 600e^{-t/400} \quad \rightarrowtail \quad e^{-t/400} = \frac{600 - 400}{600} = \frac{1}{3}$$

$$\hookrightarrow \quad t = -400\ln\left(\frac{1}{3}\right) = 400\ln 3 = 439.444915\ldots \approx 439.4 \quad .$$

10.16 a. Let $V = V(t)$ be the amount of liquid in the tank (in gallons) at time t. From the information given, we know $V(0) = 500$ and

$$\begin{aligned}\frac{dV}{dt} &= \text{change in the amount of saltwater in the tank per minute}\\ &= \text{rate saltwater is added to the tank} - \text{rate saltwater is drained from the tank}\\ &= 2 - 3 = -1 \quad .\end{aligned}$$

Integrating gives $V(t) = c - t$. From this and the initial condition, we have $500 = V(0) = c - 0$. So $c = 500$ and $V(t) = 500 - t$.

10.16 b. The differential equation is

$$\begin{aligned}\frac{dy}{dt} &= \text{change in the amount of salt in the tank per minute}\\ &= \text{rate salt is added to the tank} - \text{rate salt is drained from the tank}\end{aligned}$$

where

$$\begin{aligned}\text{rate salt is added to the tank} &= 2\left(\frac{\text{gal. of input mix}}{\text{minute}}\right) \times 2\left(\frac{\text{oz. of salt}}{\text{gal. of input mix}}\right)\\ &= 4\left(\frac{\text{oz. of salt}}{\text{minute}}\right) \quad ,\end{aligned}$$

and

$$\begin{aligned}\text{rate salt is drained from the tank} &= 3\left(\frac{\text{gal. of tank mix}}{\text{minute}}\right)\\ &\qquad \times \text{amount of salt per gal. of tank mix}\\ &= 3\left(\frac{\text{gal. of tank mix}}{\text{minute}}\right) \times \frac{y(t)\ \text{(oz. of salt)}}{V(t)\ \text{(gal. of tank mix)}}\\ &= 3\left(\frac{\text{gal. of tank mix}}{\text{minute}}\right) \times \frac{y(t)\ \text{(oz. of salt)}}{500 - t\ \text{(gal. of tank mix)}}\\ &= \frac{3}{500-t}y(t)\left(\frac{\text{oz. of salt}}{\text{minute}}\right) \quad .\end{aligned}$$

Combining the above gives $\displaystyle\frac{dy}{dt} = 4 - \frac{3}{500-t}y$.

10.16 c. The equation is linear. Getting it into standard form and finding the integrating factor:

$$\frac{dy}{dt} = 4 - \frac{3}{500-t}y \quad \rightarrowtail \quad \frac{dy}{dt} + \frac{3}{500-t}y = 4$$

$$\hookrightarrow \qquad \mu = \exp\left(\int \frac{3}{500-t}\,dt\right) = e^{-3\ln|500-t|} = |500-t|^{-3} \quad .$$

As noted in the chapter on first-order linear equations, we can use $\mu = (500-t)^{-3}$. Multiplying the differential equation by this and proceeding as usual:

$$(500-t)^{-3}\left[\frac{dy}{dt} + \frac{3}{500-t}y = 4\right]$$

$$\hookrightarrow \quad \frac{d}{dt}\left[(500-t)^{-3}y\right] = 4(500-t)^{-3}$$

$$\hookrightarrow \quad (500-t)^{-3}y = \int 4(500-t)^{-3}\,dt = 2(500-t)^{-2} + c$$

$$\hookrightarrow \quad y = 2(500-t) + c(500-t)^3 \quad .$$

Since the tank initially contained just pure water (no salt),

$$0 = y(0) = 2(500-0) + c(500-0)^3 \quad \rightarrowtail \quad c = 2(500)^{-2} \quad .$$

So $y = 2(500-t) + c(500-t)^3$ with $c = 2(500)^{-2}$. But note that

$$\begin{aligned} c(500-t)^3 &= \frac{2}{500^2}(500-t)^3 = \frac{2\cdot 500}{500^3}(500-t)^3 \\ &= 10^3\left(1-\frac{t}{500}\right)^3 = \left(10-\frac{t}{50}\right)^3 \quad . \end{aligned}$$

Hence, we can write our solution as

$$y(t) = 2(500-t) - \left(10-\frac{t}{50}\right)^3 \quad .$$

10.16 d i. $y(10) = 2(500-10) - \left(10-\frac{10}{50}\right)^3 = 980 - \left(\frac{49}{50}\right)^3 = 38.808$.

10.16 d ii. $y(60) = 2(500-60) - \left(10-\frac{60}{50}\right)^3 = 880 - \left(\frac{44}{50}\right)^3 = 198.528$.

10.16 d iii. $y(100) = 2(500-100) - \left(10-\frac{100}{50}\right)^3 = 800 - 8^3 = 288$.

10.16 e i. $1 = V(t) = 500 - t \quad \rightarrowtail \quad t = 499$.

10.16 e ii.

$$\begin{aligned} y(499) &= 2(500-499) - \left(10-\frac{499}{50}\right)^3 \\ &= 2 - \frac{1}{50^3} = \frac{249{,}999}{125{,}000} = 1.999992 \quad . \end{aligned}$$