Addendum to Chapter 35: Higher-Order Systems

In the published version of chapter 35 (*Systems of Differential Equations: A Starting Point*), we defined what was meant by a "k^{th}-order $M \times N$ system", but pretty well limited our examples to standard first-order systems (in which $k = 1$ and $M = N$, and with N usually just being 2). There were reasons for this concentration on standard first-order systems. For one thing, as we will see, most other systems of interest can be converted to standard first-order systems. Still, those other systems do arise in applications, and deserve some discussion.

35.7 Higher-Order Systems

Here is a simple example:

!►Example 35.10: *Letting x, y and z be three unknown functions of t, the two differential equations*

$$x'' + 3x' - y' + \sin(t)\,[y - x] = z$$

and

$$y'' - t^2 z' + xy = 0$$

make up a second-order system of two differential equations with three unknown functions; that is, a second-order, 2×3 system.

In practice, the number of equations is often equal to the number of unknown functions (unlike what we had in the above example). This is illustrated in the application that follows.

Application: A Double Mass-Spring System

Consider the spring system in figure 35.4 with the assumption that there are no frictional forces. Here the "natural length" of each spring — L_1 and L_2, respectively — takes into account the horizontal dimension of the object; that is, if the springs are neither compressed or stretched, then

$$x_1 = L_1 \qquad \text{and} \qquad x_2 - x_1 = L_2 \quad .$$

(If it helps, pretend that each mass is a 'point mass'.)

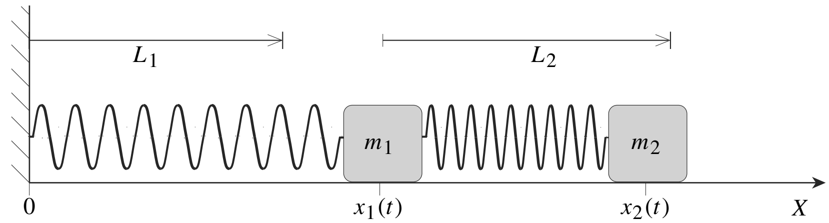

Figure 35.4: A double mass/spring system with objects of mass m_1 and m_2 located at positions $x_1(t)$ and $x_2(t)$, respectively. The first spring, which has a natural length of L_1 and spring constant κ_1, connects the object of mass m_1 to the wall. The second spring, which has a natural length of L_2 and spring constant κ_2, connects the two objects together. In this snapshot, the first spring is stretched, and the second is compressed.

Now remember, if we have a horizontal spring with spring constant κ and natural length L, then the force exerted by the spring on an object attached to its right end is

$$F_{\text{right}} = -\kappa \times \text{the "stretch" in the spring} = -\kappa \times [\text{current length of the spring} - L] \quad .$$

(The negative sign tells us that the force of the spring is in the negative direction if the spring is stretched beyond its natural length, and is positive if the spring is compressed to a length less than its normal length.)

Changing the sign then gives the corresponding force exerted by the spring at the left end,

$$F_{\text{left}} = \kappa \times \text{"stretch" in the spring} = \kappa \times [\text{current length of the spring} - L] \quad .$$

Then applying $F = ma$ to the first object and noting how the "current length" of each spring is computed from $x_1(t)$ and $x_2(t)$, we get

$$\begin{aligned} m_1 \frac{d^2 x_1}{dt^2} &= \text{force of spring 1 on object 1} + \text{force of spring 2 on object 1} \\ &= F_{1,\text{right}} + F_{2,\text{left}} \\ &= -\kappa_1 [x_1 - L_1] + \kappa_2 [(x_2 - x_1) - L_2] \\ &= -[\kappa_1 + \kappa_2] x_1 + \kappa_2 x_2 + [\kappa_1 L_1 - \kappa_2 L_2] \quad . \end{aligned}$$

Since the second object is only attached to the second spring,

$$\begin{aligned} m_2 \frac{d^2 x_2}{dt^2} &= \text{force of spring 2 on object 2} \\ &= F_{2,\text{right}} \\ &= \kappa_2 [L - (x_2 - x_1)] \\ &= \kappa_2 x_1 - \kappa_2 x_2 + \kappa_2 L_2 \quad . \end{aligned}$$

So the motion of the objects in this physical system is described by the solutions to the system

$$\begin{aligned} m_1 x_1'' &= -(\kappa_1 + \kappa_2) x_1 + \kappa_2 x_2 + (\kappa_1 L_1 - \kappa_2 L_2) \\ m_2 x_2'' &= \kappa_2 x_1 - \kappa_2 x_2 + \kappa_2 L_2 \end{aligned} \quad . \tag{35.14}$$

Equivalently,

$$\begin{aligned} x_1'' &= a_{11} x_1 + a_{12} x_2 + b_1 \\ x_2'' &= a_{21} x_1 + a_{22} x_2 + b_2 \end{aligned}$$

where

$$a_{11} = -\frac{\kappa_1 + \kappa_2}{m_1} \quad , \quad a_{12} = \frac{\kappa_2}{m_1} \quad , \quad b_1 = \frac{\kappa_1 L_1 - \kappa_2 L_2}{m_1} \quad ,$$

$$a_{21} = \frac{\kappa_2}{m_2} \quad , \quad a_{22} = -\frac{\kappa_2}{m_2} \quad \text{and} \quad b_2 = \frac{\kappa_2 L_2}{m_2} \quad .$$

In particular, suppose the first spring has a natural length of $L_1 = 1$ meter and spring constant of $\kappa_1 = 1$ kg./sec.^{2}, and is attached to an object of mass $m_1 = 1$ kg., while the second spring is shorter and stiffer with natural length $L_2 = 0.2$ meter and spring constant $\kappa_2 = 2.5$ kg./sec.^{2} and is attached on the right to an object of mass $m_2 = 0.1$ kg.. Then (in units of sec.$^{-2}$)

$$a_{11} = -\frac{1+2.5}{1} = -\frac{7}{2} \quad , \quad a_{12} = \frac{2.5}{1} = \frac{5}{2} \quad ,$$

$$a_{21} = \frac{2.5}{0.1} = 25 \quad \text{and} \quad a_{22} = -\frac{2.5}{0.1} = -25 \quad ,$$

and (in units of meters·sec.$^{-2}$)

$$b_1 = \frac{1 \cdot 1 - 2.5 \cdot 0.2}{1} = \frac{1}{2} \quad \text{and} \quad b_2 = \frac{2.5 \cdot 0.2}{0.1} = 5 \quad ,$$

and the above system governing the positions of the two objects as functions of time, $x_1(t)$ and $x_2(t)$, is

$$\begin{aligned} x_1'' &= -\tfrac{7}{2}x_1 + \tfrac{5}{2}x_2 + \tfrac{1}{2} \\ x_2'' &= 25x_1 - 25x_2 + 5 \end{aligned} \quad .$$

Converting to First-Order Systems

In section 35.3 of the published text, we saw a way to convert a single differential equation of order two or greater to a corresponding standard first-order system of differential equations. With very minor modifications, this approach can be used with higher-order systems. The biggest difficulty is simply keeping track of the unknown functions.

!▶Example 35.11: *Let us consider the system*

$$\begin{aligned} x_1'' &= -\tfrac{7}{2}x_1 + \tfrac{5}{2}x_2 + \tfrac{1}{2} \\ x_2'' &= 25x_1 - 25x_2 + 5 \end{aligned} \quad . \tag{35.15}$$

from our discussion of a double mass-spring system. Let

$$x_3 = x_1' \quad \text{and} \quad x_4 = x_2' \quad .$$

Then

$$x_3' = x_1'' = -\tfrac{7}{2}x_1 + \tfrac{5}{2}x_2 + \tfrac{1}{2}$$

and

$$x_4' = x_2'' = 25x_1 - 25x_2 + 5 \quad ,$$

allowing us to rewrite our second-order system of two equations as the first-order system

$$\begin{aligned} x_1' &= x_3 \\ x_3' &= -\tfrac{7}{2}x_1 + \tfrac{5}{2}x_2 + \tfrac{1}{2} \\ x_2' &= x_4 \\ x_4' &= 25x_1 - 25x_2 + 5 \end{aligned} \quad .$$

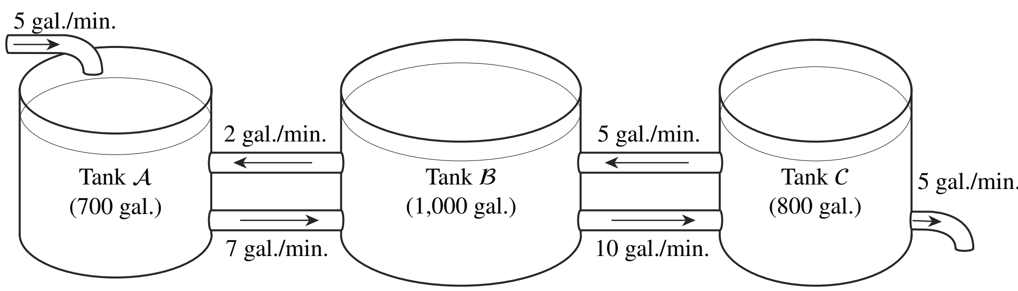

Figure 35.5: The system of three tanks containing water/alcohol mixtures for exercise 35.13. In this scenario, each tank contains a mixture of water and alcohol, and each minute five gallons of mix is added from the upper spigot, with 40 % of that added mix being alcohol.

Additional Exercises

Note: Some of the following exercises concern higher-order systems of differential equations and refer to the material in this addendum, while others are exercises that could have been in the published version of chapter 35, but were excluded to save space.

35.13. *Consider the tank system illustrated in figure 35.5. Let* x , y *and* z *be, respectively, the amount of alcohol in tanks* $\mathcal{A}$, $\mathcal{B}$ *and* $\mathcal{C}$ *at time* t *(measured in minutes), and find the first-order system of three differential equations describing how* x , y *and* z *varies over time.*

35.14. *Consider the mass/spring system illustrated in figure 35.6. Assume there are no frictional forces, and let* κ_j *and* L_j *be, respectively, the spring constant and natural length for the* j^{th} *spring (for* $j = 1, 2,$ *and* 3*).*

a. *Derive the second-order system of two differential equations describing how* x_1 *and* x_2 *vary in time. (As in the derivation of system (35.14) on page 35–2, assume the widths of the two objects are both zero.)*

b. *What, in particular, is the system just derived when* $W = 3$ *meters* ,

$$m_1 = m_2 = \frac{1}{2}\,(\textit{kilogram}) \quad ,$$

$$L_1 = L_3 = 1\,(\textit{meter}) \quad , \quad L_2 = \frac{1}{5}\,(\textit{meter}) \quad ,$$

$$\kappa_1 = \kappa_3 = 1\left(\frac{\textit{kilogram}}{\textit{second}^2}\right) \quad \textit{and} \quad \kappa_2 = \frac{5}{2}\left(\frac{\textit{kilogram}}{\textit{second}^2}\right) \quad ?$$

35.15. *Rewrite the following differential equations as systems of first order equations:*

a. $4t^2 y'' + y = 0$ **b.** $y^{(4)} + y^4 = 0$

35.16. *Rewrite each of the following second-order systems as first-order systems:*

a.
$$\begin{aligned} x' - 7y' &= tx^2 \\ y'' + 4y &= 3x \end{aligned}$$

b.
$$\begin{aligned} x_1'' + 2x_2 x_1' + 3x_1 x_2' &= 0 \\ x_2'' - 4x_2' + 8x_2 &= (x_1)^2 \end{aligned}$$

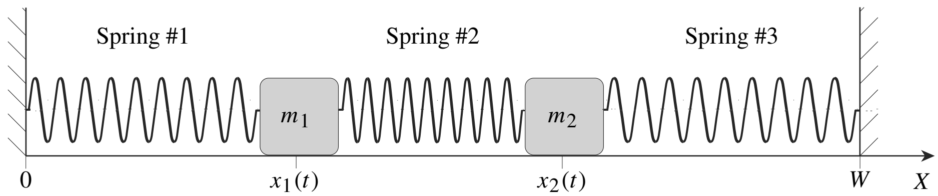

Figure 35.6: The mass/spring system for exercise 35.14 consisting of two objects with masses m_1 and m_2 located at positions $x_1(t)$ and $x_2(t)$, respectively, and attached to each other and to walls at $x = 0$ and $x = W$ by three springs as indicated.

35.17 a. *In section 35.3 of the published text, we saw that we can convert any second-order differential equation of the form*

$$ay'' + by' + cy = 0$$

to a first order system after introducing a new function x related to y by $x = y'$. While this is the "standard" approach, it is not the only approach. In particular, convert each of the following second-order Euler equations to a first-order system by introducing an a new function x related to y by $x = ty'$. (Also, compare the resulting systems to those obtained for the same equations in exercise 35.9 of the published text and exercise 35.15, above.)

i. $t^2y'' - 5ty' + 8y = 0$ **ii.** $t^2y'' - ty' + 10y = 0$

iii. $4t^2y'' + y = 0$

b. *Show that, by introducing a new function x related to y by $x = ty'$, any second-order Euler equation*

$$\alpha t^2y'' + \beta t^2y' + \gamma y = 0$$

can be converted to the first-order system

$$\begin{aligned} tx' &= \left[1 - \frac{\beta}{\alpha}\right]x - \frac{\gamma}{\alpha}y \\ ty' &= x \end{aligned} \quad .$$

35.18 a. *Convert each of the following third-order Euler equations to a first-order system by introducing new functions x and z satisfying $x = ty'$ and $z = tx'$:*

i. $t^3y''' + 2t^2y'' - 4ty' + 4y = 0$

ii. $t^3y''' + 4t^2y'' + 2ty' - 3y = 0$

b. *Show that, by introducing new functions x and z satisfying $x = ty'$ and $z = tx'$, any third-order Euler equation*

$$\alpha t^3y''' + \beta t^2y'' + \gamma ty' + \omega y = 0$$

can be converted to the first-order system

$$\begin{aligned} tx' &= z \\ ty' &= x \\ tz' &= \left(\frac{\beta - \gamma}{\alpha} - 2\right)x - \frac{\omega}{\alpha}y + \left(3 - \frac{\beta}{\alpha}\right)z \end{aligned} \quad .$$

Some Answers to Some of the Exercises

WARNING! Most of the following answers were prepared hastily and late at night. They have not been properly proofread! Errors are likely!

13.
$$x' = 2 - \frac{1}{100}x + \frac{1}{500}y$$
$$y' = \frac{1}{100}x - \frac{3}{250}y + \frac{1}{160}z$$
$$z' = \frac{1}{100}y - \frac{1}{80}z$$

14a.
$$m_1 x_1'' = -(\kappa_1 + \kappa_2)x_1 + \kappa_2 x_2 + (\kappa_1 L_1 - \kappa_2 L_2)$$
$$m_2 x_2'' = \kappa_2 x_1 - (\kappa_2 + \kappa_3)x_2 + \kappa_2 L_2 + \kappa_3(W - L_3)$$

14b.
$$x_1'' = -7x_1 + 5x_2 + 1$$
$$x_2'' = 5x_1 - 7x_2 + 5$$

15a.
$$x' = -\frac{1}{4t^2}y$$
$$y' = x$$

15b.
$$y_1' = y_2 \qquad (\text{with } y_1 = y)$$
$$y_2' = y_3$$
$$y_3' = y_4$$
$$y_4' = -(y_1)^4$$

16a.
$$x' = 7z + tx^2$$
$$y' = z$$
$$z' = -4y + 3x$$

16b.
$$x_1' = x_3$$
$$x_2' = x_4$$
$$x_3' = -2x_2x_3 - 3x_1x_4$$
$$x_4' = 4x_4 - 8x_2 + (x_1)^2$$

17a i.
$$tx' = 6x - 8y$$
$$ty' = x$$

17a ii.
$$tx' = 2x - 10y$$
$$ty' = x$$

17a iii.
$$tx' = x - \frac{1}{4}y$$
$$ty' = x$$

18a i.
$$tx' = z$$
$$ty' = x$$
$$tz' = 4x - 4y + z$$

18a ii.
$$
\begin{aligned}
tx' &= z \\
ty' &= x \\
tz' &= 3y - z
\end{aligned}
$$