43

Coordinate Curves for Trajectories

The material on linearizations and Jacobian matrices developed in the last chapter certainly expanded our ability to deal with nonlinear systems of differential equations. Unfortunately, those tools were not quite precise enough to completely determine the nature of the trajectories about critical points which were centers for the corresponding linear systems. We'll develop some useful tools for handling these situations in this chapter and the next.

In this chapter, we will concentrate on some relatively easily visualized geometry to develop an approach to identifying when a possible center truly is a center with the nearby trajectories actually forming closed loops about the critical point. Not only will this approach be useful in itself, it will provide a good background for developing the more general methods (the "Liapunov method") in the next chapter.

By the way, for several reasons, we will limit ourselves to 2×2 systems in this chapter.

43.1 Coordinate Equations for Trajectories

Remember, a trajectory for a solution $\mathbf{x}(t) = [x(t), y(t)]^{\mathsf{T}}$ to a 2×2 system $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ is the oriented curve in the XY–plane traced out by the point $(x(t), y(t))$ as t varies. Accordingly, the curve of any trajectory in the XY–plane can be described, in theory, using a coordinate equation for that curve. This equation may be an explicit formula for one coordinate in terms of the other coordinate, as in

$$y = x^2 + 3 \qquad \text{or} \qquad x = \cos(y) + 4 \quad ,$$

or the relation between the two coordinates may be related implicitly through some more general equation such as

$$x^2 + y^2 = 9 \qquad \text{or} \qquad xe^{2x} = ye^{3y} \quad .$$

Finding Coordinate Equations for Trajectories

So let us consider finding such an coordinate equation for a trajectory corresponding to an arbitrary solution $\mathbf{x}(t) = [x(t), y(t)]^{\mathsf{T}}$ to a 2×2 regular autonomous system of differential equations $\mathbf{x}' = \mathbf{F}(\mathbf{x})$, which we will write more explicitly as

$$\begin{aligned} \frac{dx}{dt} &= f(x, y) \\ \frac{dy}{dt} &= g(x, y) \end{aligned} \quad . \tag{43.1}$$

Using this system and the chain rule, we have

$$g(x, y) = \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx} f(x, y) \quad .$$

In other words, the coordinates of each trajectory of our system satisfy the single ordinary differential equation

$$f(x, y)\frac{dy}{dx} = g(x, y) \quad , \tag{43.2}$$

which we can rewrite as

$$\frac{dy}{dx} = \frac{g(x, y)}{f(x, y)} \quad . \tag{43.3}$$

After recalling what g and f represent, we see that this is simply

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad . \tag{43.4}$$

In practice, it may be easier to remember equation (43.4), and then note that equation (43.3) follows from the original system of differential equations.

If we are very lucky, we can solve ordinary differential equation (43.2) or (43.3), obtaining an explicit formula for x or y in terms of the other. If we are less lucky (but still reasonably lucky), we can at least obtain an implicit solution for the ordinary differential equation— something that looks like

$$\phi(x, y) = c$$

where $\phi(x, y)$ is some formula of x and y, and c is an arbitrary constant (but possibly, as illustrated in the next example, not a completely arbitrary constant).

!►Example 43.1: *Consider the simple linear system*

$$\frac{dx}{dt} = 4y$$
$$\frac{dy}{dt} = -x \quad .$$

You can easily verify that the origin $(0, 0)$ *is a center for this system, and you surely recall being assured that the trajectories are ellipses centered at the origin. Ignoring this, however, we can still form our single ordinary differential equation:*

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-x}{4y} \quad .$$

Cutting out the middle leaves us with a very easily solved separable first-order ordinary differential equation:

$$\frac{dy}{dx} = \frac{-x}{4y}$$

$$\hookrightarrow \qquad 4y\frac{dy}{dx} = -x$$

$$\hookrightarrow \qquad \int 4y\frac{dy}{dx}\,dx = -\int x\,dx$$

$$\hookrightarrow \qquad 2y^2 = -\frac{1}{2}x^2 + c \quad .$$

Now we could solve the last equation for y in terms of x, but it is more informative to observe that it can be rewritten as

$$\frac{x^2}{4} + \frac{y^2}{1} = C \qquad \text{where} \quad C \geq 0 \quad ,$$

because this equation is readily recognized as a standard equation for an ellipse centered at the origin and with vertices at $(0, \pm\sqrt{C})$ and $(\pm 2\sqrt{C}, 0)$.

Thus, not only do we now know that the trajectories for our system are ellipses, we have an easily used equation for drawing these ellipses.

We should observe that a derivation analogous to that which led to equations (43.3) and (43.4) will also yield

$$\frac{dx}{dy} = \frac{dx/dt}{dy/dt} = \frac{f(x,y)}{g(x,y)} \tag{43.5}$$

as a differential equation for the coordinate equations of the trajectories for system (43.1). A straightforward application of the basic theorem on existence and uniqueness of solution to first-order equations (theorem 3.1 on page 44) then assures us that, in theory, either differential equation (43.3) or (43.5) can be solved to find a coordinate equation for the trajectory through any point that is not a critical point. And, of course, we already know that the trajectory through any critical point is simply that one point.

We should note the method we've just developed is mainly suited for just 2×2 systems. It does not readily generalize to larger systems of differential equations. And even with 2×2 systems, there are limitations. After all, we already know from our studies of first-order equations that while, in theory,

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$$

may have solutions, actually finding a formula for these solutions may be a significant challenge. And even if we can obtain a general solution $\phi(x,y) = c$ to this differential equation, it might not be immediately obvious if this equation describes closed loops about some given point, or describes some other set of curves. Fortunately, we do have tools that may help deal with this last issue.

43.2 Trajectories as Level Curves

Let's assume that we have determined that the trajectories in some region $\mathcal{R}$ of the XY–plane are given by

$$\phi(x,y) = c$$

where ϕ is at least a continuous function on $\mathcal{R}$ and c is an arbitrary constant.

Now recall that the set of all (x,y,z) satisfying

$$z = \phi(x,y)$$

is a surface in three-dimensional space. Replacing the variable z with a constant c gives us the equation of all points on this surface with $z = c$. That is,

$$\phi(x,y) = c$$

is the equation of the level curve(s) for $z = \phi(x,y)$ corresponding to $z = c$. So we can treat trajectories given by such equations as level curves for some surface. This is particularly useful

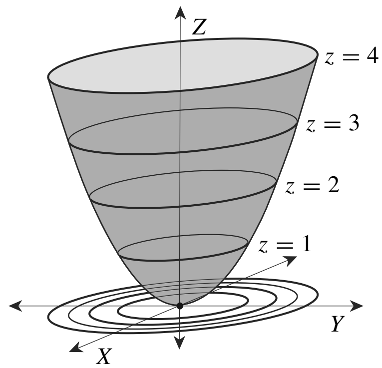

Figure 43.1: The level Curves (in the XY–plane) corresponding to the cross-sections at $z = 1$, $z = 2$, $z = 3$ and $z = 4$ of the surface given by $z = \frac{1}{4}x^2 + y^2$.

when this surface has an isolated maximum or minimum at a particular point (x_0, y_0), because then this point corresponds to either the peak of a hill or the lowest point in a valley, and the nearby level curves must be closed loops about this point, as illustrated in figure 43.1 (which uses the function ϕ from example 43.1). This means that we have the following theorem:

Theorem 43.1
Let $\mathcal{R}$ be an open region of the XY–plane containing a point (x_0, y_0), and suppose there is a continuous function ϕ on $\mathcal{R}$ having either an isolated maximum or isolated minimum at at (x_0, y_0). Assume further that, in $\mathcal{R}$, the trajectories of the solutions to some system $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ are level curves for ϕ (i.e., given by $\phi(x, y) = c$). Then (x_0, y_0) is a center for the system $\mathbf{x}' = \mathbf{F}(\mathbf{x})$.

Recall that we could not determine the stability of a critical point (x_0, y_0) for a nonlinear system $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ just from the fact that this point is a center for this system's linearization about (x_0, y_0). The above theorem gives us a tool for identifying at least some of these critical points as being centers for the nonlinear system.

So how do we go about determining if a given point is a maximum or minimum for a given function? Well, why not use the standard first- and second-derivative tests from calculus. Remember, if h is a sufficiently differentiable function of one variable, and $h'(x_0) = 0$, then

$$h''(x_0) > 0 \quad \Longrightarrow \quad h \text{ has an isolated minimum at } x_0$$

and

$$h''(x_0) < 0 \quad \Longrightarrow \quad h \text{ has an isolated maximum at } x_0 \quad .$$

For functions of two variables, we have the following analog of the first- and second-derivative tests:

Theorem 43.2
Let ϕ be a function of two variables which is continuous and has continuous first and second partial derivatives in some open region about a point (x_0, y_0). Assume, further, that these partial derivatives

satisfy[1]

$$\phi_x(x_0, y_0) = 0 \quad , \quad \phi_y(x_0, y_0) = 0$$

and

$$\phi_{xx}(x_0, y_0)\phi_{yy}(x_0, y_0) - \left[\phi_{xy}(x_0, y_0)\right]^2 > 0 \quad .$$

Then ϕ *has either an isolated maximum or an isolated minimum at* (x_0, y_0) *. In particular:*

1. *If* $\phi_{xx}(x_0, y_0) > 0$ *or* $\phi_{yy}(x_0, y_0) > 0$ *, then* ϕ *has an isolated minimum at* (x_0, y_0) *.*
2. *If* $\phi_{xx}(x_0, y_0) < 0$ *or* $\phi_{yy}(x_0, y_0) < 0$ *. then* ϕ *has an isolated maximum at* (x_0, y_0) *.*

You probably at least vaguely recall this theorem from the later part of your calculus courses. The proof should be in your old calculus text.

We'll illustrate the use of the above in the next two applications.

43.3 Application: Foxes in the Rabbit Ranch

The Classic Predator-Prey Model

In chapter 10, we considered how the number of rabbits in a large ranch varies with time. Let us now assume foxes have entered the fields and, not being vegetarians, have begun dining on the rabbits they can catch, and raising little pups of their own. Our interest is in determining both

$$R(t) = \text{number of rabbits at time } t$$

and

$$F(t) = \text{number of foxes at time } t \quad .$$

For convenience, we'll again take the basic unit of time to be months.

Recall that the basic equation describing the rate of change in the number of rabbits is

$$\frac{dR}{dt} = \beta_R R - \delta_R R$$

where β_R is the monthly birthrate per rabbit, and δ_R is the monthly deathrate per rabbit (that is, δ_R is the fraction of the rabbit population that dies each month). Under ideal conditions — plenty of food and no predators — β_R and δ_R are constants, $\beta_{R,0}$ and $\delta_{R,0}$ (according to the information in chapter 10, $\beta_{R,0} \approx 5/4$ and $\delta_{R,0} \approx 0$). For simplicity, let's take $\delta_{R,0} = 0$ assume there is plenty of food for the rabbits (so $\beta_R = \beta_{R,0}$). But with foxes around, the deathrate can not be assumed constant; it will be a function of the number of foxes,

$$\delta_R = \delta_R(F) \quad .$$

In particular, as the number of foxes increases, so does the deathrate δ_R . With a little thought, you will probably agree that the simplest formula describing a death rate δ_R that increases from the ideal rate of 0 as the number of foxes increases from 0 is

$$\delta_R = \delta_R(F) = \delta_{R,1} F$$

[1] Remember, if $\phi = \phi(x, y)$, then we may use the shorthand subscript notation for partial derivatives:

$$\phi_x = \frac{\partial\phi}{\partial x} \quad , \quad \phi_{xy} = \frac{\partial^2\phi}{\partial x\partial y} \quad , \quad \phi_{xx} = \frac{\partial^2\phi}{\partial^2 x} \quad , \quad \dots \quad .$$

where $\delta_{R,1}$ is some positive constant. Thus,

$$\begin{aligned} \frac{dR}{dt} &= \beta_R R - \delta_R R \\ &= \beta_{R,0} R - \left[\delta_{R,1} F\right] R = \left(\beta_{R,0} - \delta_{R,1} F\right) R \quad . \end{aligned} \tag{43.6}$$

Likewise, the basic equation describing the rate of change in the number of foxes is

$$\frac{dF}{dt} = \beta_F F - \delta_F F$$

where β_F is the monthly birthrate per fox, and δ_F is the monthly deathrate per fox. In this case, however, both the birthrate and deathrate will depend on the amount of food available for the foxes (i.e., the number of rabbits in the fields). In other words, β_F and δ_F should be treated as functions of R,

$$\beta_F = \beta_F(R) \qquad \text{and} \qquad \delta_F = \delta_F(R) \quad .$$

If $R = 0$, there are no rabbits, and, hence, no food for the foxes, which, in turn, means no new foxes are born, and a large portion of the fox population dies from starvation each month. That is, we should have

$$\beta_F(0) = 0 \qquad \text{and} \qquad \delta_F(0) = \delta_{F,0} \quad .$$

where $\delta_{F,0}$ is the fraction of the fox population that would die from starvation each month if there is no food for them (hence, $\delta_{F,0}$ is some positive number less than or equal to one — $^{95}/_{100}$ would seem reasonable). As the number of rabbits increases, there is more food and the birthrate will increase while the deathrate will decrease. The simplest formulas describing this are

$$\beta_F(R) = \beta_{F,1} R \qquad \text{and} \qquad \delta_F(R) = \delta_{F,0} - \delta_{F,1} R$$

where $\beta_{F,1}$ and $\delta_{F,1}$ are positive constants. Thus,

$$\begin{aligned} \frac{dF}{dt} &= \beta_F F - \delta_F F \\ &= \left[\beta_{F,1} R\right] F - \left[\delta_{F,0} - \delta_{F,1} R\right] F \\ &= \left(\left[\beta_{F,1} + \delta_{F,1}\right] R - \delta_{F,0}\right) F \quad . \end{aligned}$$

Combining the last equation with (43.6) (and letting $\gamma = \beta_{F,1} + \delta_{F,1}$) we now have the system

$$\begin{aligned} R' &= \left(\beta_{R,0} - \delta_{R,1} F\right) R \\ F' &= \left(\gamma R - \delta_{F,0}\right) F \end{aligned} \tag{43.7}$$

where $\beta_{R,0}$, $\delta_{R,1}$, $\gamma_{F,1}$ and $\delta_{F,0}$ are positive constants that would have to be determined by experiment. This is the *classic predator-prey model*. In particular, with

$$\beta_{R,0} = {}^{5}/_{4} \quad , \quad \delta_{R,1} = {}^{1}/_{16} \quad , \quad \gamma_{F,1} = {}^{1}/_{300} \quad \text{and} \quad \sigma_{F,0} = {}^{95}/_{100} \quad ,$$

our system is

$$\begin{aligned} R' &= \frac{5}{4} R - \frac{1}{16} F R \\ F' &= \frac{1}{300} R F - \frac{95}{100} F \end{aligned} \quad . \tag{43.8}$$

Analyzing the Basic Model

To simplify the notation in our analysis, let us replace the symbols $\beta_{R,0}$, $\delta_{F,1}$ and $\delta_{F,0}$ in system (43.7) with β , δ and σ , respectively, giving us the system

$$\begin{aligned} \frac{dR}{dt} &= (\beta - \delta F)R \\ \frac{dF}{dt} &= (\gamma R - \sigma)F \end{aligned} \tag{43.9}$$

to describe the changes over time of a population of rabbits in a large field and a population of foxes (who prey on the rabbits). In the following, it is important to remember that β , δ , γ and σ are positive constants (with β being the birth rate per rabbit under ideal conditions, and σ being the death rate per fox when there are no rabbits).

The Jacobian matrix for this system is

$$\mathbf{J}(R, F) = \begin{bmatrix} \beta - \delta F & -\delta R \\ \gamma F & \gamma R - \sigma \end{bmatrix} \quad ,$$

and the algebraic system for the critical points is

$$\begin{aligned} 0 &= (\beta - \delta F)R \\ 0 &= (\gamma R - \sigma)F \end{aligned} \quad .$$

From this, we get that the critical points of our predator-prey system are

$$(R, F) = (0, 0) \qquad \text{and} \qquad (R, F) = \left(\frac{\sigma}{\gamma}, \frac{\beta}{\delta}\right) \quad .$$

Plugging $(0, 0)$ into the Jacobian matrix yields

$$\mathbf{J}(0, 0) = \begin{bmatrix} \beta & 0 \\ 0 & -\sigma \end{bmatrix}$$

which clearly has eigenpairs

$$\left(\beta, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \qquad \text{and} \qquad \left(-\sigma, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \quad .$$

And from this we see that our system has a saddle point at $(0, 0)$. (More precisely, this tells us that the linearization of our system about $(0, 0)$ has a saddle point at $(0, 0)$. And theorem 42.3 then tells us that our nonlinear system also has a saddle point at $(0, 0)$.)

For the other critical point, we have

$$\mathbf{J}\left(\frac{\sigma}{\gamma}, \frac{\beta}{\delta}\right) = \begin{bmatrix} \beta - \delta \cdot \dfrac{\beta}{\delta} & -\delta \cdot \dfrac{\sigma}{\gamma} \\ \gamma \cdot \dfrac{\beta}{\delta} & \gamma \cdot \dfrac{\sigma}{\gamma} - \sigma \end{bmatrix} = \begin{bmatrix} 0 & -\dfrac{\delta\sigma}{\gamma} \\ \dfrac{\gamma\beta}{\delta} & 0 \end{bmatrix}$$

The eigenvalues for this matrix are easily found to be

$$r_{\pm} = \pm i\sqrt{\beta\delta} \quad .$$

Hence, the linearization of our system about this critical point has a center. Unfortunately, as noted in theorem 42.4, this does not assure us that this critical point is a center for our nonlinear system. In fact, it does not even tell us about the stability of this point.

Clearly, this is a point where we want to try the tools discussed in the previous section. But before doing so, let us note that, from system (43.9) it is immediately clear that:

1. If $R = 0$ and $F > 0$, then
$$\frac{dR}{dt} = 0 \qquad \text{and} \qquad \frac{dF}{dt} = -\sigma F < 0 \quad .$$
telling us that the positive F–axis is a trajectory for our system, and that the direction of travel along this trajectory is towards the origin.

2. If $F = 0$ and $R > 0$, then
$$\frac{dF}{dt} = 0 \qquad \text{and} \qquad \frac{dR}{dt} = \beta R > 0 \quad .$$
telling us that the positive R–axis is a trajectory for our system, and that the direction of travel along this trajectory is away from the origin.

The above assures us that, as in the competing species model of section 42.5, that we can restrict our attention to the first octant of the RF–plane. Moreover, according to this model:

1. The foxes die out if there are no rabbits to eat.
2. The rabbit population increases without bound if there are no foxes to eat them.

Now, let's attempt to construct and solve a differential equation for coordinate formulas for out trajectories in the first quadrant. Proceeding as described in the previous section (with $x = R$ and $y = F$), we have
$$\frac{dF}{dR} = \frac{dF/dt}{dR/dt} = \frac{(\gamma R - \sigma)F}{(\beta - \delta F)R} \quad ,$$
which we can rewrite as
$$\frac{dF}{dR} = \frac{\gamma R - \sigma}{R} \cdot \frac{F}{(\beta - \delta F)} \quad .$$
So, this is a separable first-order differential equation— not as simple as in our previous example, but still not particularly difficult to solve. Separating the variables, integrating with respect to R gives us
$$\frac{\beta - \delta F}{F}\frac{dF}{dR} = \frac{\gamma R - \sigma}{R}$$
$$\hookrightarrow \qquad \left[\frac{\beta}{F} - \delta\right]\frac{dF}{dR} = \gamma - \frac{\sigma}{R}$$
$$\hookrightarrow \qquad \int \left[\frac{\beta}{F} - \delta\right]\frac{dF}{dR}\,dR = \int \left[\gamma - \frac{\sigma}{R}\right] dR$$
$$\hookrightarrow \qquad \beta \ln F - \delta F = \gamma R - \sigma \ln R + c$$
which we can write as
$$\phi(R,F) = c \qquad \text{with} \quad \phi(R,F) = \beta \ln F - \delta F + \sigma \ln R - \gamma R \quad .$$

The first and second partial derivatives of ϕ are easily computed:
$$\phi_R = \frac{\partial \phi}{\partial R} = \frac{\sigma}{R} - \gamma \quad , \quad \phi_F = \frac{\partial \phi}{\partial F} = \frac{\beta}{F} - \delta \quad ,$$
$$\phi_{RR} = \frac{\partial^2 \phi}{\partial R^2} = -\frac{\sigma}{R^2} \quad , \quad \phi_{FF} = \frac{\partial^2 \phi}{\partial F^2} = -\frac{\beta}{F^2} \qquad \text{and} \qquad \phi_{RF} = \frac{\partial^2 \phi}{\partial R \partial F} = 0 \quad .$$

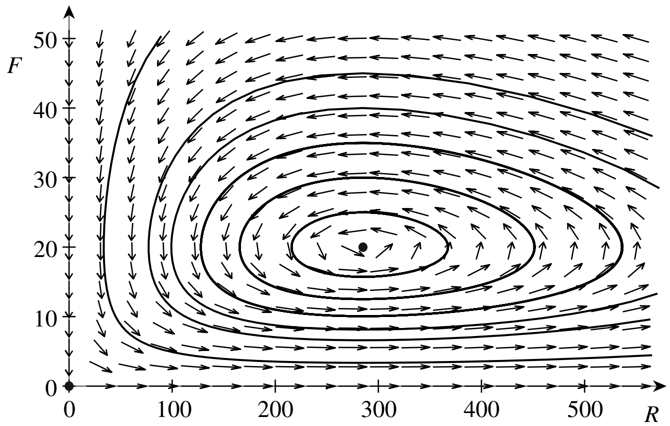

Figure 43.2: Phase portrait (with direction field) for a predator-prey system with a stable center $(r, f) = (285, 20)$.

Note that, for every (R, F) in the first octant,

$$\phi_{RR} < 0 \quad , \quad \phi_{RR} < 0 \quad \text{and} \quad \phi_{RR}\phi_{FF} - [\phi_{RF}]^2 > 0 \quad .$$

And at the critical point of our system (system (43.9)),

$$(R, F) = \left(\frac{\sigma}{\gamma}, \frac{\beta}{\delta}\right) \quad ,$$

we have

$$\phi_R = \frac{\sigma}{\sigma/\gamma} - \gamma = 0 \quad \text{and} \quad \phi_F = \frac{\beta}{\beta/\delta} - \delta = 0 \quad .$$

The first and second derivative tests of theorem 43.2 now tells us that ϕ has an isolated maximum at this critical point, and this, as noted in theorem 43.1, tells us that this critical point is actually a center for our nonlinear system of differential equations. Hence, this critical point is stable, and the trajectories near it are closed loops about this point.

In fact, it isn't that hard to continue this analysis, computing the maximum value C of ϕ and verifying that every trajectory in the first quadrant is a closed loop about the system's center. We'll leave that verification to the more dedicated reader, and simply note that the result yields the phase portrait in figure 43.2. This particular phase portrait was generated using our basic predator-prey system

$$\frac{dR}{dt} = (\beta - \delta F)R$$
$$\frac{dF}{dt} = (\gamma R - \sigma)F$$

with

$$\beta = {}^{5}\!/_{4} \quad , \quad \delta = {}^{1}\!/_{16} \quad , \quad \gamma = {}^{1}\!/_{300} \quad \text{and} \quad \sigma = {}^{95}\!/_{100} \quad .$$

For this predator-prey system, the stable critical point about which the trajectories loop is

$$(R, F) = \left(\frac{\sigma}{\gamma}, \frac{\beta}{\delta}\right) = \left(\frac{{}^{95}\!/_{100}}{{}^{1}\!/_{300}}, \frac{{}^{5}\!/_{4}}{{}^{1}\!/_{16}}\right) = (285, 20) \quad .$$

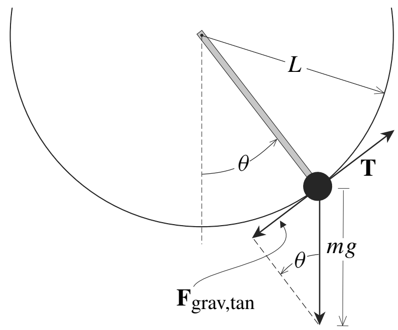

Figure 43.3: The pendulum system with a weight of mass m attached to a massless rod of length L swinging about a pivot point under the influence of gravity.

43.4 Application: The Ideal Pendulum

Let's again consider the pendulum illustrated in figure 43.3. We first considered this pendulum in chapter 35 where we derived the first-order system

$$\begin{aligned}\frac{d\theta}{dt} &= \omega \\ \frac{d\omega}{dt} &= -\gamma\sin(\theta) - \kappa\omega\end{aligned}$$

as a model to describe the pendulum's motion. Here:

$$\theta(t) = \text{the angular position of pendulum at time } t \text{ measured counterclockwise from the vertical line below the pivot point}$$

and

$$\omega(t) = \frac{d\theta}{dt} = \text{the angular velocity of the pendulum at time } t \quad .$$

In addition, γ is a positive constant given by $\gamma = {}^{g}/_{L}$ where L is the length of the pendulum and g is the acceleration of gravity, and κ is the "drag coefficient", a nonnegative constant describing the effect friction has on the motion of the pendulum. The greater the effect of friction on the system, the larger the value of κ, with $\kappa = 0$ when there is no friction slowing down the pendulum.

In section 42.8, we expanded on the work done in chapter 36 concerning the phase portrait for this system and determined the general motion of the pendulum assuming $\kappa > 0$. Let us now assume an "ideal" pendulum in which $\kappa = 0$. In this case, our system reduces to

$$\begin{aligned}\frac{d\theta}{dt} &= \omega \\ \frac{d\omega}{dt} &= -\gamma\sin(\theta)\end{aligned} \quad . \tag{43.10}$$

Going through the same sort of analysis as done in section 42.8, you can easily verify that:

1. The direction field (sketched in figure 43.4a) is periodic in the θ-direction with a period of 2π.

2. The critical points of this system are given by

$$(\theta, \omega) = (n\pi, 0) \qquad \text{with} \quad n = 0, \pm 1, \pm 2, \ldots \quad .$$

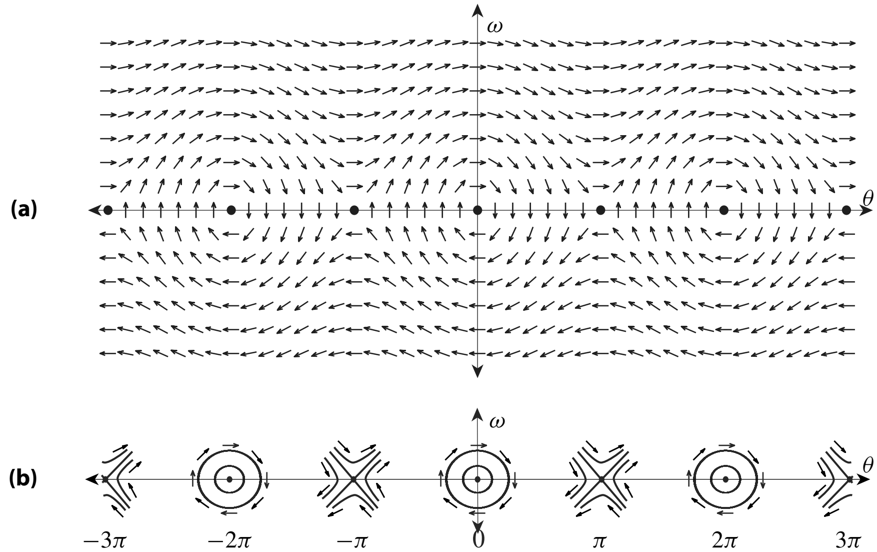

Figure 43.4: The ideal pendulum: **(a)** a (computer generated) direction field, and **(b)** rough sketches of the trajectories of the linearizations about the critical points.

3. The Jacobian matrix for the system at each critical point $(n\pi, 0)$ is

$$\mathbf{J}(n\pi, 0) = \begin{bmatrix} 0 & 1 \\ -\gamma\cos(n\pi) & 0 \end{bmatrix}$$

4. At each critical point $(n\pi, 0)$ with n being odd, the Jacobian matrix has eigenpairs

$$\left(\sqrt{\gamma}\,,\ \begin{bmatrix} 1 \\ \sqrt{\gamma} \end{bmatrix}\right) \qquad \text{and} \qquad \left(-\sqrt{\gamma}\,,\ \begin{bmatrix} 1 \\ -\sqrt{\gamma} \end{bmatrix}\right)$$

which means that the linearizations of our system about these critical points each has a saddle point at the critical point, as illustrated in figure 43.4b.

5. At each critical point $(n\pi, 0)$ with n being even, the Jacobian matrix has imaginary eigenvalues

$$\sqrt{\gamma}\,i \qquad \text{and} \qquad -\sqrt{\gamma}\,i \quad ,$$

and the linearizations of our system about these critical points each has a center at the critical point, as illustrated in figure 43.4b.

From theorem 42.3 we know that the nonlinear system for the pendulum will also have the above derived saddle points, but, as noted in theorem 42.4, we cannot be certain as to the nature of the critical points at any critical point $(n\pi, 0)$ where n is even.

Still, based on the sketched direction field and possibly intuitive notions on the motion of an ideal pendulum, it should seem likely that these critical points are, indeed, centers for our nonlinear system. So let's try forming the differential equation for the coordinate formulas of the trajectories of system (43.10):

$$\frac{d\omega}{d\theta} = \frac{d\omega/dt}{d\theta/dt} = \frac{-\gamma\sin(\theta)}{\omega} \quad .$$

Again, we are blessed with a rather easily solved separable differential equation:

$$\frac{d\omega}{d\theta} = \frac{-\gamma\sin(\theta)}{\omega} \tag{43.11}$$

$$\hookrightarrow \qquad \omega\frac{d\omega}{d\theta} = -\gamma\sin(\theta) \tag{43.12}$$

$$\hookrightarrow \qquad \int\omega\frac{d\omega}{d\theta}\,d\theta = -\gamma\int\sin(\theta)\,d\theta \tag{43.13}$$

$$\hookrightarrow \qquad \frac{1}{2}\omega^2 = \gamma\cos(\theta) + c \tag{43.14}$$

Our solution can be written either implicitly as

$$\omega^2 - 2(\gamma\cos(\theta) + c) = 0$$

or explicitly as

$$\omega = \pm\sqrt{2(\gamma\cos(\theta) + c)} \quad .$$

Note that the arbitrary constant, c, must be at least $-\gamma$ to ensure that ω is real. Moreover, as you can readily verify, the basic nature of the curves given by the above equations depend on whether $c > \gamma$, $c = \gamma$, $|c| \leq 1$, or $c = -\gamma$:

1. If $c > \gamma$, then $2(\gamma\cos(\theta) + c) > 0$ for all real θ. From this it follows that

$$\omega = +\sqrt{2(\gamma\cos(\theta) + c)} \qquad \text{and} \qquad \omega = -\sqrt{2(\gamma\cos(\theta) + c)}$$

each gives a periodically oscillating smooth curve (with period 2π about the straight lines $\omega = \sqrt{2c)}$ and $\omega = -\sqrt{2c}$.

2. If $c = \gamma$, the above formulas for ω again yield periodically oscillating curves with a period of 2π. These curves touch the Θ–axis at the odd integers, and, thus, the two curves form closed loops about the points on the Θ–axis where θ is an even integer. What's more, by computing the derivatives of these curves as θ approaches any odd integer, you can easily verify that these curves are tangent to the straightline trajectories of the corresponding linearized systems at the saddle points of the system.

3. If $|c| < \gamma$, then

$$\omega = +\sqrt{2(\gamma\cos(\theta) + c)} \qquad \text{and} \qquad \omega = -\sqrt{2(\gamma\cos(\theta) + c)}$$

only defines curves in regions where $\gamma\cos(\theta) + c \geq 0$. In particular, on the strip with $-\pi \leq \theta \leq \pi$, these two formulas for ω yield real values only if $-\omega_c \leq \omega \leq \omega_c$ where

$$\omega_c = \operatorname{Arccos}\left(\frac{c}{\gamma}\right) \quad .$$

Over this interval, the above formulas for ω correspond to the top and bottom halves of a loop about $(0,0)$ which is tangent to vertical lines at $(-\theta_c, 0)$ and $(\theta_c, 0)$.

4. If $c = -\gamma$, then the only solutions to

$$\omega = \pm\sqrt{2(\gamma\cos(\theta) + c)}$$

are the points $(n\pi, 0)$ with $n = 0, \pm 2, \pm 4, \ldots$.

Thus, the points we suspected to be centers for our nonlinear idea pendulum system are, indeed, centers.

A phase portrait for the pendulum is sketched in figure 43.5.

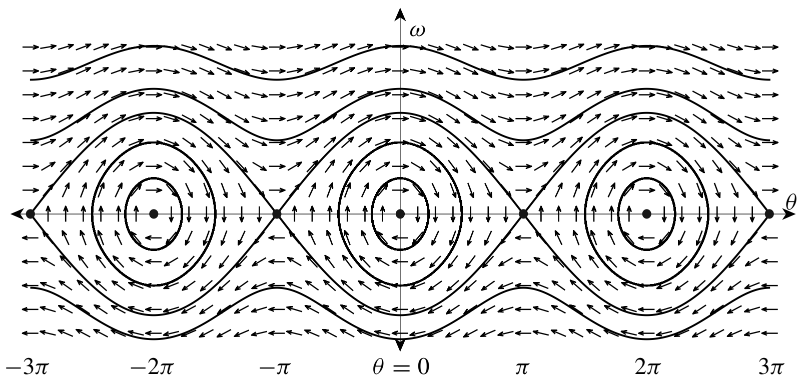

Figure 43.5: Phase portrait (with direction field) for an ideal pendulum.

Additional Exercises

To Be Written