Chapter 1: The Starting Point: Basic Concepts and Terminology

1.3 a i. Replacing y with e^{3x} in the differential equation, we get

$$\frac{d}{dx}\left[e^{3x}\right] = 3\left[e^{3x}\right] \quad \rightarrowtail \quad 3e^{3x} = 3e^{3x} \quad .$$

Since both sides of the last equation are identical (and defined for all real values of x), this equation is true on the entire real line. Hence, $y = e^{3x}$ is a solution to the differential equation.

1.3 a ii. Replacing y with x^3 in the differential equation, we get

$$\frac{d}{dx}\left[x^3\right] = 3\left[x^3\right] \quad \rightarrowtail \quad 3x^2 = 3x^3$$

$$\hookrightarrow \qquad 3x^2 - 3x^3 = 0 \quad \rightarrowtail \quad 3x^2(1-x) = 0 \quad ,$$

which is only true if $x = 0$ or $x = 1$. But for $y = x^3$ to be a solution to the differential equation on the real line, the last equation must hold for every real value of x. So, $y = x^3$ is not a solution to the differential equation.

1.3 a iii. Replacing y with $\sin(3x)$ in the differential equation, we get

$$\frac{d}{dx}[\sin(3x)] = 3[\sin(3x)] \quad \rightarrowtail \quad 3\cos(3x) = 3\sin(3x) \quad ,$$

which is not true for all real values of x. For example, letting $x = 0$, the last equation gives

$$3 = 3\cos(3\cdot 0) = 3\sin(3\cdot 0) = 0 \quad ,$$

which certainly is not true. So $\sin(3x)$ is not a solution to the differential equation.

1.3 c i. Replacing y with e^{3x} in the differential equation, we get

$$\frac{d^2}{dx^2}\left[e^{3x}\right] = 9\left[e^{3x}\right] \quad \rightarrowtail \quad \frac{d}{dx}\left[3e^{3x}\right] = 9e^{3x}$$

$$\hookrightarrow \qquad 9e^{3x} = 9e^{3x} \quad .$$

Since both sides of the last equation are identical, $y(x) = e^{3x}$ is a solution to the differential equation.

1.3 c ii. Replacing y with x^3 in the differential equation, we get

$$\frac{d^2}{dx^2}\left[x^3\right] = 9\left[x^3\right] \quad \rightarrowtail \quad \frac{d}{dx}\left[3x^2\right] = 9x^3$$

$$\hookrightarrow \qquad 6x = 9x^3 \quad ,$$

which does not hold for all real values of x. So, $y(x) = x^3$ is not a solution to the differential equation.

1.3 c iii. Replacing y with $\sin(3x)$ in the differential equation, we get

$$\frac{d^2}{dx^2}[\sin(3x)] = 9[\sin(3x)] \quad \rightarrowtail \quad \frac{d}{dx}[3\cos(3x)] = 9\sin(3x)$$

$$\hookrightarrow \qquad -9\sin(3x) = 9\sin(3x) \quad \rightarrowtail \quad 0 = 18\sin(3x) \quad ,$$

which is only true if $3x$ is an integer multiple of π. So, $\sin(3x)$ is not a solution to the differential equation.

1.3 e i. Replacing y with x^4 in the differential equation yields

$$x\frac{d}{dx}\left[x^4\right] - 2\left[x^4\right] = 6x^4 \quad \rightarrowtail \quad x\left[4x^3\right] - 2x^4 = 6x^4$$

$$\hookrightarrow \qquad 2x^4 = 6x^4 \quad \rightarrowtail \quad 0 = 4x^4 \quad ,$$

which does not hold if $x \neq 0$. So $y(x) = x^4$ is not a solution to the differential equation.

1.3 e ii. Replacing y with $3x^4$ in the differential equation yields

$$x\frac{d}{dx}\left[3x^4\right] - 2\left[3x^4\right] = 6x^4 \quad \rightarrowtail \quad x\left[12x^3\right] - 6x^4 = 6x^4$$

$$\hookrightarrow \qquad 6x^4 = 6x^4 \quad ,$$

which is true no matter what value x is. So $y(x) = 3x^4$ is a solution to the differential equation.

1.3 e iii. Letting $y(x) = 3x^4 + 5x^2$ in the differential equation yields

$$x\frac{d}{dx}\left[3x^4 + 5x^2\right] - 2\left[3x^4 + 5x^2\right] = 6x^4$$

$$\hookrightarrow \qquad x\left[12x^3 + 10x\right] - 6x^4 - 10x^2 = 6x^4$$

$$\hookrightarrow \qquad 6x^4 = 6x^4 \quad .$$

Since both sides of the last equation are identical, $y(x) = 3x^4 + 5x^2$ is a solution to the differential equation.

1.3 g i. For convenience, we first compute the derivatives of the given y:

$$y = \sin(2x) \quad \rightarrowtail \quad \frac{dy}{dx} = -2\cos(2x) \quad \rightarrowtail \quad \frac{d^2y}{dx^2} = -4\sin(2x) \quad .$$

So we have

$$\frac{d^2y}{dx^2} + 4y = 12x \quad \rightarrowtail \quad -4\sin(2x) + 4\sin(2x) = 12x$$

$$\hookrightarrow \qquad 0 = 12x \quad ,$$

which only holds if $x \neq 0$. Hence $y(x) = \sin(2x)$ is not a solution to the differential equation.

1.3 g ii. Here, $y = 3x \quad \rightarrowtail \quad \frac{dy}{dx} = 3 \quad \rightarrowtail \quad \frac{d^2y}{dx^2} = 0$.

So
$$\frac{d^2y}{dx^2} + 4y = 12x \quad \rightarrowtail \quad 0 + 4[3x] = 12x \quad \rightarrowtail \quad 12x = 12x \quad .$$

which does holds for every real value of x. Hence $y(x) = 3x$ is a solution to the differential equation.

1.3 g iii. Replacing y with $\sin(2x) + 3x$ in the differential equation yields

$$\frac{d^2}{dx^2}[\sin(2x) + 3x] + 4[\sin(2x) + 3x] = 12x$$

$$\hookrightarrow \quad -4\sin(2x) + 0 + 4\sin(2x) + 12x = 12x$$

$$\hookrightarrow \quad 12x = 12x \quad .$$

So $y(x) = \sin(2x) + 3x$ is a solution to the differential equation.

1.4 a i. To be a solution to the initial-value problem, $y(x) = e^{4x}$ must satisfy both the differential equation and the initial condition. Checking each:

D. E.: $\frac{dy}{dx} = 4y \quad \rightarrowtail \quad \frac{d}{dx}\left[e^{4x}\right] = 4\left[e^{4x}\right] \quad \rightarrowtail \quad 4e^{4x} = 4e^{4x}$ (YES)

I. C.: $5 = y(0) = e^{4\cdot 0} = 1$ (NO)

Since the initial condition is not satisfied, $y(x) = e^{4x}$ is not a solution to the initial-value problem.

1.4 a ii. D. E.: $\frac{dy}{dx} = 4y \quad \rightarrowtail \quad \frac{d}{dx}\left[5e^{4x}\right] = 4\left[5e^{4x}\right] \quad \rightarrowtail \quad 20e^{4x} = 20e^{4x}$ (YES)

I. C.: $5 = y(0) = 5e^{4\cdot 0} = 5 \cdot 1 = 5$ (YES)

So $y(x) = e^{4x}$ is satisfies both the differential equation and the initial condition, and, hence, is a solution to the initial-value problem.

1.4 a iii. Checking just the differential equation, we see that

$$\frac{dy}{dx} = 4y \quad \rightarrowtail \quad \frac{d}{dx}\left[e^{4x} + 1\right] = 4\left[e^{4x} + 1\right] \quad \rightarrowtail \quad 4e^{4x} = 4e^{4x} + 4 \quad .$$

Thus the differential equation is not satisfied, and this means $y(x) = e^{4x} + 1$ is not a solution to the initial-value problem (and there is no need to check to see if the initial condition is satisfied).

Alternative Approach: In this case, we also have

I. C.: $5 = y(0) = e^{4\cdot 0} + 1 = 1 + 1 = 2$ (NO)

Since the initial condition is not satisfied, $y(x) = e^{4x} + 1$ is not a solution to the initial-value problem (and there is no need to check to see if the differential equation is satisfied).

1.4 c i. With second-order initial-value problems, it often helps to first compute the derivatives of the given y :

$$y(x) = 2e^{3x} - e^{-3x} \quad \rightarrowtail \quad \frac{dy}{dx} = 6e^{3x} + 3e^{-3x} \quad \rightarrowtail \quad \frac{d^2y}{dx^2} = 18e^{3x} - 9e^{-3x} \quad .$$

So:

D. E.:
$$\frac{d^2y}{dx^2} - 9y = 0$$

$$\hookrightarrow \quad \left[18e^{3x} - 9e^{-3x}\right] - 9\left[2e^{3x} - e^{-3x}\right] = 0$$

$$\hookrightarrow \quad 0 = 0 \text{ (YES)}$$

I. C. 1: $1 = y(0) = 2e^{3\cdot 0} - e^{-3\cdot 0} = 2 \cdot 1 - 1 = 1$ (YES)

I. C. 2: $9 = y'(0) = 6e^{3\cdot 0} + 3e^{-3\cdot 0} = 6 + 3 = 9$ (YES)

Since $y(x) = 2e^{3x} - e^{-3x}$ satisfies the differential equation and both initial conditions, it is a solution to the initial-value problem.

1.4 c ii. Here $y = e^{3x} \rightarrowtail \frac{dy}{dx} = 3e^{3x} \rightarrowtail \frac{d^2y}{dx^2} = 9e^{3x}$.

Testing the differential equation and initial conditions:

D. E.: $\frac{d^2y}{dx^2} - 9y = 0 \rightarrowtail 9e^{3x} - 9\left[e^{3x}\right] = 0 \rightarrowtail 0 = 0$ (YES)

I. C. 1: $1 = y(0) = e^{3\cdot 0} = 1$ (YES)

I. C. 2: $9 = y'(0) = 3e^{3\cdot 0} = 9$ (NO)

Since one initial condition is not satisfied, the given function is not a solution to the initial-value problem.

1.4 c iii. Here, $y = e^{3x} + 1 \rightarrowtail \frac{dy}{dx} = 3e^{3x} \rightarrowtail \frac{d^2y}{dx^2} = 9e^{3x}$.

Testing the differential equation and initial conditions:

D. E.: $\frac{d^2y}{dx^2} - 9y = 0 \rightarrowtail 9e^{3x} - 9\left[e^{3x} + 1\right] = 0 \rightarrowtail 1 = 0$ (NO)

Since differential equation is not satisfied, the given function is not a solution to the initial-value problem. There is no need to also test the initial conditions (which, in this case, are also not satisfied).

1.6 a. Using $y = Ae^{x^2} - 3$ in the given differential equation, we get

$$\frac{d}{dx}\left[Ae^{x^2} - 3\right] - 2x\left[Ae^{x^2} - 3\right] = 6x$$

$$\hookrightarrow \quad A2xe^{x^2} - A2xe^{x^2} + 6x = 6x$$

$$\hookrightarrow \quad 6x = 6x \quad ,$$

verifying that $y = Ae^{x^2} - 3$ is a solution to the differential equation.

1.6 b i. We simply solve for the value of A that makes the initial condition true:

$$1 = y(0) = Ae^{0^2} - 3 = A \cdot 1 - 3 = A - 3$$

$$\hookrightarrow \qquad A = 1 + 3 = 4 \quad ,$$

Hence, the solution is $y(x) = Ae^{x^2} - 3$ with $A = 4$; that is, $y = 4e^{x^2} - 3$.

1.6 b ii. Here, $0 = y(0) = Ae^{0^2} - 3 = A - 3 \rightarrowtail A = 3$.
So the solution is $y(x) = Ae^{x^2} - 3$ with $A = 3$; that is, $y = 3e^{x^2} - 3$.

1.8 a i. Combining the differential equation with the given initial condition:

$$y'(1) = 3 \cdot 1 \cdot y(1) + 1^2 = 3 \cdot 2 + 1 = 7 \quad .$$

1.9 a. Since $y(t) = 0$ when $t = T_{\text{hit}}$,

$$0 = y(T_{\text{hit}}) = -4.9\,(T_{\text{hit}})^2 + 1{,}000 \quad \rightarrowtail \quad (T_{\text{hit}})^2 = \frac{1{,}000}{4.9} = \frac{10{,}000}{49} \quad .$$

Taking the positive square root (since T_{hit} is supposed to be positive) and recalling that time is being measured in seconds, we get

$$T_{\text{hit}} = \sqrt{\frac{10{,}000}{49}} = \frac{100}{7} \quad \text{(seconds)} \quad .$$

1.9 b. Remember that the velocity v is the derivative of y,

$$v(t) = \frac{dy}{dt} = \frac{d}{dt}\left[-4.9t^2 + 1{,}000\right] = -9.8t \quad ,$$

and that distance and time are in terms of meters and seconds, respectively. So, using this and the answer from the previous exercise,

$$v\,(T_{\text{hit}}) = -9.8T_{\text{hit}} = -\frac{98}{10} \cdot \frac{100}{7} = -140 \quad \text{(meters/sec.)} \quad .$$

1.9 c ii. The full initial-value problem is

$$\frac{d^2y}{dx^2} = -9.8 \qquad \text{with} \quad y(0) = 1{,}000 \quad \text{and} \quad y'(0) = 2 \quad .$$

As in the example, integrating the differential equation once and then again yields

$$\frac{dy}{dx} = -9.8t + c \qquad \text{and} \qquad y = -4.9t^2 + ct + C \quad .$$

Applying the initial conditions, we get

$$2 = y'(0) = -9.8 \cdot 0 + c = c$$

and

$$1{,}000 = y(0) = -4.9 \cdot 0^2 + c \cdot 0 + C = C \quad .$$

So the solution is

$$y = -4.9t^2 + ct + C = -4.9t^2 + 2t + 1{,}000 \quad .$$

1.9 c iii. The time T_{hit} when the object hits the ground satisfies

$$0 \ = \ y(T_{\text{hit}}) \ = \ -4.9\,(T_{\text{hit}})^2 \ + \ 2T_{\text{hit}} \ + \ 1{,}000 \quad .$$

Using the quadratic formula to solve this, we get

$$T_{\text{hit}} \ = \ \frac{-2 \pm \sqrt{(2)^2 - 4(-4.9)(1{,}000)}}{2(-4.9)} \ = \ \frac{1 \mp \sqrt{4901}}{4.9} \quad .$$

Since T_{hit} should be positive, we take

$$T_{\text{hit}} \ = \ \frac{1 + \sqrt{4901}}{4.9}$$

to compute the velocity (in meters/second) at impact:

$$\begin{aligned} v(T_{\text{hit}}) \ = \ y'(T_{\text{hit}}) \ &= \ -9.8T_{\text{hit}} \ + \ 2 \\ &= \ -9.8\left[\frac{1 + \sqrt{4901}}{4.9}\right] \ + \ 2 \ = \ -2\sqrt{4901} \quad . \end{aligned}$$