44

The Liapunov Method for Determining Stability (DRAFT)

44.1 The Liapunov Method, Naively Developed

In the last chapter, we discussed describing trajectories of a 2×2 autonomous system $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ as level curves for some function $\phi(x, y)$ over some open region $\mathcal{R}$ of the XY–plane. This was especially useful when those level curves where clearly closed loops about a critical point in $\mathcal{R}$ for our system of differential equations.

That approach can be expanded to a more general method — the "Liapunov method" — for determining the stability of critical points of systems of differential equations. In this section, we will develop the basic ideas behind the method in a relatively naive manner using pictures and ignoring some technical details in our definitions and results. Those details will be added in the next section.

Trajectories and Regions

We will be discussing whether given trajectories on the XY–plane "enter" or "leave" particular regions of the plane.[1] So we had better state, as precisely as practical, just what these terms mean. Oddly enough, we'll base our definitions on what is usually meant by these terms in everyday language.

A *region* $\mathcal{R}$ is simply a set of points in the plane bounded by some curve or collection of curves $\mathcal{C}$ (the *boundary* of $\mathcal{R}$). The bounding curve(s) will be assumed to be "reasonably smooth", and the region $\mathcal{R}$ will usually be assumed to be "open", that is, no point in the boundary $\mathcal{C}$ will be considered a point in $\mathcal{R}$. Note that $\mathcal{C}$ divides the XY–plane into three disjoint sets of points: the curve or set of curves $\mathcal{C}$ itself, the open region $\mathcal{R}$, and the open region of all points in neither $\mathcal{C}$ nor $\mathcal{R}$. We will, unimaginatively, call this last region the *(open) region outside of* $\mathcal{R}$.

Now suppose we have some open region $\mathcal{R}$ with boundary $\mathcal{C}$, along with a trajectory given by $(x(t), y(t))$ where $x(t)$ and $y(t)$ are smooth functions of t, and assume this trajectory intersects $\mathcal{C}$ at some point (x_1, y_1). If we say that *this trajectory enters* $\mathcal{R}$ *at* (x_1, y_1) (or anything similar), then we mean there are real numbers t_0, t_1 and t_2 with $t_0 < t_1 < t_2$ such that

$$(x(t_1), y(t_1)) = (x_1, y_1) \quad ,$$

$$(x(t), y(t)) \text{ is in the open region outside } \mathcal{R} \text{ when } t_0 < t < t_1$$

and

$$(x(t), y(t)) \text{ is in } \mathcal{R} \text{ when } t_1 < t < t_2 \quad .$$

[1] We are using the "XY–plane" for convenience, but, of course, the symbols being used, "X" and "Y", are irrelevant and can be replaced by any other reasonable pair of symbols.

And if we say that *this trajectory leaves* $\mathcal{R}$ *at* (x_1, y_1) (or anything similar), then we mean there there are real numbers t_0, t_1 and t_2 with $t_0 < t_1 < t_2$ such that

$$(x(t_1), y(t_1)) = (x_1, y_1) \quad ,$$

$$(x(t), y(t)) \text{ is in } \mathcal{R} \text{ when } t_0 < t < t_1$$

and

$$(x(t), y(t)) \text{ is in the open region outside } \mathcal{R} \text{ when } t_1 < t < t_2 \quad .$$

And if we simply say that a given trajectory "enters" or "leaves" some open region, then we mean that it enters or leaves at some point on the boundary of that region.

Thus, when we say that a trajectory "enters" or "leaves" an open region $\mathcal{R}$ bounded by $\mathcal{C}$, then the trajectory actually crosses the boundary and goes from one region bounded by $\mathcal{C}$ to the other region bounded by $\mathcal{C}$. It does not 'linger' on the boundary, nor does it bend so that it touches the boundary tangentially, without actually going from one region bounded by $\mathcal{C}$ into a different region bounded by $\mathcal{C}$.

By the way, it is possible to define a trajectory so that it retraces part of its path and both enters and leaves a given region at the same point. However, since we are only interested in the trajectories of regular autonomous systems of differential equations (which follow well-defined direction fields), this will not be an issue for us.

"Potential Liapunov Functions", Trajectories and Autonomous Systems

"Potential Liapunov Functions"

For the rest of this section (x_0, y_0) will be some point of interest in the XY–plane, and $\Psi(x, y)$ will be a reasonably smooth function of two variables with the following two properties:

1. The minimum value z_0 of $\Psi(x, y)$ occurs at (x_0, y_0) and only at (x_0, y_0). (Hence $z_0 = \Psi(x_0, y_0) < \Psi(x, y)$ whenever $(x, y) \neq (x_0, y_0)$.)

2. The surface $\mathcal{S}$ given by $z = \Psi(x, y)$ is roughly "bowl shaped" with (x_0, y_0, z_0) being the bottom , and with the level curves of Ψ being closed loops about (x_0, y_0), as indicated in figure 44.1a.

This function, Ψ, is a "potential Liapunov function" (a more complete definition will be given in the next section). Observe that an immediate consequence of the minimum value z_0 of $\Psi(x, y)$ occuring only at (x_0, y_0) is that

$$\Psi(x, y) = z_0 \quad \Longleftrightarrow \quad (x, y) = (x_0, y_0) \quad . \tag{44.1}$$

"Potential Liapunov Functions" and Trajectories

Now suppose we have a reasonably smooth trajectory in the XY–plane given by $(x(t), y(t))$ as t varies. Corresponding to this is the trajectory on the surface $\mathcal{S}$ traced out by $(x(t), y(t), z(t))$ where

$$z(t) = \Psi(x(t), y(t)) \quad ,$$

as illustrated in figure 44.1b. Let t_1 be some value for t, and set

$$(x_1, y_1, z_1) = (x(t_1), y(t_1), z(t_1)) \quad .$$

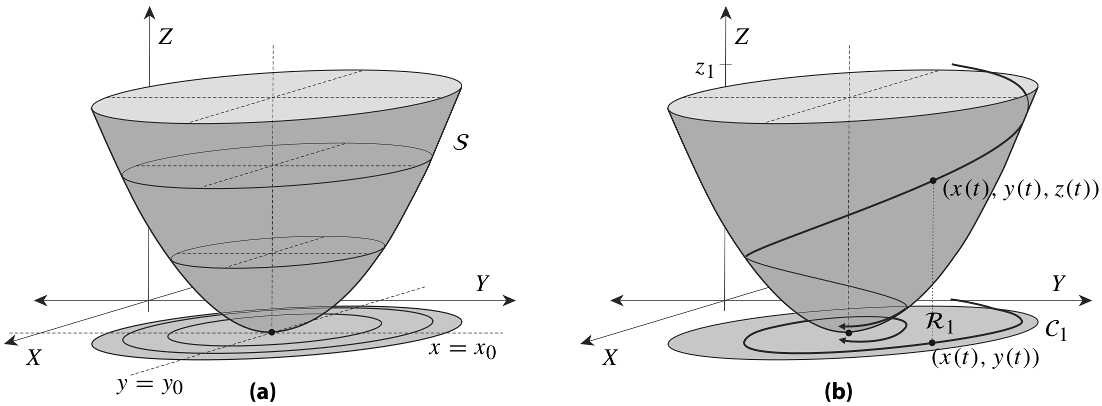

Figure 44.1: **(a)** A portion of the surface $\mathcal{S}$ given by $z = \Psi(x, y)$ where Ψ is a "potential Liapuov function" about (x_0, y_0) (with $z_0 = 0$), and **(b)** a portion of a trajectory (in the XY–plane) and the corresponding trajectory on $\mathcal{S}$.

Note that, since z_0 is the minimum value of Ψ, we must have $z_1 \geq z_0$. For now, let's assume that, in fact, $z_1 > z_0$.

To continue setting things up: Let $\mathcal{C}_1$ be the level curve (in the XY–plane) given by

$$\Psi(x, y) = z_1 \quad ,$$

and let $\mathcal{R}_1$ be the open region in the XY–plane enclosed by $\mathcal{C}_1$. Observe (see figure 44.1b) that we've set things up so that $\mathcal{C}_1$ is a simple loop containing (x_1, y_1) about the point (x_0, y_0), and that $\mathcal{R}_1$ is an open region with nonzero area containing (x_0, y_0).

Now look at what $z'(t)$, the derivative of $z(t)$, tells us when $t = t_1$:

1. If $z'(t_1) < 0$, then $z(t)$ is a decreasing function at $t = t_1$. Thus, as t is increasing past t_1, the point $(x(t), y(t), z(t))$ is moving *downwards* on $\mathcal{S}$ to the part of $\mathcal{S}$ below $z = z_1$. This, in turn, means that, as t increases past t_1, the point $(x(t), y(t))$ on the XY–plane is crossing the level curve $\mathcal{C}_1$ and entering $\mathcal{R}_1$.

2. If $z'(t_1) > 0$, then $z(t)$ is an increasing function at $t = t_1$. Consequently, as t is increasing past t_1, the point $(x(t), y(t), z(t))$ is moving *upwards* on $\mathcal{S}$ to the part of $\mathcal{S}$ above $z = z_1$. Thus, in turn, as t increases past t_1, the point $(x(t), y(t))$ on the XY–plane is crossing the level curve $\mathcal{C}_1$ and leaving $\mathcal{R}_1$.

Very little can be said if all we know about z' is that $z'(t_1) = 0$. The point $(x(t), y(t))$ may then remain on $\mathcal{C}_1$, move into $\mathcal{R}_1$ or move to the region outside of $\mathcal{R}_1$ depending on whether $z'(t)$ is zero, negative or positive for $t > t_1$.

But, of course, if we happen to know that the values of $z'(t)$ are consistently zero, negative or positive for all $t \geq t_1$, then we can say much more:

3. If $z'(t) = 0$ for $t \geq t_1$, then $z(t)$ is constant on $[t_1, \infty)$. Hence

$$\Psi(x(t), y(t)) = z(t) = z(t_1) = z_1 \qquad \text{whenever} \quad t \geq t_1 \quad ,$$

telling us that $(x(t), y(t))$ remains on the level curve $\mathcal{C}_1$ as t continues to increase.

4. If $z'(t) < 0$ whenever $t \geq t_1$, then $z(t) = (x(t), y(t))$ is a decreasing function of t on (t_1, ∞). So, as t increases, the point $(x(t), y(t), z(t))$ moves down the surface $\mathcal{S}$ towards

the bottom where $(x, y) = (x_0, y_0)$. Consequently, on the XY–plane, the point $(x(t), y(t))$ is continually crossing level curves bounding smaller and smaller regions about (x_0, y_0) as t increases (see figure 44.1b).

In fact, it should be clear that,

$$\text{if} \quad \lim_{t\to\infty} z(t) = z_0 \quad \text{then} \quad \lim_{t\to\infty} (x(t), y(t)) = (x_0, y_0) \quad .$$

5. If $z'(t) > 0$ for $t \geq t_1$, then $z(t)$ is an increasing function of t on (t_1, ∞) , and, as $t \to \infty$, the point $(x(t), y(t), z(t))$ moves up the surface $\mathcal{S}$. Consequently, $(x(t), y(t))$ is continually moving outside of larger and larger regions about (x_0, y_0) .

Now consider the special case where z_1 is z_0 , the minimum possible value of $z = \Psi(x, y)$. Since this minimum only occurs at (x_0, y_0) , this is equivalent to having

$$(x(t_1), y(t_1)) = (x_0, y_0) \quad .$$

If we also know that, for some positive value ϵ ,

$$z'(t) \leq 0 \qquad \text{whenever} \quad t_1 \leq t < t_1 + \epsilon \quad ,$$

then we know $z(t)$ either decreases or remains the same as t increases from t_1 to $t_1 + \epsilon$. But since $z(t)$ is already at its possible minimum value when $t = t_1$, it cannot further decrease, and, so, we must have

$$\Psi(x(t), y(t)) = z(t) = z_0 \qquad \text{whenever} \quad t_1 \leq t < t_1 + \epsilon \quad ,$$

which, as noted in implication (44.1), means that

$$(x(t), y(t)) = (x_0, y_0) \qquad \text{whenever} \quad t_1 \leq t < t_1 + \epsilon \quad .$$

Differentiating then gives us

$$(x'(t), y'(t)) = (0, 0) \qquad \text{whenever} \quad t_1 \leq t < t_1 + \epsilon \quad ,$$

a fact that will lead to a significant result in the next section.

By very similar arguments, we can also show that, if $z(t_1) = z_0$ and, for some $\epsilon > 0$,

$$z'(t) \geq 0 \qquad \text{whenever} \quad t_1 - \epsilon < t \leq t_1 \quad ,$$

then

$$(x(t), y(t)) = (x_0, y_0) \quad \text{and} \quad (x'(t), y'(t)) = (0, 0) \qquad \text{whenever} \quad t_1 - \epsilon < t \leq t_1 \quad .$$

In fact, our interest in the next section will only be in the derivatives of $x(t)$ and $y(t)$ at $t = t_1$. So let us summarize what we've just derived as:

6. Suppose $(x(t_1), y(t_2)) = (x_0, y_0)$, and that there is an $\epsilon > 0$ such that either

$$z'(t) \leq 0 \qquad \text{whenever} \quad t_1 \leq t < t_1 + \epsilon$$

or

$$z'(t) \geq 0 \qquad \text{whenever} \quad t_1 - \epsilon < t \leq t_1 \quad .$$

Then $(x'(t_1), y'(t_1)) = (0, 0)$.

"Potential Liapunov Functions" and Autonomous Systems

Our interest is in the case where the components of $(x(t), y(t))$ are the components of a solution to a 2×2 regular autonomous system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= f(x, y) \\ \frac{dy}{dt} &= g(x, y) \end{aligned} \quad . \tag{44.2}$$

In practice, we rarely have explicit solutions to our system, and cannot actually find a formula in terms of t for the above $z(t) = \Psi(x(t), y(t))$. But look at what we can compute using the chain rule for multi-variable functions along with the fact that $x(t)$ and $y(t)$ satisfy the above system:

$$\begin{aligned} z'(t) &= \frac{d}{dt}[\Psi(x(t), y(t))] \\ &= \frac{\partial \Psi}{\partial x}\frac{dx}{dt} + \frac{\partial \Psi}{\partial y}\frac{dy}{dt} \\ &= \Psi_x(x, y) f(x, y) + \Psi_y(x, y) g(x, y) \end{aligned} \tag{44.3}$$

where, in the last line,

$$x = x(t) \qquad \text{and} \qquad y = y(t) \quad .$$

For brevity, let $D\Psi$ be the function on the XY–plane given by the above derivative,

$$D\Psi(x, y) = \Psi_x(x, y) f(x, y) + \Psi_y(x, y) g(x, y) \quad .$$

This allows us to rewrite equation (44.3) as

$$z'(t) = \frac{d}{dt}\Psi(x(t), y(t)) = D\Psi(x, y) \tag{44.4}$$

where

$$x = x(t) \qquad \text{and} \qquad y = y(t) \quad .$$

In other words, $D\Psi(x, y)$ gives the "$z'(t)$" of any trajectory of our system passing through a given point (x, y) at the time t it passes through that point. The important thing is that we only need to know the position, not the time t nor the functions $x(t)$ and $y(t)$.

The relation between $D\Psi(x, y)$ and z' given in equation (44.4) allows us to expand the observations regarding all trajectories with points on that level curve $\mathcal{C}_1$ of Ψ and the region $\mathcal{R}_1$ enclosed, as follows:

1. If $D\Psi(x, y) < 0$ at every point (x, y) on $\mathcal{C}_1$, then every trajectory for system (44.2) that intersects $\mathcal{C}_1$ is entering the enclosed region $\mathcal{R}_1$ at that intersection point. From this, it clearly follows that, if $[x(t), y(t)]^{\mathsf{T}}$ is any solution to system (44.2), and

   $$(x(t_1), y(t_1)) \text{ is in } \mathcal{C}_1 \text{ for some } t_1 \quad ,$$

   then

   $$(x(t), y(t)) \text{ is in } \mathcal{R}_1 \text{ whenever } t > t_1 \quad .$$

2. If $D\Psi(x, y) > 0$ at every point (x, y) on $\mathcal{C}_1$, then every trajectory for system (44.2) that intersects with $\mathcal{C}_1$ is leaving the enclosed region $\mathcal{R}_1$ at that intersection point. Consequently, it also follows that, if $[x(t), y(t)]^{\mathsf{T}}$ is any solution to system (44.2), and

   $$(x(t_1), y(t_2)) \text{ is in } \mathcal{C}_1 \text{ for some } t_1 \quad ,$$

   then

   $$(x(t), y(t)) \text{ is in the open region outside of } \mathcal{R}_1 \text{ whenever } t > t_1 \quad .$$

The relation between $D\Psi(x, y)$ and z' given in equation (44.4) also allows us to expand the other observations made in the previous section regarding all trajectories through points in the region $\mathcal{R}_1$. Let us start by looking at the point (x_0, y_0), assuming that either

$$D\Psi(x, y) \leq 0 \qquad \text{for all } (x, y) \text{ in } \mathcal{R}_1$$

or

$$D\Psi(x, y) \geq 0 \qquad \text{for all } (x, y) \text{ in } \mathcal{R}_1 \quad .$$

More precisely, let's assume either of the above inequalities, and let $[x(t), y(t)]^{\mathsf{T}}$ be a solution to system (44.2) satisfying

$$(x(t_0), y(t_0)) = (x_0, y_0) \qquad \text{for some} \quad t_0 \quad .$$

Then, depending on which of the two inequalities hold, there clearly must be an $\epsilon > 0$ such that either

$$z'(t) = D\Psi(x(t), y(t)) \leq 0 \qquad \text{whenever} \quad t_0 \leq t < t_0 + \epsilon$$

or

$$z'(t) = D\Psi(x(t), y(t)) \geq 0 \qquad \text{whenever} \quad t_0 - \epsilon \leq t < t_0 \quad .$$

Either way, the last observation made in the previous section, immediately tells us that

$$(x'(t), y'(t)) = (0, 0) \qquad \text{for} \quad t = t_0 \quad .$$

And since $[x(t), y(t)]^{\mathsf{T}}$ is a solution to system (44.2), we then must then have

$$\begin{aligned} f(x_0, y_0) &= 0 \\ g(x_0, y_0) &= 0 \end{aligned} \quad . \tag{44.5}$$

In other words, (x_0, y_0) must be a critical point for system (44.2).

Could there be other critical points in $\mathcal{R}_1$? Well, if $(\hat{x}, \hat{y})$ is any critical point for our system, then

$$\begin{aligned} f(\hat{x}, \hat{y}) &= 0 \\ g(\hat{x}, \hat{y}) &= 0 \end{aligned} \quad , \tag{44.6}$$

and, thus,

$$D\Psi(\hat{x}, \hat{y}) = \Psi_x(\hat{x}, \hat{y}) f(\hat{x}, \hat{y}) + \Psi_y(\hat{x}, \hat{y}) g(\hat{x}, \hat{y}) = 0 \quad .$$

So there might be other critical points, but only if there are other points at which $D\Psi$ is zero (however, the vanishing of $D\Psi$ does not guarantee a point being critical).

Keeping in mind the above observations regarding critical points, and how we expanded the observations regarding trajectories through the level curve C_1, let's finish expanding on the observations made in the previous section:

3. If $D\Psi(x, y) = 0$ at every point (x, y) in $\mathcal{R}_1$, then (x_0, y_0) is a critical point for system (44.2), and every trajectory for the system through any point in $\mathcal{R}_1$ must lie on the level curve of Ψ about (x_0, y_0) through that point.

   Moreover, if (x_0, y_0) is the only critical point in $\mathcal{R}_1$, then no trajectory through any other point in $\mathcal{R}_1$ can consist of a single point. With a little thought, this should lead you to strongly suspect that all the other trajectories trace out complete level curves enclosing (x_0, y_0), which would mean that (x_0, y_0) is, in fact, a center for system (44.2).

4. If $D\Psi(x, y) < 0$ at every point (x, y) in $\mathcal{R}_1$ other than (x_0, y_0), then, once a trajectory enters $\mathcal{R}_1$ it continues moving "deeper into" $\mathcal{R}_1$. More precisely, if $[x(t), y(t)]^{\mathsf{T}}$ is a solution to system (44.2), and $(x(\tau), y(\tau))$ is in $\mathcal{R}_1$ for some τ, then $(x(t), y(t))$ is continually crossing level curves bounding smaller and smaller regions about (x_0, y_0) as t increases beyond τ.

   This, then, rather strongly suggests that critical point (x_0, y_0) is at least a stable critical point. It may even be suspected that it must be an asymptotically stable critical point.

5. On the other hand, if $D\Psi(x, y) > 0$ at every point (x, y) in $\mathcal{R}_1$ other than (x_0, y_0), then each trajectory through any point in $\mathcal{R}_1$ (other than (x_0, y_0) is continually crossing level curves bounding larger and larger regions about (x_0, y_0) as t increases τ.

   And this strongly suggests that critical point (x_0, y_0) is an unstable critical point.

44.2 The Liapunov Theory: Rigorous Definitions and Results

Basic Definitions and Results

We will say that Ψ is a *potential Liapunov function on a region* $\mathcal{R}$ *about a point* (x_0, y_0) if and only if the following all hold:

1. $\mathcal{R}$ is an open region of the XY–plane containing the point (x_0, y_0).

2. Ψ is a continuous function of two variables on $\mathcal{R}$ whose first partial derivatives exist and are continuous on $\mathcal{R}$.

3. The minimum value of Ψ on $\mathcal{R}$ occurs at (x_0, y_0) and only at (x_0, y_0). That is, if (x, y) is any point in $\mathcal{R}$ other than (x_0, y_0), then

$$\Psi(x_0, y_0) < \Psi(x, y) \quad .$$

On occasion, we may not need to specify the region $\mathcal{R}$ and may say something like

Ψ is a potential Liapunov function about (x_0, y_0)

when we mean Ψ is a potential Liapunov function on $\mathcal{R}$ about (x_0, y_0) for some region $\mathcal{R}$. Likewise,

Ψ is a potential Liapunov function on $\mathcal{R}$

will mean Ψ is a potential Liapunov function on $\mathcal{R}$ about (x_0, y_0) for some point (x_0, y_0).

The region $\mathcal{R}$ need not be the entire domain of Ψ. In fact, we will often want to restrict the choice of $\mathcal{R}$ so that all the following additional conditions hold:

1. The boundary of $\mathcal{R}$
   (a) consists of a single simple closed loop, and
   (b) is a level curve for Ψ.

2. At no point in $\mathcal{R}$ other than (x_0, y_0) do we have both $\partial\Psi/\partial x$ and $\partial\Psi/\partial y$ being zero.

The first condition simply means that we have no "holes" in our region of interest, while the last ensures that no plane tangent to the surface $\mathcal{S}$ given by

$$z = \Psi(x, y) \qquad \text{for} \quad (x, y) \in \mathcal{R}$$

is horizontal except for the tangent plane at the lowest point of the surface, where $(x, y) = (x_0, y_0)$. Together, these conditions ensure that the surface $\mathcal{S}$ is "bowl shaped" as in the earlier discussions, with the the upper edge of that "bowl-shaped" surface being parallel to the XY–plane.

For want of better terminology, let us say that if these additional conditions are satisfied, then $\mathcal{R}$ is a *basin region* for our potential Liapunov function (about (x_0, y_0)). An important point is that eachpotential Liapunov function must have a corresponding basin region.

Lemma 44.1

Let Ψ be a potential Liapunov function about a point (x_0, y_0). Then there is a basin region $\mathcal{R}$ for Ψ about (x_0, y_0). Moreover, each point in $\mathcal{R}$ other than (x_0, y_0) is on the boundary of smaller basin region for Ψ about (x_0, y_0).

The proof of this lemma will be discussed later.

Now given both a potential Liapunov function Ψ over a region $\mathcal{R}$, and a 2×2 regular autonomous system $\mathbf{x}' = \mathbf{F}(\mathbf{x})$, we will define the corresponding *differential function* $D\Psi$ by

$$D\Psi(x, y) = \Psi_x(x, y) f(x, y) + \Psi_y(x, y) g(x, y) \qquad \text{for each } (x, y) \text{ in } \mathcal{R}$$

where

$$\begin{aligned} \frac{dx}{dt} &= f(x, y) \\ \frac{dy}{dt} &= g(x, y) \end{aligned}$$

is the system $\mathbf{x}' = \mathbf{F}(\mathbf{x})$.

Do remember that, in the previous section, we saw that if $z(t) = \Psi(x(t), y(t))$ for any particular solution $[x(t), y(t)]^{\mathsf{T}}$ to our system, then

$$z'(t) = D\Psi(x(t), y(t)) \quad ,$$

and from the sign of this we could gain insight on the behavior of the trajectory passing through any point in $\mathcal{R}$. In particular, from our discussion there, it should be clear that we now have the following lemma:

Lemma 44.2

Assume Ψ is a potential Liapunov function on an open region $\mathcal{R}$, and $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ is a 2×2 regular autonomous system. Let C_1 be any level curve for Ψ in $\mathcal{R}$ bounding a basin region for Ψ, and let $[x(t), y(t)]^{\mathsf{T}}$ be any solution to $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ such that $(x(t_1), y(t_1))$ is on C_1 for some t_1. Then:

1. *If*

$$D\Psi(x, y) < 0 \qquad \text{for every } (x, y) \text{ in } C_1 \quad ,$$

then $(x(t), y(t))$ is in the open region enclosed by C_1 whenever $t > t_1$.

2. *If*

$$D\Psi(x, y) > 0 \qquad \text{for every } (x, y) \text{ in } C_1 \quad ,$$

then $(x(t), y(t))$ is in the open region outside of C_1 whenever $t > t_1$.

Liapunov Functions and Stability

Traditionally, a *Liapunov function about a point* (x_0, y_0) *for a* 2×2 *regular autonomous system* $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ is any potential Liapunov function on some open region $\mathcal{R}$ about (x_0, y_0) such that

$$D\Psi(x, y) \leq 0 \qquad \text{for each } (x, y) \text{ in } \mathcal{R} \quad .$$

We will follow tradition.

Given the observations made near the end of section 44.1, the next theorem should seem almost obvious:

Theorem 44.3 (Liapunov's basic test for stability)
Let $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ *be a* 2×2 *regular autonomous system of differential equations, and let* (x_0, y_0) *be some point in the plane. If there is a Liapunov function about this point for this system, then* (x_0, y_0) *is a stable critical point for* $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ *.*

Our observations also strongly suggested that the following refinements can be made to this basic test.

Theorem 44.4 (Liapunov's test for asymptotic stability)
Let $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ *be a* 2×2 *regular autonomous system of differential equations, and let* (x_0, y_0) *be some point in the plane. Suppose, further, that there is a Liapunov function* Ψ *for this system about* (x_0, y_0) *, and that, for every* (x, y) *in some open region* $\mathcal{R}$ *containing* (x_0, y_0) *,*

$$D\Psi(x, y) < 0 \qquad \text{whenever} \quad (x, y) \neq (x_0, y_0) \quad .$$

Then (x_0, y_0) *is an asymptotically stable critical point for* $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ *. Moreover, if* $\mathcal{R}$ *is a basin region for* Ψ *, then* $\mathcal{R}$ *is contained in the basin of attraction for* (x_0, y_0) *.*

Theorem 44.5 (Liapunov's test for centers)
Let $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ *be a* 2×2 *regular autonomous system of differential equations, and assume there is a Liapunov function* Ψ *for this system about some point* (x_0, y_0) *. Suppose further that, for some open region* $\mathcal{R}$ *containing* (x_0, y_0) *and no other critical points for* $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ *,*

$$D\Psi(x, y) = 0 \qquad \text{for each } (x, y) \text{ in } \mathcal{R} \quad .$$

Then (x_0, y_0) *is a center for* $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ *. Moreover, if* $\mathcal{R}$ *is a basin region for* Ψ *, then the trajectories through the points in* $\mathcal{R}$ *all form closed loops about* (x_0, y_0) *.*

You probably also expect a test for instability, say:

Theorem 44.6 (Liapunov's test for instability)
Let $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ *be a* 2×2 *regular autonomous system of differential equations, and let* (x_0, y_0) *be some point in the plane. Suppose, further, that there is a potential Liapunov function* Ψ *for this system about* (x_0, y_0) *, and that, for every* (x, y) *in some open region* $\mathcal{R}$ *containing* (x_0, y_0) *,*

$$D\Psi(x, y) > 0 \qquad \text{whenever} \quad (x, y) \neq (x_0, y_0) \quad .$$

Then (x_0, y_0) *is an unstable critical point for* $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ *.*

The above theorems are all true, and the validity of each pretty well follows from our previous discussions. There are a few issues that should concern the more thoughtful reader. We'll address those issues in an appendix.

The Liapunov (direct) method for identifying whether a given critical point is stable or asymptotically stable consists of first finding a suitable Liapunov function and applying the suitable theorem above. In theory, it can be applied when the methods discussed in previous chapters are difficult or impossible to apply, and it is easily extended to deal with critical points of more general $N \times N$ nonlinear systems. That's what makes the method so useful. Moreover, it can help identify the basin of attraction for an asymptotically stable critical point.

By the way, you might have observed that it was not necessary to know that (x_0, y_0) was a critical point of the system to apply the Liapunov method. In theory, the fact that (x_0, y_0) is a critical point of the system follows from the fact that you can find a Liapunov function about this point for the system. In practice, it is silly to try to construct such a function about an arbitrary point, hoping that the resulting function is also a potential Liapunov function for the system. So, in practice, we normally first find the critical points of the system (which, you should recall, is a relatively easy task), and then see if we can construct a (potential) Liapunov function for the system about each critical point — provided, of course, we've at least verified that the point is not a saddle point for the system.

44.3 Finding Liapunov Functions

The biggest difficulty in using the Liapunov method is finding a Liapunov function for a given system and critical point. Unfortunately, there is no single approach to finding these functions.

In this section, we will discuss one fairly general approach to finding Liapunov functions that is often worth trying, at least when the system of differential equations is relatively simple. We will then describe three types of systems of differential equations for which Liapunov functions can be found by relatively straightforward means.

The General Approach

One basic approach to finding a Liapunov function Ψ for a given system $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ and critical point (x_0, y_0) is to write out a general possible formula for $\Psi(x, y)$ and then 'tweek it' until the formula yields a valid Liapunov function for the system. One possible choice for $\Psi(x, y)$ is

$$\Psi(x, y) = A(x - x_0)^2 + B(y - y_0)^2$$

where A and B are positive values 'to be determined'. You can easily verify that this function is at least a potential Liapunov function about (x_0, y_0) on the entire XY–plane. If we can then determine specific positive values for A and B so that

$$D\Psi(x, y) \leq 0$$

for every (x, y) in some open region containing (x_0, y_0), then this Ψ on $\mathcal{R}$ is a Liapunov function for our system and critical point, and we can apply our stability theorems.

!►Example 44.1: *Consider the regular autonomous system*

$$\begin{aligned} \frac{dx}{dt} &= 2y - x^3 \\ \frac{dy}{dt} &= -3x - 2y^3 - x^2 y \end{aligned} \quad . \tag{44.7}$$

Clearly, $(0,0)$ is a critical point for this system, and you can easily verify that this point is a center for the corresponding linearized system. Whether $(0,0)$ is also a center for our nonlinear system remains to be determined. To do so, let's see if a function of the form

$$\Psi(x,y) = A(x-0)^2 + B(x-0)^2 = Ax^2 + By^2$$

can be a corresponding Liapunov function. As noted above, this will be a Liapunov function if A and B are positive, and $D\Psi(x,y) \leq 0$ in some region $\mathcal{R}$ about $(0,0)$. Now,

$$\begin{aligned} D\Psi(x,y) &= \Psi_x(x,y)f(x,y) + \Psi_y(x,y)g(x,y) \\ &= 2Ax\left(2y - x^3\right) + 2By\left(-3x - 2y^3 - x^2y\right) \\ &= (4A - 6B)xy - \left(2Ax^4 + 4By^4 + 2Bx^2y^2\right) \quad . \end{aligned}$$

Clearly, the last expression would be "≤ 0" if A and B are chosen to be any positive values such that the above xy term vanishes. So let us choose A and B accordingly, say,

$$A = 3 \qquad \text{and} \qquad B = 2 \quad .$$

Then

$$\Psi(x,y) = 3x^2 + 2y^2$$

and

$$\begin{aligned} D\Psi(x,y) &= (4\cdot 3 - 6\cdot 2)xy - \left(2\cdot 3x^4 + 4\cdot 2y^4 + 2\cdot 2x^2y^2\right) \\ &= -\left(6x^4 + 8y^4 + 4x^2y^2\right) \quad . \end{aligned}$$

Clearly, for every (x,y) in the XY–plane,

$$D\Psi(x,y) \leq 0$$

with

$$D\Psi(x,y) < 0 \qquad \text{whenever} \quad (x,y) \neq (0,0) \quad .$$

Thus,

1. *This Ψ is a Liapunov function (on $\mathcal{R} =$ the XY–plane) for our system and critical point.*
2. *Liapunov's test for asymptotic stability (Theorem 44.4) applies, telling us that $(0,0)$ is an asymptotically stable critical point for our system (and* not *a center as we may have suspected).*

To help determine the basin of attraction, first observe that all the level curves of Ψ are given by the ellipses

$$3x^2 + 2y^2 = C$$

with C being any positive value. And if $\mathcal{R}_C$ (with $C > 0$) is the region enclosed by the ellipse

$$3x^2 + 2y^2 = C$$

then $\mathcal{R}_C$ contains $(0,0)$ and is contained in $\mathcal{R}$. Hence, theorem 44.4 also tells us that $\mathcal{R}_C$ is contained in the basin of attraction for the critical point $(0,0)$. Now let (x_1,y_1) be any point in the XY–plane. Given this point, we can certainly choose C large enough so that $\mathcal{R}_C$ contains the point, which means this arbitrarily chosen point is in the basin of attraction of the critical point $(0,0)$. This means that every point in the plane is in the basin of attraction. In other words, the entire XY–plane is the basin of attraction for the critical point $(0,0)$. That is, the trajectory of our system through an given point on the plane will converge to $(0,0)$.

Total-Energy Liapunov Functions

In many applications arising from physics, there is a "total energy", E, associated with the physical process being modeled by some autonomous system of differential equations, $\mathbf{x}' = \mathbf{F}(\mathbf{x})$. It often turns out that the formula for the total energy of the system can be used as the formula for a Liapunov function. Just what that "total energy function" is depends on the physics governing the application. For example, when the application just concerns the motion through space of some object of mass m, then physics tells us that

$$E = E_{\mathrm{P}} + E_{\mathrm{K}}$$

where E_{P} and E_{K} are, respectively, the kinetic energy of the object and the potential energy of the object due to the forces causing the motion. The kinetic energy of an object of mass m moving with a velocity of v is always

$$E_K = \frac{1}{2}mv^2 \quad .$$

The potential energy depends on the forces involved. If the only force causing the motion is that of Earth's gravity near the surface of the Earth, then we can take the potential energy to be

$$E_{\mathrm{P}} = mgh \quad ,$$

where g is the standard acceleration due to gravity (9.8 meters/second^2) and h is the vertical position of the object above some fixed height (say, the lowest point of the object path of motion). This gives the kinetic energy of a object of mass m after falling a distance of h after having been dropped (if there were no air resistance).

!►*Example 44.2:* *Let's consider, for one last time, the undamped pendulum system illustrated in figure 43.3 on page 43–10. The system of differential equations for this pendulum is still*

$$\frac{d\theta}{dt} = \omega$$
$$\frac{d\omega}{dt} = -\gamma \sin(\theta)$$

where $\gamma = {}^{g}/_{L}$.

At any given time, the velocity of the mass is the angular velocity ω *multiplied by the length of the pendulum's rod,* L. *So the kinetic energy of the moving pendulum is*

$$E_{\mathrm{K}} = \frac{1}{2}mv^2 = \frac{1}{2}m(\omega L)^2 \quad .$$

By simple trigonometry, we can see that the height h *of the pendulum above its lowest point is given by*

$$h = L - L\cos(\theta) = L[1 - \sin(\theta)]$$

Thus, the potential energy is given by

$$E_P = mgh = mgL[1 - \cos(\theta)] \quad .$$

This means that the total energy in the motion of the pendulum at any given time is given by

$$E = mgh = mgL[1 - \cos(\theta)] + \frac{1}{2}m(\omega L)^2 \quad .$$

So let $\Psi(\theta, \omega)$ *be given by this formula for energy,*

$$\Psi(\theta, \omega) = mgh = mgL[1 - \cos(\theta)] + \frac{1}{2}m(\omega L)^2 \quad .$$

It is easily verified that $\Psi(\theta, \omega)$ *has a minimum value at the critical point* $(\theta, \omega) = (0, 0)$ *.*

Computing $D\Psi$ *we see that, for every* θ *and* ω *,*

$$\begin{aligned} D\Psi(\theta, \omega) &= \Psi_\theta(\theta, \omega) f(\theta, \omega) + \Psi_\omega(\theta, \omega) g(\theta, \omega) \\ &= (mgL[\sin(\theta)])(\omega) + \left(mL^2\omega\right)(-\gamma \sin(\theta)) \\ &= (mgL[\sin(\theta)])(\omega) + \left(mL^2\omega\right)\left(-\frac{g}{L}\sin(\theta)\right) \\ &= 0 \quad . \end{aligned}$$

Since $D\Psi = 0$ *everywhere, we can apply Liapunov's test for centers (theorem 44.5 on page 44–9) and conclude that the critical point* $(0, 0)$ *is a center for our system.*

Gradient Systems

To Be Written

Hamiltonian Systems

To Be Written

44.4 Appendix – Technical Issues

To Be Written

Additional Exercises

To Be Written