36

Critical Points, Direction Fields and Trajectories—ALT

In the last chapter, we saw that standard first-order systems of differential equations naturally arise in many applications. We also saw that many other differential equations and systems of differential equations can often be rewritten as standard first-order systems. Consequently, if we have tools to effectively deal with these first-order systems, then we can also deal with a wide variety of other differential equations and systems by converting them to first-order systems and then using those tools. This suggests narrowing our discussions to those first-order systems, and that's just what we will do in this chapter.

In particular, we will identify more precisely the types of systems we will be studying for the next several chapters, develop more terminology and notation, and discuss the idea of "graphing" solutions of these systems. All of this will be used repeatedly in the following chapters.

36.1 Important Terminology and Convenient Notation

Regular and Autonomous Systems

Recall that a standard first-order $N \times N$ system of differential equations can be written as

$$\begin{aligned} x_1{}' &= f_1(t, x_1, x_2, \dots, x_N) \\ x_2{}' &= f_2(t, x_1, x_2, \dots, x_N) \\ &\vdots \\ x_N{}' &= f_N(t, x_1, x_2, \dots, x_N) \end{aligned} \tag{36.1}$$

where N is some positive integer, the x_j's are (presumably unknown) real-valued functions of t (hence $x' = {}^{dx}/_{dt}$), and the $f_k(t, x_1, x_2, \dots, x_N)$'s (the component functions) are known functions of $N+1$ variables.

As in the previous chapter, we will use whatever symbols are convenient for the unknown functions and the component functions, and, if $N = 2$ or $N = 3$, we'll usually avoid subscripts and write our generic system as

$$\begin{aligned} x' &= f(t, x, y) \\ y' &= g(t, x, y) \end{aligned} \quad \text{or} \quad \begin{aligned} x' &= f(t, x, y, z) \\ y' &= g(t, x, y, z) \\ z' &= h(t, x, y, z) \end{aligned} \quad , \tag{36.2}$$

as appropriate.

As already noted, many systems of interest either are or can be converted to standard first-order systems. In addition, many of these systems are also "regular" and/or "autonomous", and just how we deal with a particular system may depend on which of these terms apply. So let us define them:

- A "regular" system is simply a standard system having "reasonably nice" component functions. More precisely, when we refer to a system as being *regular*, we mean that it is a standard first-order system whose component functions and all their first partial derivatives exist and are continuous for all real values of their variables. If you check, you see that the existence and uniqueness theorems of the previous chapter automatically apply when the system is regular, assuring us that solutions exist and are "reasonably well behaved".

- A standard system is *autonomous* if and only if no component function explicitly depends on t. When we limit ourselves to autonomous systems, the k^{th} equation in system (36.1) reduces to
$$x_k{}' = f_k(x_1, x_2, \ldots, x_N) \quad ,$$
and the systems in set (36.2) reduce to
$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned} \qquad \text{or} \qquad \begin{aligned} x' &= f(x,y,z) \\ y' &= g(x,y,z) \\ z' &= h(x,y,z) \end{aligned} \qquad .$$

  Autonomous systems naturally arise in applications. If you check, all the first-order systems derived in the previous chapter from applications were autonomous.

Matrix/Vector Notation for Standard Systems

We can express our generic systems much more concisely if we view the functions in an ordered set of N functions $(x_1(t),\ x_2(t),\ \ldots,\ldots,\ x_N(t))$ as components in the $N \times 1$ column matrix

$$\mathbf{x} = \mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix} \quad .$$

Following standard conventions and to avoid rather cumbersome terminology later, we will (unless otherwise indicated) use the terms *column vector* or just *vector* as synonyms for "$N \times 1$ matrix".[1] If each component of a given column vector $\mathbf{x}$ is a constant, then we will refer to $\mathbf{x}$ as a *constant vector*. Of particular importance, of course, is the (N-dimensional) *zero vector* $\mathbf{0}$, which is the column vector whose components are all 0. Thus,

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{if} \quad N = 2 \quad ,$$

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{if} \quad N = 3 \quad ,$$

and so on.

[1] Warning: This definition of a vector as a column matrix is a much more limited definition of "vector" than typically found in, say, texts on linear algebra or physics.

More generally, the components of $\mathbf{x}$ will be functions on some given interval. In this case, we may refer to $\mathbf{x}$ as a *vector-valued function* (on that interval), and we will further classify $\mathbf{x}$ as being, respectively, continuous or differentiable on an interval if and only if all the components are continuous or differentiable on that interval. In practice, we will usually assume this continuity and differentiability (often without comment).[2] Under this viewpoint, we can treat our sets of functions as (column) vector-valued functions with

$$\mathbf{x}' = \frac{d\mathbf{x}}{dt} = \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix} = \begin{bmatrix} {}^{dx_1}/_{dt} \\ {}^{dx_2}/_{dt} \\ \vdots \\ {}^{dx_N}/_{dt} \end{bmatrix} = \begin{bmatrix} x_1{}'(t) \\ x_2{}'(t) \\ \vdots \\ x_N{}'(t) \end{bmatrix} \quad .$$

We can then express system (36.1) as

$$\mathbf{x}'(t) = \mathbf{F}(t,\mathbf{x}) \qquad \text{or even} \qquad \mathbf{x}' = \mathbf{F} \tag{36.1$'$}$$

with the understanding that

$$\mathbf{F} = \mathbf{F}(t,\mathbf{x}) = \begin{bmatrix} f_1(t, x_1, x_2, \ldots, x_N) \\ f_2(t, x_1, x_2, \ldots, x_N) \\ \vdots \\ f_N(t, x_1, x_2, \ldots, x_N) \end{bmatrix}$$

or, if the system is autonomous,

$$\mathbf{F} = \mathbf{F}(\mathbf{x}) = \begin{bmatrix} f_1(x_1, x_2, \ldots, x_N) \\ f_2(x_1, x_2, \ldots, x_N) \\ \vdots \\ f_N(x_1, x_2, \ldots, x_N) \end{bmatrix} \quad .$$

In the future, we will use whatever bold-faced letters or symbols seem convenient to denote vectors, just as we will use whatever symbols are convenient for the unknown and component functions. In particular, when $N = 2$ or $N = 3$, we will often use, respectively, the index-free notation

$$\mathbf{x} = \mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \qquad \text{and} \qquad \mathbf{F} = \mathbf{F}(t,\mathbf{x}) = \begin{bmatrix} f(t,x,y) \\ g(t,x,y) \end{bmatrix}$$

or

$$\mathbf{x} = \mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \qquad \text{and} \qquad \mathbf{F} = \mathbf{F}(t,\mathbf{x}) = \begin{bmatrix} f(t,x,y,z) \\ g(t,x,y,z) \\ h(t,x,y,z) \end{bmatrix} \quad .$$

When not using this index-free notation, though, let us at agree that (unless otherwise indicated) whenever a particular bold-faced letter or symbol is used for a particular vector or matrix, then the same letter or symbol — lower-case, nonbold-faced and appropriately subscripted — will denote the components of that vector. So if we have vector $\mathbf{y}$, then (unless otherwise indicated)

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} \quad .$$

[2] The validity of this assumption of differentiability will be discussed further in section 36.6.

Along these lines, recall that the *transpose* of a matrix $\mathbf{A}$ — denoted $\mathbf{A}^{\mathsf{T}}$ — is the matrix constructed by interchanging the rows and columns of $\mathbf{A}$. In particular,

$$[x_1\,,\, x_2\,,\, x_3\,,\, \dots\,,\, x_N]^{\mathsf{T}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \end{bmatrix} \quad .$$

For now, the main value in using the transpose is to save a little space by writing column vectors as transposed row matrices.

Some of the advantages of the matrix/vector notation are obvious. It saves a great deal of effort and space, especially when discussing generic systems. Later, when we start dealing with "linear" systems of differential equations and begin to incorporate elements of linear algebra in nontrivial ways, the benefits of this notation will become even more significant.

!►Example 36.1: *Consider the system*

$$\begin{aligned} x' &= x + 2y \\ y' &= 5x - 2y \end{aligned}$$

along with the initial conditions

$$x(0) = 0 \quad \text{and} \quad y(0) = 1 \quad .$$

In example 35.3 we saw that a solution to this initial-value problem is the pair

$$x(t) = \frac{2}{7}e^{3t} - \frac{2}{7}e^{-4t} \quad \text{and} \quad y(t) = \frac{2}{7}e^{3t} + \frac{5}{7}e^{-4t} \quad .$$

In matrix/vector form, this initial-value problem can be written as either

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + 2y \\ 5x - 2y \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

or even as

$$\mathbf{x}' = \mathbf{F} \quad \text{with} \quad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

with

$$\mathbf{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix} \quad \text{and} \quad \mathbf{F} = \begin{bmatrix} x + 2y \\ 5x - 2y \end{bmatrix} \quad .$$

The given solution can then be written as either

$$\mathbf{x}(t) = \begin{bmatrix} \frac{2}{7}e^{3t} - \frac{2}{7}e^{-4t} \\ \\ \frac{2}{7}e^{3t} + \frac{5}{7}e^{-4t} \end{bmatrix} \quad \text{or} \quad \mathbf{x}(t) = \left[\frac{2}{7}e^{3t} - \frac{2}{7}e^{-4t} \;,\; \frac{2}{7}e^{3t} + \frac{5}{7}e^{-4t} \right]^{\mathsf{T}}$$

depending on how wasteful of space we care to be. While we are at it, let us also observe that the above formula for $\mathbf{x}$ *can also be written as*

$$\mathbf{x}(t) = \frac{2}{7}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + \frac{1}{7}\begin{bmatrix} 2 \\ 5 \end{bmatrix} e^{-4t} \quad .$$

(The significance of this last observation will become more apparent in the next few chapters.)

In the future, we will use, or not use, the matrix/vector notation as convenient. We will also expand this notation in the next chapter when we start dealing with "linear" systems of differential equations.

36.2 Constant (or Equilibrium) Solutions

A solution

$$\mathbf{x} = \mathbf{x}(t) = \left[x_1(t)\,,\ x_2(t)\,,\ \ldots\,,\ x_N(t)\right]^{\mathsf{T}}$$

to a system $\mathbf{x}' = \mathbf{F}(t, \mathbf{x})$ is a *constant solution* if each $x_k(t)$ is simply some single constant value x_k^0 for all t. Now remember,

$$x_k(t) = \text{some constant for all } t \quad \Longleftrightarrow \quad {x_k}'(t) = 0 \qquad \text{for all} \quad t \quad .$$

Thus, $\mathbf{x}$ is a constant solution if and only if

$$\mathbf{x}'(t) = \mathbf{0} \qquad \text{for all} \quad t$$

where $\mathbf{0}$ is (as defined a few pages ago) the zero vector. Clearly, then, $\mathbf{x}(t)$ is a constant solution of $\mathbf{x}' = \mathbf{F}(t, \mathbf{x})$ if and only if $\mathbf{x}$ is a constant vector satisfying

$$\mathbf{0} = \mathbf{F}(t, \mathbf{x}) \qquad \text{for all} \quad t \quad .$$

That is,

$$\begin{aligned} 0 &= f_1(t, x_1, x_2, \ldots, x_N) \\ 0 &= f_2(t, x_1, x_2, \ldots, x_N) \\ &\ \vdots \\ 0 &= f_N(t, x_1, x_2, \ldots, x_N) \end{aligned} \tag{36.3}$$

By solving this algebraic system, you can then find all the constant solutions for a given system of differential equations. (Do remember that we are insisting our solutions be real valued. So the constants must be real numbers.)

Constant solutions will be especially important when we study autonomous systems. And, when the system is autonomous, it is traditional to call any constant solution an *equilibrium solution*. We will follow tradition.

!►*Example 36.2:* *Let's try to find every equilibrium solution for the autonomous system*

$$\begin{aligned} x' &= x(y^2 - 9) \\ y' &= (x - 1)(y^2 + 1) \end{aligned} \quad .$$

The constant/equilibrium solutions are all obtained by setting both x' *and* y' *equal to* 0*, and then solving the resulting algebraic system,*

$$\begin{aligned} 0 &= x(y^2 - 9) \\ 0 &= (x - 1)(y^2 + 1) \end{aligned} \quad . \tag{36.4}$$

Consider the first equation, first:

$$0 = x(y^2 - 9)$$

$$\hookrightarrow \qquad x = 0 \quad \text{or} \quad y^2 = 9$$

$$\hookrightarrow \qquad x = 0 \quad \text{or} \quad y = 3 \quad \text{or} \quad y = -3 \quad .$$

If $x = 0$, then the second equation equation in system (36.4) reduces to

$$0 = (0-1)(y^2+1) \quad ,$$

which means that $y^2 = -1$, and, hence, $y = \pm i$. But these are not real numbers, as required. So we do not have an equilibrium solution

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x^0 \\ y^0 \end{bmatrix} \qquad \text{with} \quad x^0 = 0 \quad .$$

On the other hand, if the first equation in system (36.4) is satisfied because $y = 3$, then the second equation in that system reduces to

$$0 = (x-1)\left(3^2+1\right) = (x-1)\cdot 10 \quad ,$$

telling us that $x = 1$. Thus, one equilibrium solution for our system of differential equations is

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad \text{for all} \quad t \quad .$$

Finally, if the first equation in system (36.4) holds because $y = -3$, then the second equation in that system becomes

$$0 = (x-1)\cdot 10 \quad .$$

Hence, again, $x = 1$, and the corresponding equilibrium solution is

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \qquad \text{for all} \quad t \quad .$$

In summary, then, our system of differential equations has two equilibrium solutions:

$$\mathbf{x}(t) = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}(t) = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \quad .$$

“Graphing” Standard Systems
True Graphs and Trajectories

Let us briefly discuss two ways of graphically representing solutions to standard first-order systems, starting with a simple example.

!►*Example 36.3:* *Consider “graphically representing” the solution to the initial-value problem*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + 2y \\ 5x - 2y \end{bmatrix} \qquad \text{with} \qquad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad .$$

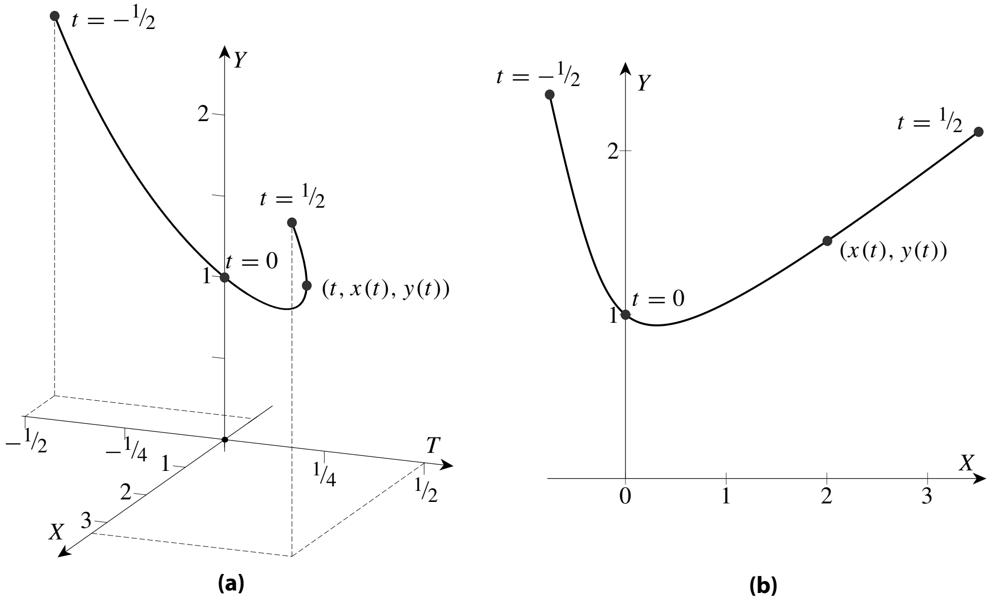


Figure 36.1: Two graphical representations of the solution from example 36.3 with $-1/2 \leq t \leq 1/2$: **(a)** The actual graph, and **(b)** The curve traced out by $(x(t), y(t))$ in the XY–plane.

From example 36.1 we know a solution to this initial-value problem is the pair

$$x(t) = \frac{2}{7}e^{3t} - \frac{2}{7}e^{-4t} \qquad \text{and} \qquad y(t) = \frac{2}{7}e^{3t} + \frac{5}{7}e^{-4t} \quad .$$

To construct the actual graph of this solution, we need to plot each point $(t, x(t), y(t))$ in TXY–space, using the above formulas for $x(t)$ and $y(t)$. This results in a curve in TXY–space. Part of this curve has been sketched in figure 36.1a.

As an alternative to constructing the graph, we can sketch the curve in the XY–plane that is traced out by $(x(t), y(t))$ as t varies. That is what what was sketched in figure 36.1b.

Take a look at the two figures. Both were easily done using a computer math package.

In general, the graph of a solution to an $N \times N$ system requires a coordinate system with $N + 1$ axes. Sketching such a graph is do-able if $N = 2$, as in the above example, especially if you have a decent computer math package. Then you can even rotate the image to see the graph from different views. Unfortunately, when you are limited to just one view, as in figure 36.1a, it may be somewhat difficult to properly interpret the figure. Because of this, and because of the very serious problems we would have if $N > 2$, we will rarely, if ever again, actually attempt to sketch true graphs of our solutions.

On the other hand, we will find the approach illustrated by figure 36.1b quite useful, especially for 2×2 systems. Moreover, the sketches that we will generate for 2×2 systems will give us insight as to the behavior of solutions to larger systems.

Observe that the curve in figure 36.1b has a natural "direction of travel" corresponding the way the curve is traced out as t increases. If you start at the point where $t = -1/2$ and travel the curve "in the positive direction" (that is, in the direction in which $(x(t), y(t))$ travels along the curve as t increases), then you would pass through the point where $t = 0$ and then through the point where $t = +1/2$. That makes this an *oriented* curve. We will call this oriented curve a *trajectory* for our system.

Just to be a bit more compleat: Suppose we have a solution

$$\mathbf{x}(t) = \left[x_1(t)\,,\ x_2(t)\,,\ \dots\,,\ x_N(t)\right]^{\mathsf{T}}$$

to a standard $N \times N$ system of first-order differential equations $\mathbf{x}' = \mathbf{F}(t,\mathbf{x})$. We will view the components of $\mathbf{x}(t)$, — $x_1(t)$, $x_2(t)$, ... and $x_N(t)$ — as the coordinates of a point in N-dimensional space using a Cartesian coordinate system, and we will refer to the oriented curve traced by out by this point as t increases as this solution's *trajectory*, with the orientation being the direction of travel along the curve given by $(x_1(t), x_2(t), \dots, x_N(t))$ as t increases. We will also refer to this oriented curve as a *trajectory for the system* $\mathbf{x}' = \mathbf{F}(t,\mathbf{x})$. Much of the analysis in next several chapters will be devoted to determining just what the trajectories of various system look like.

This raises a minor issue of notation and terminology. Traditionally, most people use an N-tuple, such as

$$(x, y) \qquad \text{or} \qquad (x_1, x_2, \dots, x_N)$$

to describe position, and not a column vector

$$\mathbf{x} = [x, y]^{\mathsf{T}} \qquad \text{or} \qquad \mathbf{x} = [x_1, x_2, \dots, x_N]^{\mathsf{T}}$$

where the x, y and x_k's are the coordinates of position using some coordinate system. Still, either notation is simply an ordered listing of the coordinates and so, either can be used to describe position. Let us go ahead and do so. This means, for example, that we will accept the phrase

The trajectory traced out by $\mathbf{x}(t)$ *(as* t *varies)*

as meaning

The trajectory traced out by $(x_1(t), y_2(t), \dots, x_N(t))$ *(as* t *varies)*
where $[x_1(t), x_2(t), \dots, x_N(t)]^{\mathsf{T}} = \mathbf{x}(t)$

Sometimes, when we want to emphasize the fact that $\mathbf{x}$ is describing position, we will refer to $\mathbf{x}$ as a *position vector* or even as simply a "point (in N-dimensional space)".

"Arrows", Velocity Vectors and the Direction of Travel

Traditionally, a vector $\mathbf{a}$ is often thought of as an "arrow" of a particular length and pointing in a particular direction. We, however, have defined a vector $\mathbf{a}$ to simply be a column matrix

$$\mathbf{a} = \left[a_1\,,\ a_2\,,\ \dots\,,\ a_N\right]^{\mathsf{T}} \quad .$$

Still, you should be well-acquainted with the "arrow" associated with such a column vector when using a Cartesian coordinate system. In particular, a position vector

$$\mathbf{x} = [x, y]^{\mathsf{T}} \qquad \text{or} \qquad \mathbf{x} = [x_1, x_2, \dots, x_N]$$

can be viewed, respectively, as an arrow from the origin

$$(0, 0) \qquad \text{or} \qquad (0, 0, \dots, 0)$$

to the position with coordinates

$$(x, y) \qquad \text{or} \qquad (x_1, x_2, \dots, x_N) \quad .$$

Now recall that, if $\mathbf{x}$ is a vector-valued function $\mathbf{x}(t)$, then

$$\mathbf{v}(t) = \mathbf{x}'(t) = \left[{x_1}'(t)\,,\ {x_2}'(t)\,,\ \dots\,,\ {x_N}'(t)\right]^{\mathsf{T}}$$

is the *velocity vector* $\mathbf{v}$ at time t of an object whose position at time t is $\mathbf{x}(t)$.[3] As you should recall from elementary multivariable calculus, this vector $\mathbf{v}(t)$ is an 'arrow' pointing in the direction the object is traveling at time t. So, if we pick some value t_0 for t and draw $\mathbf{v}^0 = \mathbf{v}(t_0)$ at position $\mathbf{x}^0 = \mathbf{x}(t_0)$, then $\mathbf{v}^0$ will be tangent to and pointing in the direction of travel of the trajectory of $\mathbf{x}(t)$ at $\mathbf{x}^0$, thus giving some idea of what that trajectory looks like near that point. And if $\mathbf{x}(t)$ is known to be a solution to

$$\mathbf{x}'(t) = \mathbf{F}(t, \mathbf{x}) \quad ,$$

then we can actually compute $\mathbf{v}^0 = \mathbf{x}'(t_0)$ for each choice of t_0 and $\mathbf{x}^0$ without solving the system for $\mathbf{x}(t)$. All we need to do is to compute $\mathbf{F}(t_0, \mathbf{x}^0)$.

!►*Example 36.4:* *Consider the trajectories of two objects both of whose positions at time t, $(x(t), y(t))$ satisfy the nonautonomous and nonlinear system*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} t(x+y) \\ y - tx \end{bmatrix} \quad .$$

Assume the first object passes through the point $(x, y) = (1, 2)$ at time $t = 0$. At that time, its velocity is

$$\mathbf{v} = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0(1+2) \\ 2 - 0 \cdot 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad .$$

However, if the other object also passes through the point $(x, y) = (1, 2)$, but at time $t = 2$, then its velocity at that time is

$$\mathbf{v} = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2(1+2) \\ 2 - 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} \quad .$$

Note the directions of travel for each of these objects as they pass through the point $(x, y) = (1, 2)$: The first is moving parallel to the Y–axis, while the second in moving parallel to the X–axis.

As the last example illustrates, the direction of travel for a solution's trajectory through a given point may depend on "when" it passes through the point. However, this is only for *non*autonomous systems. If our first-order system $\mathbf{x}' = \mathbf{F}$ is autonomous, then $\mathbf{F}$ does not depend on t, only on the components of $\mathbf{x}$. Consequently,

$$\mathbf{x}' = \mathbf{F}(\mathbf{x})$$

depends only on position. We will use this fact in the next section.

[3] This formula for velocity, $\mathbf{v} = \mathbf{x}'$, assumes we are using Cartesian coordinates. It is not true, for example, if we are using polar or spherical coordinates.

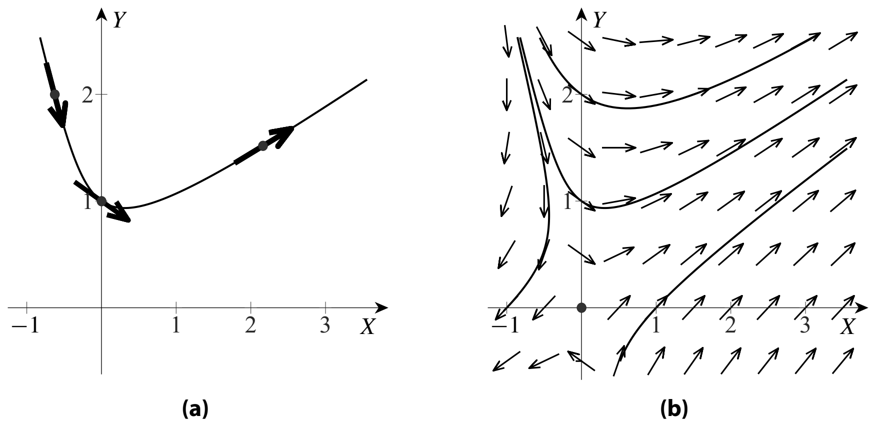


Figure 36.2: Direction arrows for trajectories for the system in example 36.5, with **(a)** being a few direction arrows tangent to the trajectory passing through (0, 1) (drawn oversized for clarity), and **(b)** being a more complete direction field, along with a few more trajectories.

36.3 Sketching Trajectories for Autonomous Systems
The Two-Dimensional Case
Direction Fields

Suppose we are given a regular 2×2 autonomous first-order system of differential equations

$$\begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned} \quad .$$

Now, pick a point (x_0, y_0) on the XY–plane, and, using the given system, compute the 'velocity' at that point for the trajectory through that point,

$$\mathbf{v} = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} f(x_0, y_0) \\ g(x_0, x_0) \end{bmatrix} \quad .$$

This gives us a vector tangent at the point (x_0, x_0) to any trajectory passing through this point, and pointing in the direction of travel along the trajectory as t increases. So, if we draw a short arrow at (x_0, y_0) in the same direction as this velocity vector, we then have a short arrow tangent to the trajectory through this point and pointing in the "direction of travel" along this curve. We will call this short arrow a *direction arrow*. (This assumes $\mathbf{x}'$ is nonzero at the point. If it is zero, we have a "critical point". We'll discuss critical points in just a little bit.) In figure 36.2a, we've sketched a few of these direction arrows at points along one particular trajectory.

If we sketch a corresponding direction arrow at every point (x_j, y_k) in a grid of points, then we have a *direction field*, as illustrated (along with a few trajectories) in figure 36.2b. This direction field tells us the general behavior of the system's trajectories. To sketch a trajectory using a direction field, simply start at any chosen point in the plane, and sketch a curve following the directions indicated by the nearby direction arrows. "Go with the flow", do not attempt to "connect the dots". The goal is to draw, as well as practical, a curve whose tangent at each point on the curve is lined up with the direction arrow that would be sketched at that point. That curve (if perfectly drawn), oriented in the direction given by the tangent direction arrows, is a trajectory of the system. (In practice, of course, it's as good an approximation to a trajectory as our drafting skills allow.)

The construction and use of a direction field for a 2×2 first-order autonomous system of differential equations is analogous to the construction and use of a slope field for a first-order differential equation (see chapter 8). It's not exactly the same — we are now sketching trajectories instead of true graphs of solutions (which would require a T–axis), but the mechanics are very much the same. And, just as with slope lines for a slope fields, it is good for your understanding to practice sketching the direction arrows for a few small direction fields, and, for the sake of your sanity, it is a good idea to learn how to construct direction fields (and trajectories) using a good computer math package such as Maple or Mathematica.[4]

Critical Points

When constructing a direction field, it is important to note each point (x_0, y_0) such that

$$\begin{bmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad .$$

Any such point (x_0, y_0) is said to be a *critical point* for the system. Since the zero vector has no well-defined direction, we cannot sketch a direction arrow at a critical point. Instead, plot a clearly visible dot at this point. After all, if (x_0, y_0) is a critical point for our system, then

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad ,$$

which, in turn, means we have the constant/equilibrium solution

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \qquad \text{for all} \quad t \quad ,$$

and, if you think about it a moment, you'll realize that the point (x_0, y_0) is the entire trajectory of this solution.

In fact, this gives us an alternate definition for critical point; namely, that a *critical point* for an autonomous system of differential equations is the trajectory of an equilibrium solution for that system.

Finding critical points and determining the behavior of the trajectories in regions around them will prove rather important. The mechanics of finding critical points is identical to the mechanics of finding equilibrium solutions (as illustrated in example 36.2 on page 36–5). Issues regarding the behavior of near-by trajectories will be discussed in the next section, and in future chapters.

!►*Example 36.5:* *Consider, once again, the system*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + 2y \\ 5x - 2y \end{bmatrix} \quad .$$

Plugging in $(x, y) = (0, 1)$ *, we get*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 + 2 \cdot 1 \\ 5 \cdot 0 - 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \quad .$$

Thus, the direction arrow sketched at position $(x, y) = (0, 1)$ *should be a short arrow (which we center at the point) parallel to and pointing in the same direction as the vector from the origin* $(0, 0)$ *to position* $(2, -2)$ *. That is what was sketched at point* $(0, 1)$ *in figure 36.2a, along with the trajectory through that point.*

[4] You might also find online programs for constructing direction fields.

A more complete direction field, along with three other trajectories, is illustrated in figure 36.2b. It was drawn using Maple (and touched up in a graphics program). Note the dot at $(0, 0)$ *. This is a critical point for the system, and is the trajectory of the one equilibrium solution*

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{for all} \quad t \quad .$$

Trajectories and Solutions

Keep in mind that a trajectory through a point (a, b) for a regular autonomous system

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix}$$

is not a solution to the system, it is the path traced out by a solution to the initial-value problem

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix} \qquad \text{with} \qquad \begin{bmatrix} x(t_0) \\ y(t_0) \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

for any choice of t_0 in $(-\infty, \infty)$. And since there are infinitely many possible values of t_0 , we should expect infinitely many solutions tracing out that one trajectory.

However, all the different solutions corresponding to a single trajectory are simply 'shifted' versions of each other. To see this, let $\left[\hat{x}(t), \hat{y}(t)\right]^{\mathsf{T}}$ be a solution to

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix} \qquad \text{with} \qquad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \quad .$$

You can then easily verify (see exercise 36.11 on page 36–33) that, for any real value t_0 , the vector-valued function

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \hat{x}(t - t_0) \\ \hat{y}(t - t_0) \end{bmatrix}$$

satisfies

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix} \qquad \text{with} \qquad \begin{bmatrix} x(t_0) \\ y(t_0) \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \quad .$$

The uniqueness theorems from the previous chapter assures us that there are no other solutions to this initial-value problem.

This, by the way, leads to a minor technical point that should be discussed briefly: When specifying a solution $\mathbf{x}(t)$, we should also specify the domain of this solution; that is, the interval (α, β) over which t varies. In practice, we often don't mention that interval, simply assuming it to be "as large as possible" (which, we'll see, often means $(\alpha, \beta) = (-\infty, \infty)$). On the few occasions where it is particularly relevant, we will refer to a trajectory as being *complete* if it is the trajectory of a solution

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \qquad \text{with} \quad \alpha < t < \beta$$

where (α, β) is "as large as possible". (That was the implicit assumption in the previous paragraph.)

Do note that, while we may be most interested in the complete trajectories of our systems, in practice, the trajectories sketched in phase portraits are often merely portions of complete trajectories simply because the complete trajectories often extend beyond the region over which we are making our drawings. One obvious exception, of course, is that a critical point $\mathbf{x}^0$ is a complete trajectory since is is the trajectory of the equilibrium solution

$$\mathbf{x}(t) = \mathbf{x}^0 \qquad \text{for} \quad -\infty < t < \infty \quad ,$$

and there certainly is no larger interval than $(-\infty, \infty)$.

Phase Portraits and Planes

A little more terminology: When dealing with direction fields and trajectories for a standard 2×2 autonomous system, we refer to the plane on which we sketch the direction field and/or trajectories as the *phase plane* (as opposed to the " XY–plane" or " X_1X_2–plane" plane or ...). If we sketch an enlightening representative sample of trajectories on the phase plane, then this sketch is said to be a *phase portrait* of the system. At this point, we are using direction fields to sketch phase portraits, so we are getting phase portraits superimposed on direction fields. If a phase portrait does not have an accompanying direction field to indicate direction of travel along the trajectories, then you should have little arrows on the trajectories to indicate the direction of travel for each trajectory.

Higher-Order Cases

The fundamental ideas just discussed regarding direction fields and trajectories for any 2×2 autonomous system extend naturally to analogous ideas for any $N\times N$ regular autonomous system of first-order differential equations

$$\mathbf{x}' = \begin{bmatrix} {x_1}' \\ {x_2}' \\ \vdots \\ {x_N}' \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2, \dots, x_N) \\ f_2(x_1, x_2, \dots, x_N) \\ \vdots \\ f_N(x_1, x_2, \dots, x_N) \end{bmatrix} = \mathbf{F}(\mathbf{x}) \quad .$$

As before, we define a *critical point* to be any point $\left(x_1^0, x_2^0, \dots, x_N^0\right)$ in N-dimensional space for which

$$\begin{bmatrix} f_1(x_1, x_2, \dots, x_N) \\ f_2(x_1, x_2, \dots, x_N) \\ \vdots \\ f_N(x_1, x_2, \dots, x_N) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad ,$$

and, as before, any such point is the trajectory of the corresponding equilibrium solution

$$\mathbf{x}(t) = \left[x_1^0, x_2^0, \dots, x_N^0\right]^{\mathsf{T}} \qquad \text{for all} \quad t$$

for our system $\mathbf{x}' = \mathbf{F}(\mathbf{x})$.

Likewise, at any point $\mathbf{a}$ other than a critical point, we can, in theory, find a short vector pointing in the direction of travel of any trajectory through that point by just taking any short vector pointing in the same direction as $\mathbf{F}(\mathbf{a})$. This gives a *direction arrow* at that point. Plotting these direction arrows on a suitable grid of points in N-dimensional space then gives us a *direction field* for the system, which can, in theory, be used to sketch trajectories by sketching the curves in N-dimensional space that "follow" the direction arrows.

Admittedly, few of us can actually sketch and use a direction field when $N > 2$ (especially if $N > 3$!). Still, we can find the critical points, and we will later discover that much of what we learn about the behavior of trajectories near critical points for 2×2 systems will apply when our systems are larger.

By the way, it is traditional to refer to the N-dimensional space in which the trajectories would, in theory, be drawn as the *phase space* (*phase plane* if $N = 2$), and any enlightening representative sample of trajectories in this space is called a *phase portrait*. Of course, visualizing a phase portrait when $N > 2$ requires a certain imagination (especially if $N > 3$!).

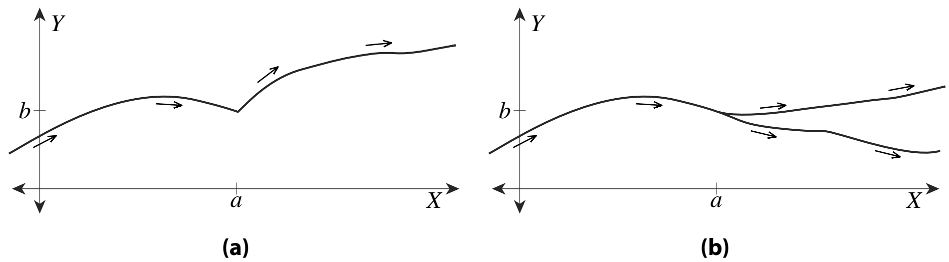


Figure 36.3: Two impossibilities regrading a trajectory of a regular autonomous system containing a point (a, b) : **(a)** A sharp "kink" at (a, b) , and **(b)** a "splitting" into two different trajectories at (a, b) .

Properties of Trajectories

It should be noted that the existence and uniqueness results regarding solutions that we developed in the previous chapter naturally imply corresponding existence and uniqueness results regarding (complete) trajectories, allowing us to say that "through each point there is one and only one (complete) trajectory". We'll say more about this in section 36.6. Also in that section, we will see that we also have the following:

1. If a trajectory contains a critical point, then that critical point is the entire trajectory. Conversely, if a trajectory has nonzero length, then it does not contain a critical point of the system. (Hence, in sketching trajectories other than critical points, make sure your trajectories do not pass through any critical points.)

2. If a complete trajectory has an endpoint, then that endpoint must be a critical point for the system.

3. Any oriented curve not containing a critical point that "follows" the system's direction field (i.e., any oriented curve whose tangent vector at each point is in the same direction as the system's direction arrow at that point) is the trajectory of some solution to the system. (This is what we intuitively expect when "sketching trajectories".)

4. Any trajectory that is not a critical point is "smooth" in that no trajectory can have a "kink" or "corner" (as in figure 36.3a).

5. No trajectory can "split" into two or more trajectories (as in figure 36.3b), nor can two or more trajectories "merge" into one trajectory (as in figure 36.3b with the arrows reversed).

Keeping the above in mind will help in both sketching trajectories from a direction field and in interpreting your results.

36.4 Critical Points, Stability and Long-Term Behavior

A useful feature of a direction field for an autonomous system of differential equations is that it can give us some notion of the long-term behavior of the solutions to that system. All we need to do is to follow sketched trajectories.

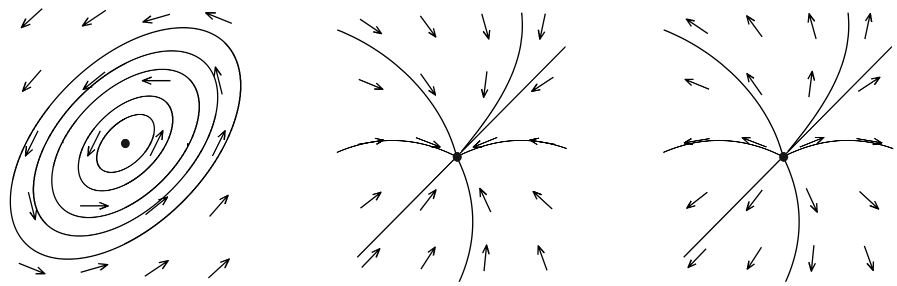


Figure 36.4: Two-dimensional direction fields and trajectories about critical points corresponding to equilibrium solutions that are **(a)** stable, but not asymptotically stable, **(b)** asymptotically stable and **(c)** unstable.

Critical Points and Stability

Stability of Equilibrium Solutions

Of particular interest will be the long-term behavior of solutions whose trajectories pass close to a critical point $\left(x_1^0, x_2^0, \ldots, x_N^0\right)$ of the system, and we will use this behavior to classify the "stability" of that critical point and the corresponding equilibrium solution

$$\mathbf{x}_{\text{eq}}(t) = \mathbf{x}^0 \qquad \text{for all} \quad t$$

where $\mathbf{x}^0 = \left[x_1^0, x_2^0, \ldots, x_N^0\right]^{\mathsf{T}}$. Loosely speaking we will classify this critical point and the corresponding equilibrium solution as being:

- *stable* if and only if each trajectory that gets close to $\left(x_1^0, x_2^0, \ldots, x_N^0\right)$ stays close to $\left(x_1^0, x_2^0, \ldots, x_N^0\right)$ afterwards. That is, this critical point and equilibrium solution are stable if and only if, whenever $\mathbf{x}$ is any other solution to the system satisfying $\mathbf{x}(t_0) \approx \mathbf{x}^0$ for some t_0, then

  $$\mathbf{x}(t) \approx \mathbf{x}^0 \qquad \text{for all} \quad t > t_0 \quad ,$$

  as illustrated in figures 36.4a and 36.4b.

- *asymptotically stable* if and only if each trajectory that gets close to $\left(x_1^0, x_2^0, \ldots, x_N^0\right)$ doesn't just stay close, but converges to $\left(x_1^0, x_2^0, \ldots, x_N^0\right)$ as $t \to +\infty$. That is, this critical point and equilibrium solution are asymptotically stable if and only if, whenever $\mathbf{x}$ is any other solution to the system satisfying $\mathbf{x}(t_0) \approx \mathbf{x}^0$ for some t_0, then

  $$\lim_{t \to +\infty} \mathbf{x}(t) = \mathbf{x}^0 \quad ,$$

  as illustrated in figure 36.4b. (Note that an asymptotically stable critical point is automatically stable.)

- *unstable* if and only if the equilibrium solution is not stable. Examples of unstable equilibrium solutions are illustrated in figure 36.4c (in which the nearby trajectories all diverge away from the critical point) and in figure 36.2b (in which trajectories approach the critical point $(0, 0)$ and then diverge away).

Be warned that the stability of an equilibrium solution is not always clear from just the direction field. The field may suggest that the nearby trajectories are loops circling the critical point (indicating that the equilibrium solution is stable) when, in fact, the nearby trajectories are either slowing spiralling in towards or out from the critical point (in which case the equilibrium solution is actually either asymptotically stable or simply unstable).

The identification of the stability of equilibrium solutions turns out to be rather important in the practical study of autonomous systems of differential equations, especially when the system is not linear. We'll return to this issue and develop more definitive ways of determining stability in a few chapters.

Precise Definitions

Of course, precise mathematics requires precise definitions. So, to be precise, we classify our equilibrium solution

$$\mathbf{x}_{\text{eq}}(t) = \mathbf{x}^0 \qquad \text{for all} \quad t$$

and the corresponding critical point as being:

- *stable* if and only if, for each $\epsilon > 0$, there is a corresponding $\delta > 0$ such that, if $\mathbf{x}$ is any solution to the system satisfying[5]

  $$\left\| \mathbf{x}(t_0) - \mathbf{x}^0 \right\| < \delta \qquad \text{for some} \quad t_0 \quad ,$$

  then

  $$\left\| \mathbf{x}(t_0) - \mathbf{x}^0 \right\| < \epsilon \qquad \text{for all} \quad t > t_0 \quad .$$

- *asymptotically stable* if and only if there is a $\delta > 0$ such that, if $\mathbf{x}$ is any solution to the system satisfying

  $$\left\| \mathbf{x}(t_0) - \mathbf{x}^0 \right\| < \delta \qquad \text{for some} \quad t_0 \quad ,$$

  then

  $$\lim_{t \to +\infty} \left\| \mathbf{x}(t_0) - \mathbf{x}^0 \right\| = 0 \quad .$$

- *unstable* if the equilibrium solution is not stable.

While we are at it, we should give precise meanings to the 'convergence/divergence' of a trajectory to/from a critical point $\mathbf{x}^0$. So assume we have a trajectory and any solution $\mathbf{x}(t)$ to the system that generates that trajectory. We'll say the the trajectory

- *converges* to critical point $\mathbf{x}^0$ if and only if $\lim_{t \to +\infty} \mathbf{x}(t) = \mathbf{x}^0$, and
- *diverges* from critical point $\mathbf{x}^0$ if and only if $\lim_{t \to -\infty} \mathbf{x}(t) = \mathbf{x}^0$.

Long-Term Behavior

A direction field may also give us some idea of the long-term behavior of the solutions to the given system at points far away from the critical points. Of course, this supposes that any patterns that appear to be evident in the direction field actually continue outside the region on which the direction field is drawn. This issue will be further examined later, at least for some systems.

---

[5] For any two vectors $\mathbf{a}$ and $\mathbf{b}$,

$$\|\mathbf{a} - \mathbf{b}\| = \text{distance between } \mathbf{a} \text{ and } \mathbf{b} = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \cdots + (a_N - b_N)^2} \quad .$$

!►Example 36.6: *Consider the direction field and sample trajectories sketched in figure 36.2b. In particular, look at the trajectory passing through the point $(x, y) = (1, 0)$, and follow it in the direction indicated by the direction field. The last part of this curve seems to be straightening out to a straight line proceeding further into the first quadrant at, very roughly, a 45 degree angle to both the positive X–axis and positive Y–axis. This suggests that, if $\mathbf{x}(t) = [x(t), y(t)]^{\mathsf{T}}$ is any solution to the direction field's system satisfying $\mathbf{x}(t_0) = [1, 0]^{\mathsf{T}}$ for some t_0, then*

$$\lim_{t\to\infty} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \infty \\ \infty \end{bmatrix}$$

with

$$y(t) \approx x(t) \qquad \text{when} \quad t \text{ is “large”} \quad .$$

On the other hand, if you follow the trajectory passing through position $(x, y) = (-1, 0)$, then you probably get the impression that, as t increases, the trajectory is heading deeper into the third quadrant of the XY–plane, suggesting that if $\mathbf{x}(t) = [x(t), y(t)]^{\mathsf{T}}$ is any solution to the direction field's system satisfying $\mathbf{x}(t_0) = [-1, 0]^{\mathsf{T}}$ for some t_0, then

$$\lim_{t\to\infty} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\infty \\ -\infty \end{bmatrix} \quad .$$

You may even suspect that, for such a solution,

$$y(t) \approx x(t) \qquad \text{when} \quad t \text{ is “large”} \quad ,$$

though there is hardly enough of the trajectory sketched to be too confident of this suspicion.

36.5 Applications

Let us try to apply the above to three applications from the previous chapter; namely, the applications involving two-tank mixing, competing species, and a swinging pendulum. In each case, we will find the critical points, attempt to use a (computer-generated) direction field to determine the stability of these points, and see what conclusions we can then derive.

A Mixing Problem

Our mixing problem is illustrated in figure 36.5. In it, we have two tanks $\mathcal{A}$ and $\mathcal{B}$ containing, respectively, 500 and 1,000 gallons of a water/alcohol mix. Each minute 6 gallons of a water/alcohol mix containing 50% alcohol is added to tank $\mathcal{A}$, while 6 gallons of the mix is drained from that tank. At the same time, 6 gallons of pure water is added to tank $\mathcal{B}$, and 6 gallons of the mix in tank $\mathcal{B}$ is drained out. In addition, the two tanks are connected by two pipes, with one pumping liquid from tank $\mathcal{A}$ to tank $\mathcal{B}$ at a rate of 2 gallons per minute, and with the other pumping liquid in the opposite direction, from tank $\mathcal{B}$ to tank $\mathcal{A}$, at a rate of 2 gallons per minute.

In the previous chapter, we found that the system describing this mixing process is

$$\begin{aligned} x' &= -\frac{8}{500}x + \frac{2}{1000}y + 3 \\ y' &= \frac{2}{500}x - \frac{8}{1000}y \end{aligned} \quad . \tag{36.5}$$

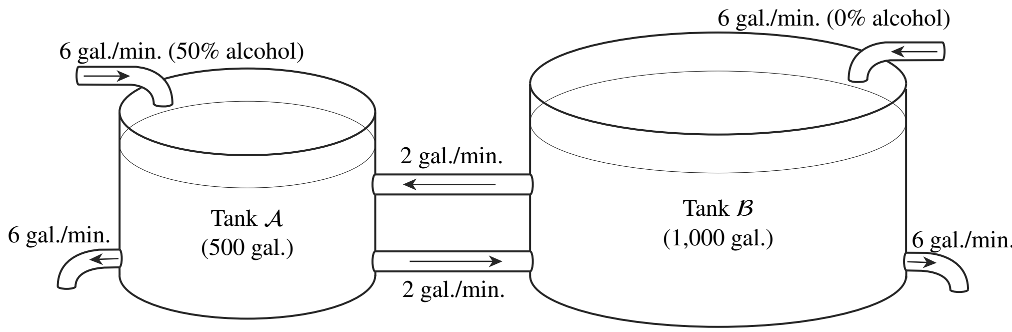


Figure 36.5: A simple system of two tanks containing water/alcohol mixtures.

where

$$t = \text{number of minutes since we started the mixing process} \quad ,$$

$$x = x(t) = \text{amount (in gallons) of alcohol in tank } \mathcal{A} \text{ at time } t \quad ,$$

and

$$y = y(t) = \text{amount (in gallons) of alcohol in tank } \mathcal{B} \text{ at time } t \quad .$$

To find any critical points of the system we first replace each derivative in system (36.5) with 0, obtaining

$$\begin{aligned} 0 &= -\frac{8}{500}x + \frac{2}{1000}y + 3 \\ 0 &= \frac{2}{500}x - \frac{8}{1000}y \end{aligned} \quad .$$

Then we solve this algebraic system. That is easily done. From the last equation we have

$$x = \frac{500}{2} \cdot \frac{8}{1000}y = 2y \quad .$$

Using this with the first equation, we get

$$0 = -\frac{8}{500} \cdot 2y + \frac{2}{1000}y + 3 = 3 - \frac{15}{500}y$$

$$\hookrightarrow \qquad y = 3 \cdot \frac{500}{15} = 100 \qquad \text{and} \qquad x = 2y = 2 \cdot 100 = 200 \quad .$$

So the one critical point for this system is

$$(x_0, y_0) = (200, 100) \quad .$$

Considering the physical process being modeled, it should seem reasonable for this critical point to describe the long-term equilibrium of the system. That is, we should expect $(200, 100)$ to be an asymptotically stable equilibrium, and that

$$\lim_{t\to\infty}(x(t), y(t)) = (200, 100) \quad .$$

Checking the computer-generated direction field in figure 36.6, we see that this is clearly the case. Around critical point $(200, 100)$, the direction arrows are all pointing toward this point. Thus, as $t \to \infty$ the concentrations of alcohol in tanks $\mathcal{A}$ and $\mathcal{B}$, respectively, approach

$$\frac{200}{500} \ \text{(i.e., 40\%)} \qquad \text{and} \qquad \frac{100}{1000} \ \text{(i.e., 10\%)} \quad .$$

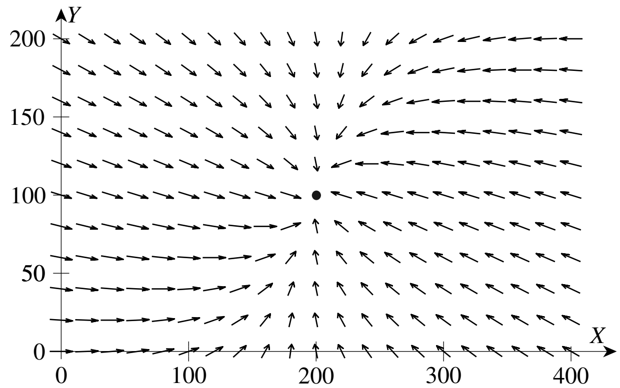


Figure 36.6: Direction field for the system of two tanks from figure 36.5.

Competing Species

The Model and Analysis

In our competing species example, we assumed that we have a large field containing rabbits and gerbils that are competing with each other for the resources in the field. Letting

$$R = R(t) = \text{number of rabbits in the field at time } t$$

and

$$G = G(t) = \text{number of gerbils in the field at time } t \quad ,$$

we derived the following system describing how the two populations vary over time:

$$\begin{aligned} R' &= (\beta_1 - \gamma_1 R - \alpha_1 G)R \\ G' &= (\beta_2 - \gamma_2 G - \alpha_2 R)G \end{aligned} \quad . \tag{36.6}$$

In this, β_1 and β_2 are the net birth rates per creature under ideal conditions for rabbits and gerbils, respectively, and the γ_k's and α_k's are positive constants that would probably have to be deterimined by experiment and measurement. In particular, the values we chose yielded the system

$$\begin{aligned} R' &= \left(\frac{5}{4} - \frac{1}{160}R - \frac{3}{1000}G\right) R \\ G' &= \left(3 - \frac{3}{500}G - \frac{3}{160}R\right) G \end{aligned} \quad . \tag{36.7}$$

Setting $R' = G' = 0$ in equation set (36.7) gives us the algebraic system we need to solve to find the critical points:

$$\begin{aligned} 0 &= \left(\frac{5}{4} - \frac{1}{160}R - \frac{3}{1000}G\right) R \\ 0 &= \left(3 - \frac{3}{500}G - \frac{3}{160}R\right) G \end{aligned} \quad . \tag{36.8}$$

The first equation in this algebraic system tells us that either

$$R = 0 \qquad \text{or} \qquad \frac{1}{160}R + \frac{3}{1000}G = \frac{5}{4} \quad .$$

If $R = 0$, the second equation reduces to

$$0 = \left(3 - \frac{3}{500}G\right)G \quad ,$$

which means that either

$$G = 0 \qquad \text{or} \qquad G = 500 \quad .$$

So two critical points are $(R, G) = (0, 0)$ and $(R, G) = (0, 500)$.

If, on the other hand, the first equation in algebraic system (36.8) holds because

$$\frac{1}{160}R + \frac{3}{1000}G = \frac{5}{4} \quad ,$$

then the system's second equation can only hold if either

$$G = 0 \qquad \text{or} \qquad \frac{3}{500}G + \frac{3}{160}R = 3 \quad .$$

If $G = 0$, then we can solve the first equation in the system, obtaining

$$R = \frac{5}{4} \cdot 160 = 200 \quad .$$

So $(R, G) = (200, 0)$ is one critical point. Looking at what remains, we see that there is one more critical point, and it satisfies the simple algebraic linear system

$$\begin{aligned} \frac{1}{160}R + \frac{3}{1000}G &= \frac{5}{4} \\ \frac{3}{160}R + \frac{3}{500}G &= 3 \end{aligned} \quad .$$

You can easily verify that the solution to this is $(R, G) = (80, 250)$.

So the critical points for our system are (R, G) equaling

$$(0, 0) \quad , \quad (0, 500) \quad , \quad (200, 0) \qquad \text{and} \qquad (80, 250) \quad .$$

The first tells us that, if we start with no rabbits and no gerbils, then the populations remain at 0 rabbits and 0 gerbils — no big surprise. The next tells us that our populations can remain constant if either we have no rabbits with 500 gerbils, or we have 200 rabbits and no gerbils. The last critical point says that the two populations can coexist at 80 rabbits and 250 gerbils.

But look at the direction field in figure 36.7a for this system. From that, it is clear that critical point $(0, 0)$ is unstable. If we have at least a few rabbits and/or gerbils, then those populations do not, together, die out. We will always have at least some rabbits or gerbils.

On the other hand, critical point $(200, 0)$ certainly appears to be an asymptotically stable critical point. In fact, from the direction field it appears that, if we have, say $R(0) > 150$ and $G(0) < 250$, then the direction of the direction arrows "near" $(200, 0)$ indicate that

$$\lim_{t \to \infty} (R(t), G(t)) = (200, 0) \quad .$$

In other words, the gerbils die out and the number of rabbits stabilizes at 200.

Likewise, critical point $(500, 0)$ also appears to be asymptotically stable. Admittedly, a somewhat more detailed direction field about $(500, 0)$, such as in figure 36.7b, may be desired to clarify this. Thus, if we start with enough gerbils and too few rabbits (say, $G(0) > 250$ and $R(0) < 80$), then

$$\lim_{t \to \infty} (R(t), G(t)) = (0, 500) \quad .$$

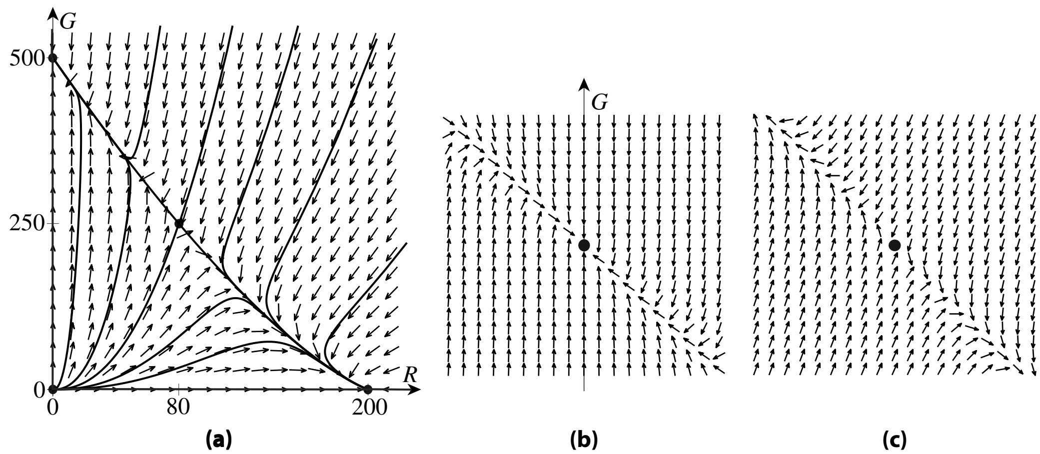


Figure 36.7: **(a)** A direction field and some trajectories for the competing species example with system (36.7), and detailed direction fields over very small regions about critical points **(b)** $(0, 500)$ and **(c)** $(80, 250)$.

In other words, the rabbits die out and the number of gerbils approaches 500 .

Finally, what about critical point $(80, 250)$? In the region about this critical point, we can see that a few direction arrows point towards this critical point, while others seem to lead the trajectories past the critical point. That suggests that $(80, 250)$ is an unstable critical point. Again, a more detailed direction field in a small area about critical point $(80, 250)$, such as in figure 36.7c, is called for. This direction field shows more clearly that critical point $(80, 250)$ is unstable. Thus, while it is possible for the populations to stabilize at 80 rabbits and 250 gerbils, it is also extremely unlikely.

To summarize: It is possible for the two competing species to coexist, but, in the long run, it is much more likely that one or the other dies out, leaving us with either a field of 200 rabbits or a field of 500 gerbils, depending on the initial number of each.

Some Notes

It turns out that different choices for the parameters in system (36.6) can lead to very different outcomes. For example, the system

$$\begin{aligned} R' &= (120 - 2R - 2G)R \\ G' &= (320 - 8G - 3R)G \end{aligned}$$

also has four critical points; namely,

$$(0, 0) \quad , \quad (60, 0) \quad , \quad (0, 40) \quad \text{and} \quad (32, 28) \quad .$$

In this case, however, the first three are unstable, and the last is asymptotically stable. Thus, if we start out with at least a few rabbits and a few gerbils, then

$$\lim_{t\to\infty} (R(t), G(t)) = (32, 28) \quad .$$

So neither population dies out. The rabbits and gerbils in this field are able to coexist, and there will eventually be approximately 32 rabbits and 28 gerbils.

It should also be noted that, sometimes, it can be difficult to determine the stability of a critical point from a given direction field. When this happens, a more detailed direction field about the critical point may be tried. A better approach involves a method we will discuss in chapter 42 using the eigenvalues of a certain matrix associated with the system and critical point.

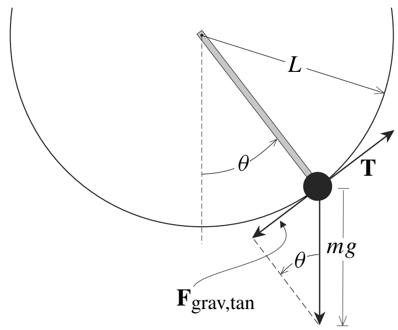


Figure 36.8: The pendulum system with a weight of mass m attached to a massless rod of length L swinging about a pivot point under the influence of gravity.

The (Damped) Pendulum

In the last chapter, we derived the system

$$\begin{aligned} \theta' &= \omega \\ \omega' &= -\gamma \sin(\theta) - \kappa\omega \end{aligned} \tag{36.9}$$

to describe the angular motion of the pendulum in figure 36.8. Here

$\theta(t)$ = the angular position of pendulum at time t measured counterclockwise from the vertical line "below" the pivot point

and

$\omega(t) = \theta'$ = the angular velocity of the pendulum at time t .

In addition, γ is the positive constant given by $\gamma = g/L$ where L is the length of the pendulum and g is the acceleration of gravity, and κ is the "drag coefficient", a nonnegative constant describing the effect friction has on the motion of the pendulum. The greater the effect of friction on the system, the larger the value of κ , with $\kappa = 0$ when there is no friction slowing down the pendulum. We will restrict our attention to the *damped* pendulum in which friction plays a role (i.e., $\kappa > 0$).[6] For simplicity, we will assume

$$\gamma = 8 \quad \text{and} \quad \kappa = 2 \quad ,$$

though the precise values for γ and κ will not play much of a role in our analysis. This gives us the system

$$\begin{aligned} \theta' &= \omega \\ \omega' &= -8\sin(\theta) - 2\omega \end{aligned} \quad . \tag{36.10}$$

Before going any further, observe that the right side of our system is periodic with period 2π with respect to θ . This means that, on the $\theta\omega$–plane, the pattern of the trajectories in any vertical strip of width 2π will be repeated in the next vertical strip of width 2π .

Setting $\theta' = 0$ and $\omega' = 0$ in system (36.10) yields the algebraic system

$$\begin{aligned} 0 &= \omega \\ 0 &= -8\sin(\theta) - 2\omega \end{aligned}$$

[6] It turns out that the undamped pendulum, in which $\kappa = 0$, is more difficult to analyze.

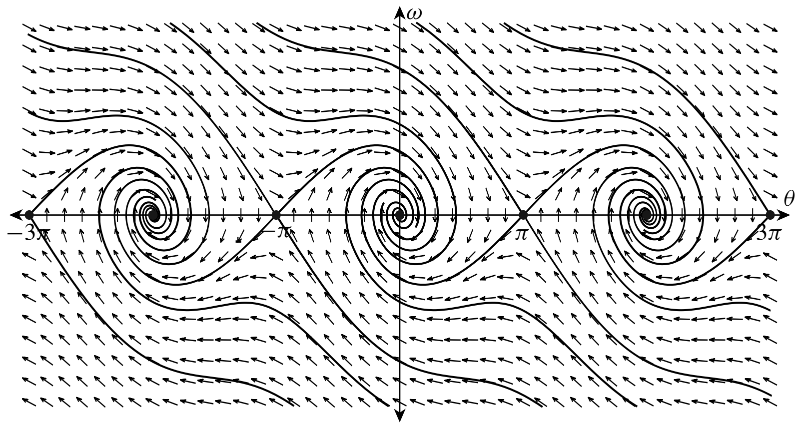


Figure 36.9: A phase portrait (with direction field) for pendulum system (36.10).

for the critical points. From this we get that there are infinitely many critical points, and they are given by

$$(\theta, \omega) = (n\pi, 0) \qquad \text{with} \quad n = 0, \pm 1, \pm 2, \ldots \quad .$$

A direction field with a few trajectories for this system is given in figure 36.9. From it, we can see that the behavior of the trajectories near a critical point $(\theta, \omega) = (n\pi, 0)$ depends strongly on whether n is an even integer or an odd integer.

If n is an even integer, then the nearby trajectories are clearly spirals "spiraling" in towards the critical point $(\theta, \omega) = (n\pi, 0)$. Hence, every critical point $(n\pi, 0)$ with n even is asymptotically stable. That is, if n is an even integer and

$$(\theta(t_0), \omega(t_0)) \approx (n\pi, 0)$$

for some t_0 , then

$$(\theta(t), \omega(t)) \rightarrow (n\pi, 0) \qquad \text{as} \qquad t \rightarrow \infty \quad .$$

This makes sense. After all, if n is even, then $(\theta, \omega) = (n\pi, 0)$ describes a pendulum hanging straight down and not moving — certainly what most of us would call a 'stable equilibrium' position for the pendulum, and certainly the position we would expect a real-world pendulum (in which there is inevitably some friction slowing the pendulum) to eventually assume.

Now consider any critical point $(\theta, \omega) = (n\pi, 0)$ when n is an odd integer. From figure 36.9 it is apparent that these critical points are unstable. This also makes sense. With n being an odd integer, $(\theta, \omega) = (n\pi, 0)$ describes a stationary pendulum balanced straight up from its pivot point, which is a physically unstable equilibrium. It may be possible to start the pendulum moving in such a manner that it approaches this configuration. But if the initial conditions are not just right, then the motion will be given by a trajectory that approaches and then goes away from that critical point. In particular, the trajectories near this critical point that pass through the horizontal axis (where the angular velocity ω is zero) are describing the pendulum slowing to a stop before reaching the upright position and then falling back down, while the trajectories near this critical point that pass over or below this point describe the pendulum traveling fast enough to reach and continue through the upright position.

From figure 36.9, it is apparent that every trajectory eventually converges to one of the critical points, with most spiraling into one of the stable critical points. This tells us that, while the pendulum may initially have enough energy to spin in complete circles about the pivot point, it eventually stops spinning about the pivot and begins rocking back and forth in smaller and smaller arcs about its stable downward vertical position. Eventually, the arcs are so small that, for all practical purposes, the pendulum is motionless and hanging straight down.

By the way, it's fairly easy to redo the above using fairly arbitrary positive choices of γ and κ in pendulum system (36.9). As long as the friction is not too strong (i.e., as long as κ is not too large compared to γ), the resulting phase portrait will be quite similar to what we just obtained. However, if κ is too large compared to γ (to be precise, if $\kappa \geq 2\sqrt{\gamma}$), then, while the critical points and their stability remain the same as above, the trajectories no longer spiral about the stable critical points but approach them more directly. The interested reader is encouraged to redo the above with a relatively large κ to see for themself.

36.6 Existence and Uniqueness of Trajectories

The existence and uniqueness of solutions to standard systems of differential equations were discussed in the last chapter. It is worth noting that the assumption of a system being regular immediately implies that the component functions all satisfy the conditions required in theorem 35.1. So we automatically have:

Corollary 36.1
Suppose $\mathbf{x}' = \mathbf{F}(t, \mathbf{x})$ is a regular system. Then any initial-value problem involving this system,

$$\mathbf{x}' = \mathbf{F}(t, \mathbf{x}) \qquad \text{with} \quad \mathbf{x}(t_0) = \mathbf{a} \quad ,$$

has exactly one solution on some open interval (α, β) containing t_0 . Moreover, the component functions of this solution and their first derivatives are continuous over that interval.

As almost immediate corollaries of corollary 36.1 (along with a few observations made in section 36.3), we have the following facts concerning trajectories of any given standard first-order system $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ that is both regular and autonomous:

1. Through each point of the plane there is exactly one complete trajectory.
2. If a trajectory contains a critical point for that system, then that entire trajectory is that single critical point. Conversely, if a trajectory for a regular autonomous system has nonzero length, then that trajectory does not contain a critical point.
3. Any trajectory that is not a critical point is "smooth" in that no trajectory can have a "kink" or "corner". (More precisely, $\mathbf{x}'(t)$ is a continuous vector-valued function for each solution $\mathbf{x}$ to the system $\mathbf{x} = \mathbf{F}(\mathbf{x})$.)

The other properties of trajectories that were claimed earlier all follow from the following theorem and its corollaries.

Theorem 36.2
Assume $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ is an 2×2 standard first-order system that is both regular and autonomous, and let C be any oriented curve of nonzero length such that all the following hold at each point (x, y) in C :

1. *The point (x, y) is not a critical point for the system.*
2. *The curve C has a unique tangent line at (x, y) , and that line is parallel to the vector $\mathbf{F}(\mathbf{x})$.*

3. *The direction of travel of C through (x, y) is in the same direction as given by the vector* $\mathbf{F}(\mathbf{x})$ *.*

Then C (excluding any endpoints) is the trajectory for some solution to the system $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ *.*

This theorem assures us that, in theory, the curves drawn "following" a direction field will be trajectories of our system (in practice, of course, the curves we actually draw will be approximations). Combining this thoerem with the existence and uniqueness results of corollary 36.1 leads to the next two corollaries regarding complete trajectories.

Corollary 36.3
Two different complete trajectories of a regular autonomous system cannot intersect each other.

Corollary 36.4
If a complete trajectory of a regular autonomous system has an endpoint, that endpoint must be a critical point.

We'll discuss the proof of the above theorem in the next section for those who are interested. Verifying the two corollaries will be left as exercises (see exercise 36.12).

36.7 Proving Theorem 36.2

The Assumptions

In all the following, let us assume we have some regular autonomous system of differential equations

$$\mathbf{x}' = \mathbf{F}(\mathbf{x}) \quad ,$$

along with an oriented curve C of nonzero length such that all the following hold at each point $\mathbf{x}^0$ in C:

1. The point $\mathbf{x}^0$ is not a critical point for the system.

2. The curve C has a unique tangent line at $\mathbf{x}^0$, and that line is parallel to the vector $\mathbf{F}(\mathbf{x}^0)$.

3. The direction of travel of C through $\mathbf{x}^0$ is in the same direction as given by the vector $\mathbf{F}(\mathbf{x}^0)$.

Note that the requirement that C has a tangent line at each point in C means that we are excluding any endpoints of this curve.

For convenience and to aid visualization, we will limit ourselves to curves on the XY–plane. Extending the analysis to curves in higher dimensional spaces spaces is easy, and mainly requires only simple changes in notation.

Preliminaries

To verify our theorem, we will need some material concerning "parametric curves" that you should recall from your calculus course.

Norms and Normalizations

The *norm* (or length) of a column vector or vector-valued function

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

is

$$\|\mathbf{v}\| = \sqrt{[v_1]^2 + [v_2]^2} \quad .$$

If $\mathbf{v}$ is a nonzero vector, then we can *normalize* it by dividing it by its norm, obtaining a vector

$$\mathbf{n} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{[v_1]^2 + [v_2]^2}} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

of unit length (i.e., $\|\mathbf{n}\| = 1$) and pointing in the same direction as $\mathbf{v}$.

Oriented Curves and Unit Tangents

If (x, y) is any point on any oriented curve at which the curve has a well-defined tangent line, then this curve has a *unit tangent vector* at (x, y) , denoted by $\mathbf{T}(x, y)$, which is simply a unit vector tangent to the curve at that point, and pointing in the direction of travel along the curve at that point. For our oriented curve, C , that tangent line is parallel to $\mathbf{F}(\mathbf{x})$, and the direction of travel is given by $\mathbf{F}(\mathbf{x})$. So the unit tangent at (x, y) must be the normalization of $\mathbf{F}(\mathbf{x})$. That is

$$\mathbf{T}(x, y) = \frac{\mathbf{F}(\mathbf{x})}{\|\mathbf{F}(\mathbf{x})\|} \qquad \text{for each } (x, y) \text{ in } C \quad . \tag{36.11}$$

Curve Parameterizations

A *parametrization* of an oriented curve C is an ordered pair of functions on some interval

$$(x(t), y(t)) \qquad \text{for} \quad t_S < t < t_E$$

that traces out the curve in the direction of travel along C as t varies from t_S to t_E . Given any such parametrization, we will automatically let

$$\mathbf{x} = \mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}' = \mathbf{x}'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} \quad .$$

If we view our parametrization $(x(t), y(t))$ as giving the position at time t of some object traveling along C , then, provided the functions are suitably differentiable,

$$\mathbf{x}'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$$

is the corresponding "velocity", of the object at time t . This is a vector pointing in the direction of travel of the object at time t , and whose length,

$$\|\mathbf{x}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2} \quad ,$$

is the speed of the object at time t (i.e., as it goes through position $(x(t), y(t))$). Recall that the integral of this speed from $t = t_1$ to $t = t_2$,

$$\int_{t_1}^{t_2} \|\mathbf{x}'(t)\| \, dt \quad , \tag{36.12}$$

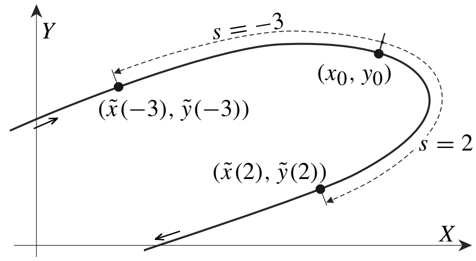


Figure 36.10: Two points given by an arclength parameterization $\tilde{\mathbf{x}}(s)$ of an oriented curve.

gives the signed distance one would travel along the curve in going from position $(x(t_1), y(t_1))$ to position $(x(t_2), y(t_2))$. This value is positive if $t_1 < t_2$ and negative if $t_1 > t_2$. Recall, also, that this distance (the *arclength*) is traditionally denoted by s.

The most fundamental parametrizations are the *arclength parametrizations*. To define one for our oriented curve C, first pick some point (x_0, y_0) on C. Then let s_S and s_E be, respectively, the negative and positive values such that s_E is the "maximal distance" that can be traveled in the positive direction along C from (x_0, y_0), and $|s_S|$ is the "maximal distance" that can be traveled in the negative direction along C from (x_0, y_0). These distances may be infinite.[7] Finally, define the arclength parametrization

$$(\tilde{x}(s), \tilde{y}(s)) \qquad \text{for} \quad s_S < s < s_E$$

as follows (and as indicated in figure 36.10):

1. For $0 \leq s < s_E$ set $(\tilde{x}(s), \tilde{y}(s))$ equal to the point on C arrived at by traveling in the positive direction along C by a distance of s from (x_0, y_0).

2. For $s_S < s \leq 0$ set $(\tilde{x}(s), \tilde{y}(s))$ equal to the point on C arrived at by traveling in the negative direction along C by a distance of $|s|$ from (x_0, y_0).

We should note that if the curve intersects itself, then the same point $(\tilde{x}(s), \tilde{y}(s))$ may be given by more than one value of s. In particular, if C is a closed loop of length L, then $(\tilde{x}(s), \tilde{y}(s))$ will be periodic with $(\tilde{x}(s+L), \tilde{y}(s+L)) = (\tilde{x}(s), \tilde{y}(s))$ for every real value s.

It should also be noted that, from arclength integral (36.12) and the fact that, by definition, s is the signed distance one would travel along the curve to go from $(\tilde{x}(0), \tilde{y}(0))$ to $(\tilde{x}(s), \tilde{y}(s))$, we automatically have

$$\int_0^s \left\| \tilde{\mathbf{x}}'(\sigma) \right\| \, d\sigma \;=\; s \quad .$$

Differentiating this yields

$$\left\| \tilde{\mathbf{x}}'(s) \right\| \;=\; \frac{d}{ds} \int_0^s \left\| \tilde{\mathbf{x}}'(\sigma) \right\| \, d\sigma \;=\; \frac{ds}{ds} \;=\; 1 \quad .$$

Hence, each $\tilde{\mathbf{x}}'(s)$ is a unit vector pointing in the direction of travel on C at $\tilde{\mathbf{x}}(s)$ — that is, $\tilde{\mathbf{x}}'(s)$ is the unit tangent vector for C at $(\tilde{x}(s), \tilde{y}(s))$. Combining this with equation (36.11) yields

$$\tilde{\mathbf{x}}\,'(s) \;=\; \mathbf{T}(\tilde{x}(s), \tilde{y}(s)) \;=\; \frac{\mathbf{F}(\tilde{\mathbf{x}}(s))}{\left\| \mathbf{F}(\tilde{\mathbf{x}}(s)) \right\|} \qquad \text{for} \quad s_S < s < s_E \quad . \tag{36.13}$$

[7] Better definitions for s_S and s_E are discussed in a technical note at the end of this subsection

Technical Note on "Maximal Distances"

We set s_E equal to "the 'maximal distance' that can be traveled in the positive direction along $\mathcal{C}$ from (x_0, y_0)". Technically, this "maximal distance" may not exist because, technically, an endpoint of a trajectory need not actually be part of that trajectory.

To be more precise, let us define a subset $\mathcal{S}$ of the positive real numbers by specifying that

$$s \quad \text{is in} \quad \mathcal{S}$$

if and only

> there is a point on $\mathcal{C}$ arrived at by traveling a distance of s in the positive direction along $\mathcal{C}$ from (x_0, y_0).

With a little thought, it should be clear that $\mathcal{S}$ must be a subinterval of $(0, \infty)$. One end point of $\mathcal{S}$ must clearly be 0. The other endpoint gives us the value s_E. In particular, letting $\mathcal{C}^+$ be that part of $\mathcal{C}$ containing all the points arrived at by traveling in the positive direction along $\mathcal{C}$ from (x_0, y_0):

1. If $\mathcal{C}^+$ is a closed loop, then $s_E = \infty$ (because we keep going around the loop as s increases).
2. If $\mathcal{C}^+$ is a curve that does not intersect itself, then s_E is the length of $\mathcal{C}^+$ (which may be infinite).

Obviously, similar comments can be made regarding the definition of s_S.

Finishing the Proof of Theorem 36.2

Let us now use the arclength parameterization $(\tilde{x}, \tilde{y})$ to define a function $\tilde{t}$ of s by

$$\tilde{t}(s) = \int_0^s \frac{1}{\|\mathbf{F}(\tilde{\mathbf{x}}(\sigma))\|} \, d\sigma \qquad \text{for} \quad s_S < s < s_E \quad .$$

Since $\mathcal{C}$ contains no critical points, the integrand is always finite and positive, and the above function is a differentiable steadily increasing function with

$$\tilde{t}\,'(s) = \frac{1}{\|\mathbf{F}(\tilde{\mathbf{x}}(s))\|} \qquad \text{for} \quad s_S < s < s_E \quad .$$

Consequently, for each s in (s_E, s_E), there is exactly one corresponding t with $t = \tilde{t}(s)$. Thus, we can invert this relationship, defining a function $\tilde{s}$ by

$$\tilde{s}(t) = s \quad \Longleftrightarrow \quad t = \tilde{t}(s) \quad .$$

The function $\tilde{s}$ is defined on the interval (t_S, t_E) where

$$t_S = \lim_{s \to s_S{}^+} \tilde{t}(s) \qquad \text{and} \qquad t_E = \lim_{s \to s_E{}^-} \tilde{t}(s) \quad .$$

By definition,

$$s = \tilde{s}\left(\tilde{t}(s)\right) \qquad \text{for} \quad s_S < s < s_E \quad .$$

From this, the chain rule and the above formula for $\tilde{t}\,'$, we get

$$1 = \frac{ds}{ds} = \frac{d}{ds}\left[\tilde{s}\left(\tilde{t}(s)\right)\right] = \tilde{s}\,'\left(\tilde{t}(s)\right)\tilde{t}\,'(s) = \tilde{s}\,'\left(\tilde{t}(s)\right) \cdot \frac{1}{\|\mathbf{F}(\tilde{\mathbf{x}}(s))\|} \quad .$$

Hence,

$$\tilde{s}'\left(\tilde{t}(s)\right) = \left\|\mathbf{F}(\tilde{\mathbf{x}}(s))\right\| \quad . \tag{36.14}$$

Now let

$$(x(t), y(t)) = (\tilde{x}(\tilde{s}(t)), \tilde{y}(\tilde{s}(t))) \qquad \text{for} \quad t_S < t < t_E \quad .$$

Observe that $(x(t), y(t))$ will trace out C as t varies from t_S to t_E, and that

$$\mathbf{F}(\mathbf{x}(t)) = \mathbf{F}\left(\tilde{\mathbf{x}}(\tilde{s}(t))\right)$$

where

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \tilde{x}(\tilde{s}(t)) \\ \tilde{y}(\tilde{s}(t)) \end{bmatrix} = \tilde{\mathbf{x}}(\tilde{s}(t)) \quad .$$

The differentiation of this (using the chain rule applied to the components), along with equations (36.13) and (36.14), then yields

$$\mathbf{x}'(t) = \frac{d}{dt}\left[\tilde{\mathbf{x}}(\tilde{s}(t))\right] = \tilde{\mathbf{x}}'(\tilde{s}(t)) \cdot \tilde{s}'(t) = \frac{\mathbf{F}(\mathbf{x}(t))}{\|\mathbf{F}(\mathbf{x}(t))\|} \cdot \|\mathbf{F}(\mathbf{x}(t)\| = \mathbf{F}(\mathbf{x}(t)) \quad ,$$

completing our proof of theorem 36.2. ▮

Additional Exercises

36.1. *Find all the constant/equilibrium solutions to each of the following systems:*

a. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2x - 5y \\ 3x - 7y \end{bmatrix}$

b. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2x - 5y + 4 \\ 3x - 7y + 5 \end{bmatrix}$

c. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3x + y \\ 6x + 2y \end{bmatrix}$

d. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} xy - 6y \\ x - y - 5 \end{bmatrix}$

e. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x^2 - y^2 \\ x^2 - 6x + 8 \end{bmatrix}$

f. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x\sin(y) \\ x^2 - 6x + 9 \end{bmatrix}$

g. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4x - xy \\ x^2y + y^3 - x^2 - y^2 \end{bmatrix}$

h. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x^2 + y^2 + 4 \\ 2x - 6y \end{bmatrix}$

36.2. *Sketch the direction field for the system*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -x + 2y \\ 2x - y \end{bmatrix}$$

a. *on the* 2×2 *grid with* $(x, y) = (0, 0)$, $(2, 0)$, $(0, 2)$ *and* $(2, 2)$.

b. *on the* 3×3 *grid with* $x = -1$, 0 *and* 1, *and with* $y = -1$, 0 *and* 1.

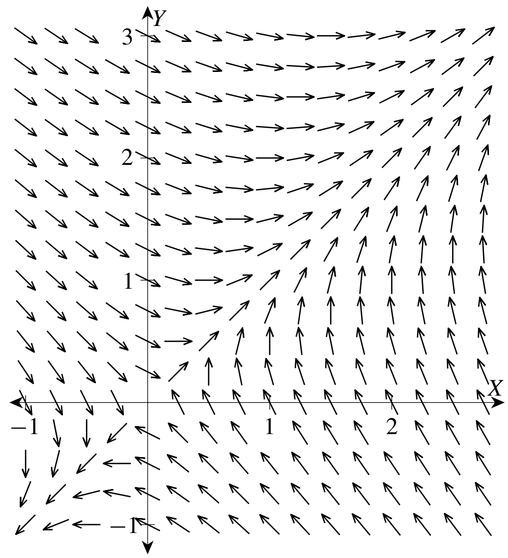
36.3. *Sketch the direction field for the system*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} (1 - 2x)(y + 1) \\ x - y \end{bmatrix}$$

on the 3×3 *grid with* $x = 0$, $1/2$ *and* 1, *and with* $y = 0$, $1/2$ *and* 1.

36.4. *A direction field for*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -x + 2y \\ 2x - y \end{bmatrix}$$



has been sketched to the right. Using this system and direction field:

a. *Find and plot all the critical points.*

b. *Sketch the trajectories that go through points* $(1, 0)$ *and* $(0, 1)$ *.*

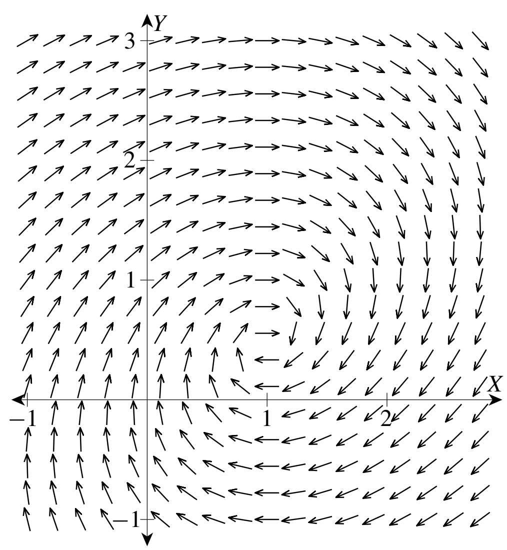
c. *Sketch a phase portrait for this system.*

d. *Suppose* $(x(t), y(t))$ *is the solution to this system satisfying* $(x(0), y(0)) = (1, 0)$ *. What apparently happens to* $x(t)$ *and* $y(t)$ *as* t *gets large?*

e. *As well as you can, decide whether the critical point found above is asymptotically stable, stable but not asymptotically stable, or unstable.*

36.5. *A direction field for*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -x + 2y \\ -2x + 2 \end{bmatrix}$$



has been sketched to the right. Using this system and direction field:

a. *Find and plot all the critical points.*

b. *Sketch the trajectories that go through points* $(-1, 0)$ *and* $(0, 2)$ *.*

c. *Sketch a phase portrait for this system.*

d. *Suppose* $(x(t), y(t))$ *is the solution to this system satisfying* $(x(0), y(0)) = (-1, 0)$ *. What apparently happens to* $x(t)$ *and* $y(t)$ *as* t *gets large?*

e. *As well as you can, decide whether the critical point found in part* **a** *is asymptotically stable, stable but not asymptotically stable, or unstable.*

36.6. *A direction field for*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ -\sin(2x) \end{bmatrix}$$

has been sketched to the right. Using this system and direction field:

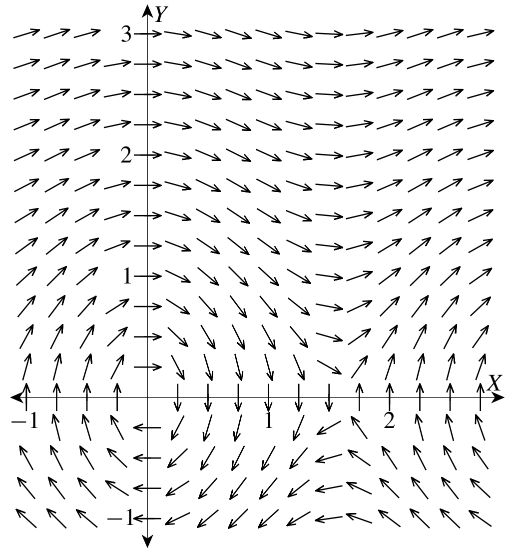
a. *Find and plot all the critical points.*

b. *All the critical points of this system are either stable (but not asymptotically stable) or unstable. Using this direction field, determine which are stable and which are unstable.*

c. *Sketch the trajectories that go through points* $(1, 0)$ *and* $(0, 2)$ *.*

d. *Sketch a phase portrait for this system.*

e. *Suppose* $(x(t), y(t))$ *is the solution to this system satisfying* $(x(0), y(0)) = (1, 0)$ *. What apparently happens to* $x(t)$ *and* $y(t)$ *as* t *gets large?*



36.7. *The system*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x - 2y - 1 \\ 2x - y - 2 \end{bmatrix}$$

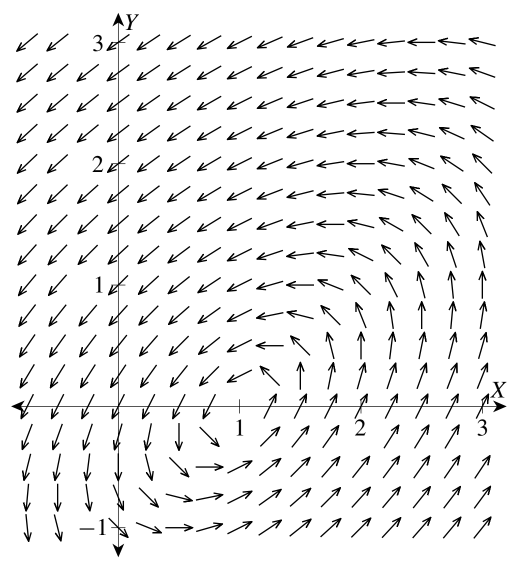
has one critical point and that point is stable, but not asymptotically stable. A direction field for this system has been sketched to the right. Using this information:

a. *Find and plot the critical point.*

b. *Sketch the trajectories that go through points* $(0, 0)$ *and* $(0, 1)$ *.*

c. *Sketch a phase portrait for this system.*

d. *Suppose* $(x(t), y(t))$ *is the solution to this system satisfying* $(x(0), y(0)) = (0, 0)$ *. What apparently happens to* $x(t)$ *and* $y(t)$ *as* t *gets large?*



36.8. *A direction field for*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -2x + y + 1 \\ -x - 4y + 5 \end{bmatrix}$$

has been sketched to the right. Using this system and direction field:

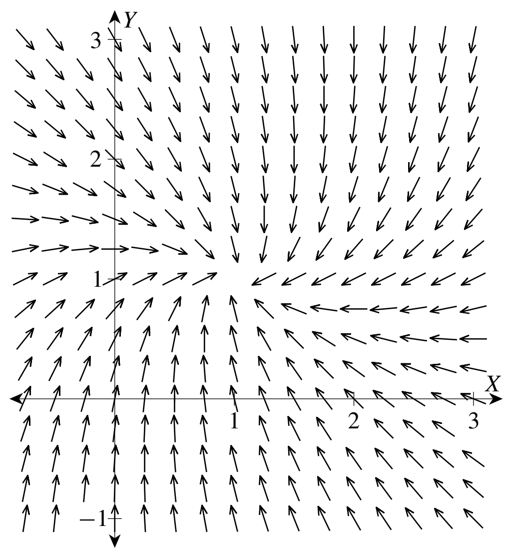
a. *Find and plot the one critical point of this system.*

b. *Decide whether this critical point is asymptotically stable, stable but not asymptotically stable, or unstable.*

c. *Sketch the trajectories that go through points* $(2, 0)$ *and* $(0, 2)$.

d. *Sketch a phase portrait for this system.*

e. *Suppose* $(x(t), y(t))$ *is the solution to this system satisfying* $(x(0), y(0)) = (1, 0)$. *What apparently happens to* $x(t)$ *and* $y(t)$ *as* t *gets large?*



36.9. *A direction field for*

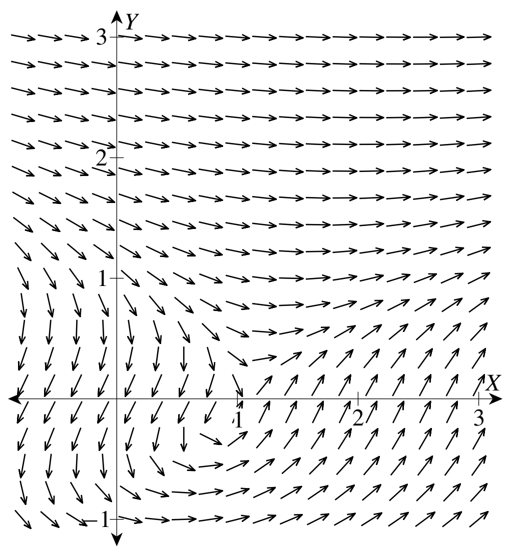
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + 4y^2 - 1 \\ 2x - y - 2 \end{bmatrix}$$

has been sketched to the right. Using this system and direction field:

a. *Find and plot all the critical points.*

b. *Sketch the trajectories that go through the points* $(0, 0)$ *and* $(1, 1)$.

c. *Sketch a phase portrait for this system.*



36.10. *Look up the commands for generating direction fields for systems of differential equations in your favorite computer math package. Then, use these commands to do the following for each problem below:*

i. *Sketch the indicated direction field for the given system.*

ii. *Use the resulting direction field to sketch (by hand) a phase portrait for the system.*

a. *The system is*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -x + 2y \\ 2x - y \end{bmatrix} \quad .$$

Use a 25×25 *grid on the region* $-1 \leq x \leq 3$ *and* $-1 \leq y \leq 3$. *(Compare the resulting direction field to the direction arrows computed in exercise 36.2.)*

b. *The system is*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} (2x-1)(y+1) \\ y-x \end{bmatrix} \quad .$$

Use a 25×25 *grid on the region* $-1 \le x \le 3$ *and* $-1 \le y \le 3$ *. (Compare the resulting direction field to the direction field found in exercise 36.3.)*

c. *The system is*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + 4y^2 - 1 \\ 2x - y - 2 \end{bmatrix} \quad .$$

Use a 25×25 *grid on the region* $0 \le x \le 2$ *and* $-1 \le y \le 1$ *. (This gives a 'close up' view of the critical points of the system in exercise 36.9.)*

d. *The system is*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + 4y^2 - 1 \\ 2x - y - 2 \end{bmatrix} \quad .$$

Use a 25×25 *grid on the region* $3/4 \le x \le 5/4$ *and* $-1/4 \le y \le 1/4$ *. (This gives an even 'closer up' view of the critical points of the system in exercise 36.9.)*

36.11. *Consider the initial-value problem*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} \qquad \text{with} \qquad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \quad .$$

Assume the system is regular and autonomous, and that $(\tilde{x}, \tilde{y})$ *is a solution to the above on an interval* (α, β) *containing* 0 *. Now let* t_0 *be any other point on the real line, and show that a solution to*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} \qquad \text{with} \qquad \begin{bmatrix} x(t_0) \\ y(t_0) \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \quad .$$

is then given by

$$(x(t), y(t)) = (\tilde{x}(t - t_0), \tilde{y}(t - t_0)) \qquad \text{for} \quad \alpha + t_0 < t < \beta + t_0 \quad .$$

(Hint: Showing that the first-order system is satisfied is a simple chain rule problem.)

36.12. *Using corollary 36.1 on page 36–24 on the existence and uniqueness of solutions to regular systems and theorem 36.2 on page 36–24 on curves being trajectories, along with (possibly) the results of exercise 36.11, verify the following:*

a. *Corollary 36.3 on page 36–25. (Hint: Start by assuming the two trajectories do intersect.)*

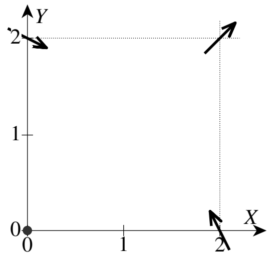
b. *Corollary 36.4 on page 36–25. (Hint: Start by assuming an endpoint of a given maximal trajectory is not a critical point.)*

Some Answers to Some of the Exercises

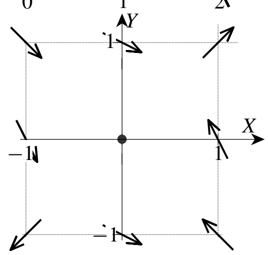
WARNING! Most of the following answers were prepared hastily and late at night. They have not been properly proofread! Errors are likely!

1a. $(0, 0)$
1b. $(3, 2)$
1c. Every (x_0, y_0) with $y_0 = -3x_0$
1d. $(6, 1)$ and $(5, 0)$
1e. $(2, 2)$, $(2, -2)$, $(4, 4)$ and $(4, -4)$
1f. $(3, n\pi)$ for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$
1g. $(0, 0)$ and $(0, 1)$
1h. No constant solutions.

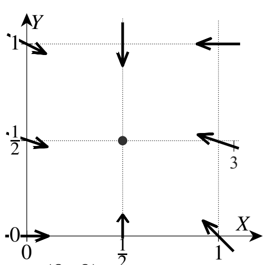
2a.



2b.

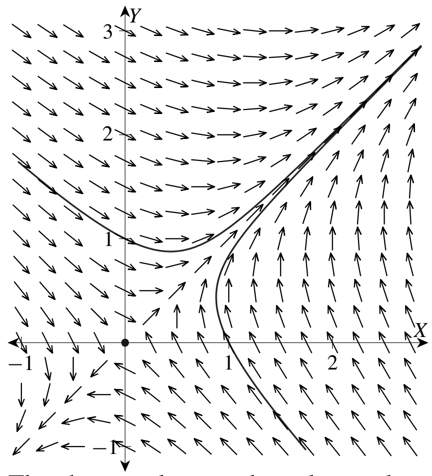


3.

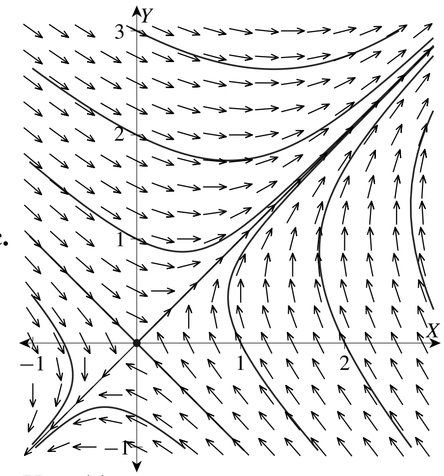


4. a. $(0, 0)$

4b.



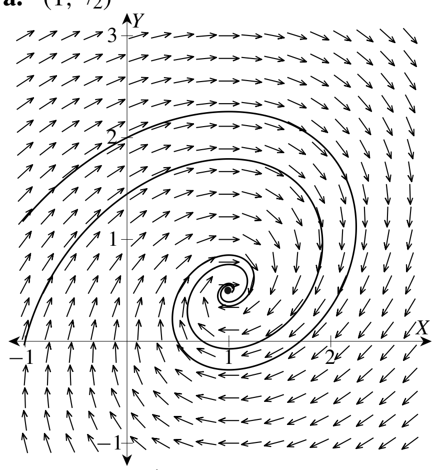
4c.



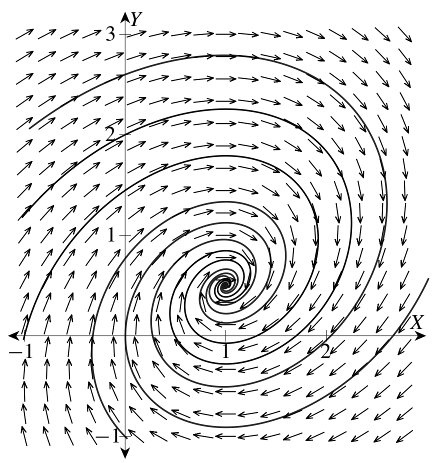
4d. They become large and nearly equal. **4e.** Unstable

5. a. $(1, {}^{1}\!/_{2})$

5b.



5c.

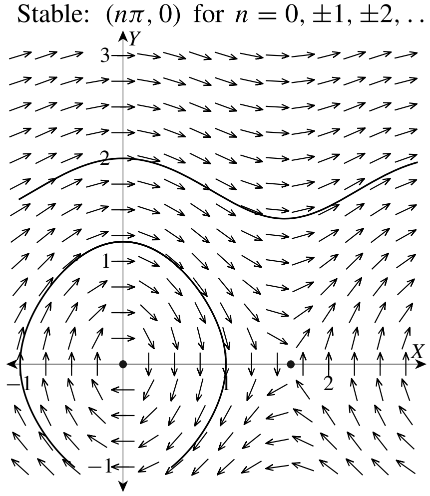


5d. $(x, y) \to (1, {}^{1}\!/_{2})$ **5e.** Asymptotically stable

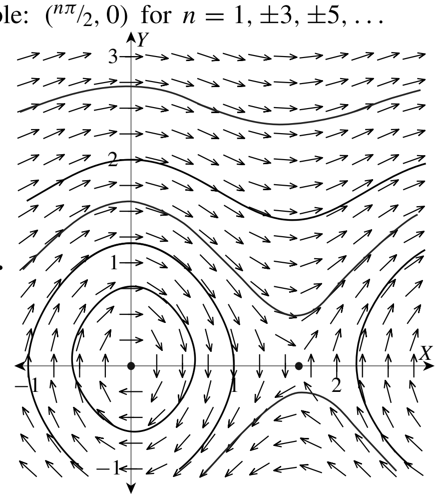
6. a. $({}^{n\pi}\!/_{2}, 0)$ for $n = 0, \pm 1, \pm 2, \ldots$

6b. Stable: $(n\pi, 0)$ for $n = 0, \pm 1, \pm 2, \ldots$; unstable: $({}^{n\pi}\!/_{2}, 0)$ for $n = 1, \pm 3, \pm 5, \ldots$

6c.



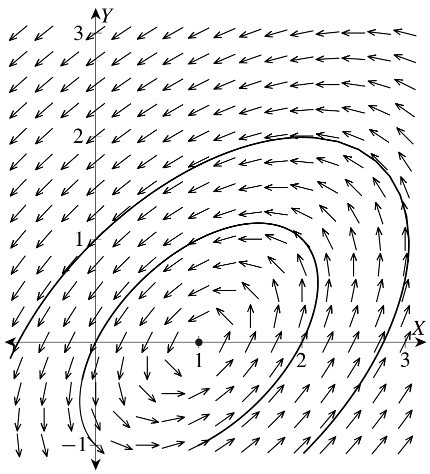
6d.



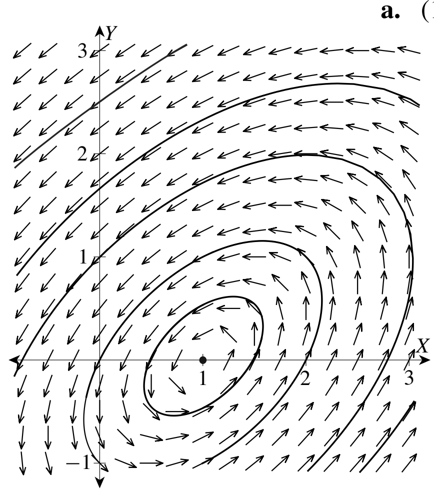
6e. (x, y) "orbits" about (0, 0) clockwise.

7. **a.** (1, 0)

7b.



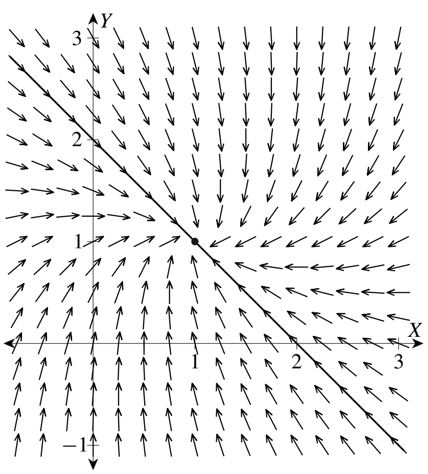
7c.



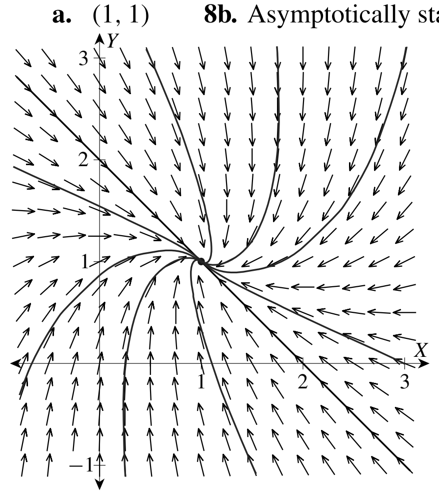
7d. (x, y) “orbits” about (1, 0) counterclockwise.

8. **a.** (1, 1) **8b.** Asymptotically stable

8c.



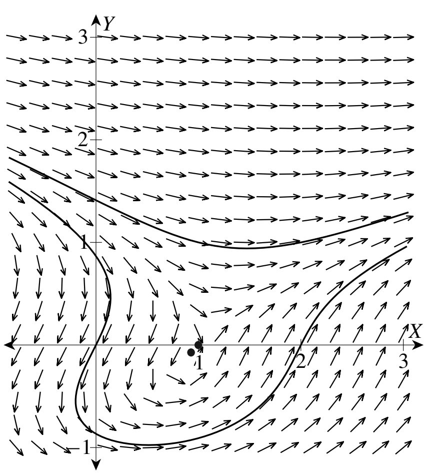
8d.



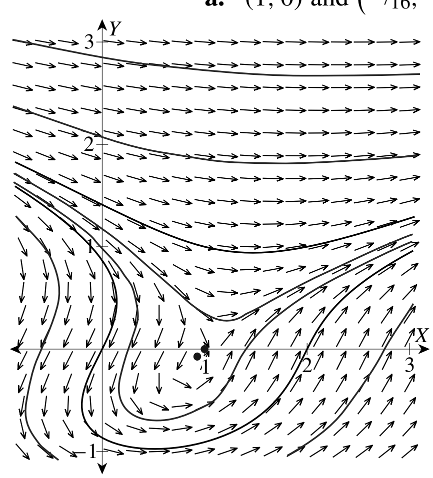
8e. $(x, y) \to (1, 1)$

9. **a.** (1, 0) and $\left({}^{15}\!/_{16}, -{}^{1}\!/_{8}\right)$

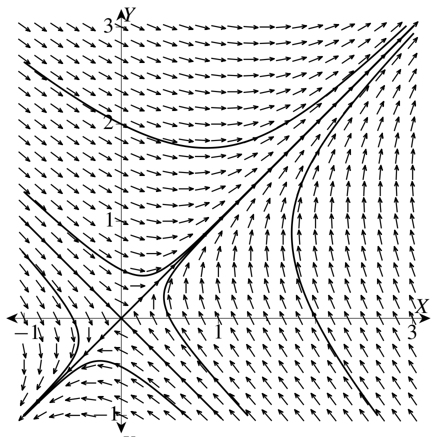
9b.



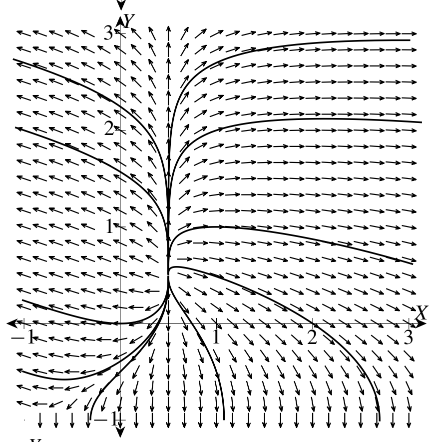
9b.



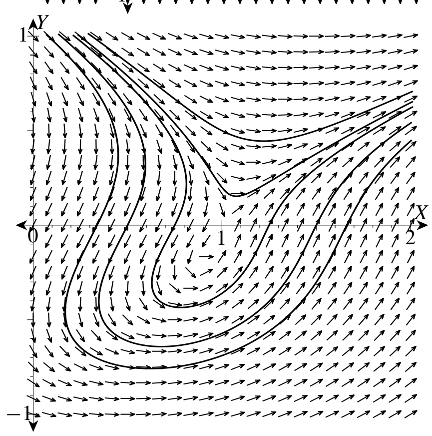
10a.



10b.



10c.



10d.