40

Miscellaneous Topics Involving Homogeneous Constant Matrix Systems

In this chapter we will discuss a variety of topics, all more-or-less related to the constant matrix systems discussed in the previous two chapters. Some of this material is of interest for its own sake, and some is developed here for use either in the chapter on nonhomogeneous linear systems or for use in discussing nonlinear systems.

40.1 Shifted Constant Matrix Systems

A *shifted constant matrix system* is simply a system of differential equations that can be written as

$$\mathbf{x}' = \mathbf{A}[\mathbf{x} - \mathbf{x}^0]$$

where $\mathbf{A}$ is a constant $N \times N$ matrix and $\mathbf{x}^0 = \left[x_1^0, x_2^0, \ldots, x_N^0\right]^\mathsf{T}$ is a constant vector. Such systems will later be important in approximating nonlinear systems about critical points.

The above shifted system looks a lot like the constant matrix system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, and we can make the shifted system look even more like the unshifted system by defining a new vector-valued function $\hat{\mathbf{x}}(t)$ by

$$\hat{\mathbf{x}}(t) = \mathbf{x}(t) - \mathbf{x}^0 \quad .$$

This is equivalent to introducing a new coordinate system that is just the original coordinate system shifted so that the new origin $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_N) = (0, 0, \ldots, 0)$ is at the point given by $(x_1, x_2, \ldots, x_N) = \left(x_1^0, x_2^0, \ldots, x_N^0\right)$ in the old coordinate system, as illustrated in figure 40.1b (in practice, though, we rarely sketch the shifted coordinate system). Since

$$\frac{d\hat{\mathbf{x}}}{dt} = \frac{d}{dt}[\mathbf{x}(t) - \mathbf{x}^0] = \frac{d\mathbf{x}}{dt} - \mathbf{0} = \frac{d\mathbf{x}}{dt}$$

and

$$\mathbf{A}[\mathbf{x} - \mathbf{x}^0] = \mathbf{A}\hat{\mathbf{x}} \quad ,$$

our system of differential equations reduces, in the shifted coordinate system, to the basic constant matrix system

$$\hat{\mathbf{x}}' = \mathbf{A}\hat{\mathbf{x}} \quad .$$

So everything we learned about solving basic constant matrix systems applies here provided we take into account the "shift by $\mathbf{x}^0$". In particular:

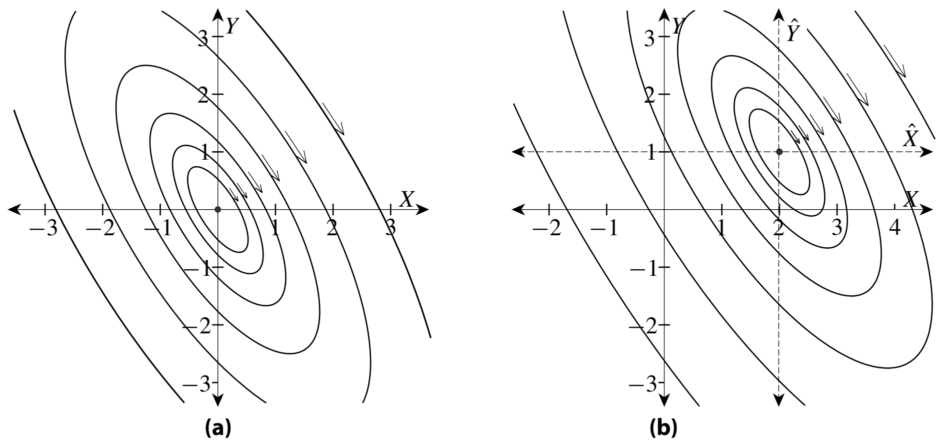


Figure 40.1: Phase portraits for **(a)** the basic system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ and **(b)** the shifted system $\mathbf{x}' = \mathbf{A}[\mathbf{x} - \mathbf{x}^0]$ from example 40.1.

1. The point $\left(x_1^0, x_2^0, \ldots, x_N^0\right)$ is a critical point for $\mathbf{x}' = \mathbf{A}[\mathbf{x} - \mathbf{x}^0]$, and is the only critical point if $\det(\mathbf{A}) \neq 0$.
2. All solutions to $\mathbf{x}' = \mathbf{A}[\mathbf{x} - \mathbf{x}^0]$ can be obtained by just adding $\mathbf{x}^0$ to all solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$.
3. The stability of the equilibrium solution $\mathbf{x}(t) = \mathbf{x}^0$ for $\mathbf{x}' = \mathbf{A}[\mathbf{x} - \mathbf{x}^0]$ is the same as the stability of the equilibrium soluton $\mathbf{x}(t) = \mathbf{0}$ for $\mathbf{x}' = \mathbf{A}\mathbf{x}$.
4. A phase portrait for $\mathbf{x}' = \mathbf{A}[\mathbf{x} - \mathbf{x}^0]$ can be obtained by just "shifting" a phase portrait of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ so that the trajectories of the shifted system about $\left(x_1^0, x_2^0, \ldots, x_N^0\right)$ match the trajectories of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ about $(0, 0, \ldots, 0)$.

!►*Example 40.1:* *Consider the shifted system*

$$\mathbf{x}' = \mathbf{A}[\mathbf{x} - \mathbf{x}^0]$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{x}^0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad .$$

The corresponding "unshifted" system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ was considered in example 39.2 on page 39–8. There, we saw that a general solution for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is given by $C\mathbf{x}^R(t - t_0)$ where

$$\mathbf{x}^R(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(t) - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin(t) \quad .$$

We also constructed a phase portrait for this system (redrawn in figure 40.1a), and observed that the equilibrium solution $\mathbf{x}(t) = \mathbf{0}$ is stable, but not asymptotically stable.

By simply adding the "shift" to the above formula, we then obtain the general solution

$$\begin{aligned} \mathbf{x}(t) &= C\mathbf{x}^R(t - t_0) + \mathbf{x}^0 \\ &= C\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(t - t_0) - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin(t - t_0)\right) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \end{aligned}$$

for the shifted system, $\mathbf{x}' = \mathbf{A}[\mathbf{x} - \mathbf{x}^0]$, and, by suitably shifting the phase portrait of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ in figure 40.1a, we get the phase portrait of the shifted system in figure 40.1b. Again, it is clear that the equilibrium solution $\mathbf{x}(t) = \mathbf{x}^0$ is stable, but not asymptotically stable.

40.2 Classifying Critical Points for 2×2 Systems

Later, we will use what we've developed in the last few chapters for constant matrix systems to help analyze solutions to 2×2 nonlinear systems of differential equations. So, for future reference, let us now

1. summarize some of what we've derived regarding stability, and
2. actually give definitions for some of the terms introduced in the previous two chapters.

In this discussion, we will assume $\mathbf{A}$ is a constant 2×2 matrix with real components, and with eigenvalues r_1 and r_2 (possibly with $r_1 = r_2$). If r_1 and r_2 are complex, then we know they are complex conjugates of each other, and we'll denote the real and imaginary parts, respectively, by λ and ω,

$$r_1 = \lambda + i\omega \quad \text{and} \quad r_2 = \lambda - i\omega \quad .$$

Stability in Constant Matrix Systems

If you go back and review the possible cases, you will see that, whenever either r_1 or r_2 is positive, or whenever r_1 and r_2 are complex with λ positive, then $\mathbf{x}(t) = \mathbf{0}$ is an unstable equilibrium for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ (and $\mathbf{x}(t) = \mathbf{x}^0$ is an unstable equilibrium for $\mathbf{x}' = \mathbf{A}[\mathbf{x} - \mathbf{x}^0]$). Otherwise, $\mathbf{x}(t) = \mathbf{0}$ is a stable equilibrium for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ (and $\mathbf{x}(t) = \mathbf{x}^0$ is a stable equilibrium for $\mathbf{x}' = \mathbf{A}[\mathbf{x} - \mathbf{x}^0]$).

Furthermore, we have that $\mathbf{x}(t) = \mathbf{0}$ is an asymptotically stable equilibrium for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ (and $\mathbf{x}(t) = \mathbf{x}^0$ is an asymptotically stable equilibrium for $\mathbf{x}' = \mathbf{A}[\mathbf{x} - \mathbf{x}^0]$) whenever both r_1 and r_2 are negative, or whenever both eigenvalues are complex with $\lambda < 0$.

Nodes, Saddle Points, Centers and Spiral Points in General

Let (x_0, y_0) be a critical point for any 2×2 system of differential equations. Then:

1. The critical point (x_0, y_0) is called a *node* if, in a region about (x_0, y_0), all of the nonequilibrium trajectories are either straight half-lines with (x_0, y_0) as an endpoint, or become tangent to such half-lines at (x_0, y_0). For the basic constant matrix system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, $(0, 0)$ is a node if and only if r_1 and r_2 are both positive or are both negative.

   Sometimes, nodes are further subdivided into being either "proper" or "improper", with the node being *proper* if and only if, for every straight half-line with (x_0, y_0) as an endpoint, there is a trajectory which is that half line or which becomes tangent to that half line at (x_0, y_0). For the basic constant matrix system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, $(0, 0)$ is a proper node if and only if $r_1 = r_2$, in which case we may also refer to (x_0, y_0) as a *star node*.

2. The critical point (x_0, y_0) is called a *saddle point* if there are two nonequilibrium solutions $\mathbf{x}^1(t)$ and $\mathbf{x}^2(t)$ such that

$$\lim_{t \to -\infty} \mathbf{x}^1(t) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \quad \text{and} \quad \lim_{t \to \infty} \mathbf{x}^2(t) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \quad .$$

   For our basic constant matrix system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, $(0, 0)$ is a saddle point if and only if r_1 and r_2 are both real, but have opposite signs.

3. The critical point (x_0, y_0) is called a *center* if each open region about the point contains infinitely many trajectories that are closed loops about (x_0, y_0). For our basic constant matrix system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, $(0, 0)$ is a center if and only if the eigenvalues of $\mathbf{A}$ are purely imaginary; that is, $r_1 = i\omega$ and $r_2 = -i\omega$ with $\omega \neq 0$.

4. The critical point (x_0, y_0) is called a *spiral point* if all the nearby nonequilibrium trajectories are spirals converging to or diverging from that point without becoming tangent at (x_0, y_0) to any straight line. For our basic constant matrix system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, $(0, 0)$ is a spiral point if and only if the eigenvalues of $\mathbf{A}$ are complex with both real and imaginary parts being nonzero; that is, $r_1 = \lambda + i\omega$ and $r_2 = \lambda - i\omega$ with both $\lambda \neq 0$ and $\omega \neq 0$.

If you check the literature, you may find other terms used in classifying critical points. For example, the terms "sink node" and "source node" (or just "sink" and "source") are often used as synonyms for stable and unstable nodes, respectively, and a spiral point may be referred to as a "focal point". In addition, you may find a point $\mathbf{x}^0$ being referred as an "attractor" if either

$$\lim_{t \to +\infty} \mathbf{x}(t) = \mathbf{x}^0 \qquad \text{for every solution } \mathbf{x} \text{ that "gets close to" } \mathbf{x}_0$$

or

$$\lim_{t \to -\infty} \mathbf{x}(t) = \mathbf{x}^0 \qquad \text{for every solution } \mathbf{x} \text{ that "gets close to" } \mathbf{x}_0 \quad .$$

40.3 Phase Portraits for Imprecisely Known Systems

Note that the basic nature of a phase portrait for a 2×2 constant matrix system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ depends strongly on the whether the real and imaginary parts of the eigenvalues of $\mathbf{A}$ are positive, negative or zero. This can be a significant issue when the matrix $\mathbf{A}$ is only approximately known, such as when the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ arises from a "real-world application" and the components of $\mathbf{A}$ are determined by "real-world measurements". Such measurements are invariably approximate. As a result, the eigenvalues obtained when solving the characteristic equation $\det[\mathbf{A} - r\mathbf{I}] = 0$ will only be approximations of the true eigenvalues for the system of real interest. And, of course the corresponding computed eigenvectors will also only be approximations of the true eigenvectors for the system.

So let us consider some of the possibilities for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ when $\mathbf{A}$ is a real 2×2 matrix, and our computed eigenvalues are are known to be approximations of the true eigenvalues. For convenience, we will denote the computed eigenvalues by r_1 and r_2 if they are real, and by $\lambda \pm i\omega$ if they are complex. We will assume each computed r_1, r_2, λ and ω is known to be "within ϵ" of the true value, where ϵ is some (hopefully small) positive value. To simplify notation, let us write $r_1 \approx r_2$ or $\omega \approx 0$ whenever these computed values are close enough that it is possible for the corresponding equalities to hold for the true values.

$\epsilon < r_1 < r_2$ with $r_1 \not\approx r_2$

According to the computed eigenvalues, the origin is an unstable node. In this case, the true values of the eigenvalues still must both be positive and different. So, using the true eigenvalues, the origin is still an unstable node. Moreover, (assuming the errors are reasonably small) the computed eigenvectors will be reasonably close to the true eigenvectors. Consequently, the phase portraits generated by the computed values will be good approximations of the true phase portraits, and will all look something like that sketched in figure 38.4a on page 38–20.

$\epsilon < r_1 < r_2$ with $r_1 \approx r_2$

Again, according to the computed eigenvalues, the origin is an unstable node, and any phase portrait based on the computed eigenvalues will be similar to that sketched in figure 38.4a on page 38–20, with the trajectories becoming tangent to $\mathbf{u}^1$, the eigenvector corresponding to r_1 as they approach the origin. In this case, however, there are four basic possibilities for the true values of the eigenvalues:

1. The true eigenvalues are two different positive numbers. In this case, the origin is truly an unstable node, and the phase portraits drawn using the computed eigenvalues and eigenvectors will be similar to those of the true phase portraits. However, the vector the trajectories become tangent to as they approach the origin will depend strongly on whether or not $\mathbf{u}^1$ approximates the eigenvector of the smaller true eigenvalue.

2. The true eigenvalues are equal and real. Then there two more possibilities:
   - *(a)* If the true eigenvalue has geometric multiplicity two, then the true trajectories are all straight half-lines, and the origin is an unstable star node.
   - *(b)* If the true eigenvalue has geometric multiplicity one, then the origin is still an unstable node, but the true trajectories will be similar to those in figure 39.4a on page 39–16.

3. The true eigenvalues are complex, with the same real parts and small (but nonzero) imaginary parts. In this case, the origin is a unstable spiral point, and the true phase portrait will be somewhat similar to that in figure 39.2a on page 39–9.

Observe that, in this case, we can be sure the equilibrium solution $\mathbf{x}(t) = \mathbf{0}$ is unstable. The actual trajectories, however, can vary radically from case to case.

$r = \lambda \pm i\omega$ with $\epsilon < \lambda$ and $\epsilon < \omega$

According to the computed eigenvalues, the origin is an unstable spiral point, and any phase portrait is similar to that in figure 39.2a on page 39–9. In this case, the computed values of λ and ω are large enought to assure us that the true eigenvalues are also complex with similar values for the real and imaginary parts. In particular, the real part of the true eigenvalues will be a single positive value. Consequently, the origin must be an unstable spiral point, and a phase portrait based on the true eigenvalues and eigenvectors will be similar to that based on the computed values.

$r = \lambda \pm i\omega$ with $\lambda \approx 0$ and $\epsilon < \omega$

Here, the imaginary parts of the computed eigenvalues are large enough to ensure that the true eigenvalues have nonzero imaginary parts, but the real parts of the computed eigenvalues are so close to 0 that we have three possibilities:

1. The real part of each true eigenvalue is positive. If so, then the origin is an unstable spiral point, and a true phase portrait will consist of spirals with the direction of travel being away from the origin.

2. The real part of each true eigenvalue is zero. Then the origin is a (stable) center, and a true phase portrait will consist of a collection of ellipses about the origin (as in figure 40.1a).

3. The real part of each true eigenvalue is negative. Hence the origin is an asymptotically stable spiral point, and a true phase portrait will consist of spirals with the direction of travel being towards the origin.

Clearly, this case is problematic. You can be sure that the trajectories go about the origin, but the true stability of the equilibrium solution cannot be determined with any certainty.

Other Cases

In exercise 40.5, you will briefly go through some of the other possible cases, comparing what the computed eigenvalues tell us with what the true eigenvalues would have told us. One thing to observe: If the eigenvalues or the real parts of the eigenvalues of the computed eigenvalues are close to zero, then you have little idea as to whether the equilibrium solution $\mathbf{x}(t) = \mathbf{0}$ is asymptotically stable, merely stable, or unstable.

40.4 Phase Portraits for Large Constant Matrix Systems

In the previous two chapters, we pretty well demonstrated how the phase portrait of a 2×2 constant matrix system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ depends on the eigenvalues and eigenvectors of $\mathbf{A}$. We won't attempt an analogous development when $\mathbf{A}$ is $N\times N$ with $N > 2$. There are just too many cases to consider, and the two-dimensional medium of this text is not adequate for representing the corresponding phase portraits. Nonetheless, the basic ideas developed assuming $\mathbf{A}$ is 2×2 still apply, and you can use what we developed to help visualize the possible trajectories when $\mathbf{A}$ is, say, 3×3.

!►Example 40.2: *Consider the rather simple 3×3 constant matrix system*

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -2 & 3 & 0 \\ -3 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad .$$

The matrix for this system has three distinct eigenvalues. Two are complex, $r_{\pm} = -2 \pm 3i$, and the third is $r_3 = -1$ (with corresponding eigenvector $[0, 0, 1]^{\mathsf{T}}$). Together, they lead to the system's general solution

$$\mathbf{x}(t) = c_1\mathbf{x}^1(t) + c_2\mathbf{x}^2(t) + c_3\mathbf{x}^3(t) \tag{40.1}$$

with

$$\mathbf{x}^1(t) = \begin{bmatrix} \cos(3t) \\ -\sin(3t) \\ 0 \end{bmatrix} e^{-2t} \quad , \quad \mathbf{x}^2(t) = \begin{bmatrix} \sin(3t) \\ \cos(3t) \\ 0 \end{bmatrix} e^{-2t}$$

and

$$\mathbf{x}^3(t) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-t} \quad .$$

The first two terms in the general solution (which came from the complex eigenvalues $-2 \pm 3i$), trace out spirals "spiraling in" towards the origin in the XY–plane. That's what you get from general solution (40.1) if $c_3 = 0$ but at least one of the other two constants in formula (40.1) is nonzero.

The last term, corresponding to the eigenpair $\left(-1, [0, 0, 1]^{\mathsf{T}}\right)$, traces out straight line trajectories along the Z–axis traveling towards the origin. That's what you get from general solution (40.1) if $c_1 = c_2 = 0$.

Finally, if $c_3 \neq 0$ and at least one of the other two constants in formula (40.1) is nonzero, then the trajectory of $\mathbf{x}$ is combination of the spiraling motion and the travel in the Z-direction towards the origin; that is, $\mathbf{x}(t)$ traces out a "three-dimensional spiral" about the Z–axis heading into the origin as t increases.

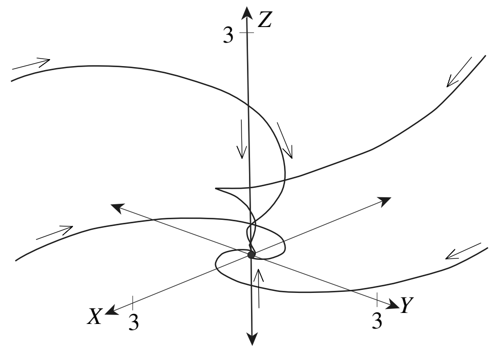


Figure 40.2: Seven trajectories for the 3×3 constant matrix system in example 40.2. Two are straight line trajectories along the Z–axis, two are spirals in the XY–plane, and two are "three-dimensional spirals" about the Z–axis. The seventh is the critical point $(0, 0, 0)$.

Examples of these trajectories have been sketched in figure 40.2. Note that $\mathbf{x}(t) = \mathbf{0}$ is still an equilibrium solution, and that, whatever the values of c_1, c_2 and c_3 are,

$$\lim_{t\to\infty} \left[c_1\mathbf{x}^1(t) + c_2\mathbf{x}^2(t) + c_3\mathbf{x}^3(t) \right] = 0 \quad .$$

So $\mathbf{x}(t) = \mathbf{0}$ is an asymptotically stable equilibrium solution for this system.

?▶*Exercise 40.1:* *Assume $\mathbf{x}^1$ and $\mathbf{x}^2$ are as in the above example, but that*

$$\mathbf{x}^3(t) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^t \quad .$$

How would a sketch of the trajectories given by

$$\mathbf{x}(t) = c_1\mathbf{x}^1(t) + c_2\mathbf{x}^2(t) + c_3\mathbf{x}^3(t)$$

differ from those sketched in figure 40.2?

40.5 Using Fundamental and Exponential Matrices

Fundamental Matrices

Let us briefly go back to a fairly general linear system of differential equations

$$\mathbf{x}' = \mathbf{P}\mathbf{x}$$

where $\mathbf{P}$ is a continuous $N \times N$ matrix-valued function on some interval (α, β). Recall (from section 37.5) that a fundamental matrix for this system is any $N \times N$ matrix-valued function on (α, β)

$$\mathbf{X} = \begin{bmatrix} x_1^1 & x_1^2 & \cdots & x_1^N \\ x_2^1 & x_2^2 & \cdots & x_2^N \\ \vdots & \vdots & \ddots & \vdots \\ x_N^1 & x_N^2 & \cdots & x_N^N \end{bmatrix}$$

whose columns

$$\mathbf{x}^1 = \begin{bmatrix} x_1^1 \\ x_2^1 \\ \vdots \\ x_N^1 \end{bmatrix} \quad , \quad \mathbf{x}^2 = \begin{bmatrix} x_1^2 \\ x_2^2 \\ \vdots \\ x_N^2 \end{bmatrix} \quad , \quad \cdots \quad \text{and} \quad \mathbf{x}^N = \begin{bmatrix} x_1^N \\ x_2^N \\ \vdots \\ x_N^N \end{bmatrix}$$

make up a fundamental set of solutions for our linear system $\mathbf{x}' = \mathbf{P}\mathbf{x}$.

!►*Example 40.3:* *From example 37.5 on page 37–15, we know that*

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} e^{2t}$$

is a general solution to the constant matrix system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \qquad \textit{with} \quad \mathbf{A} = \begin{bmatrix} 2 & -\frac{12}{5} & \frac{4}{5} \\ 2 & -3 & 1 \\ 0 & \frac{6}{5} & \frac{8}{5} \end{bmatrix} \quad .$$

This, of course, means that the set $\{\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3\}$ *with*

$$\mathbf{x}^1 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} e^{2t} \quad , \quad \mathbf{x}^2 = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} e^{-2t} \quad \textit{and} \quad \mathbf{x}^3 = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} e^{2t}$$

is a fundamental set of solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *. Hence, the matrix whose* k^{th} *columm given by* $\mathbf{x}^k$ *,*

$$\mathbf{X}(t) = \begin{bmatrix} e^{2t} & 2e^{-2t} & 3e^{2t} \\ e^{2t} & 3e^{-2t} & e^{2t} \\ 3e^{2t} & -e^{-2t} & 3e^{2t} \end{bmatrix} \quad ,$$

is a fundamental matrix for the above system of differential equations.

It is worth noting that, because each column $\mathbf{x}$ of $\mathbf{X}$ satisfies $\mathbf{x}' = \mathbf{P}\mathbf{x}$, the fundamental matrix $\mathbf{X}$, itself, satisfies the corresponding "matrix/matrix" system of differential equations

$$\mathbf{X}' = \mathbf{P}\mathbf{X}$$

where $\mathbf{X}'$, the derivative of $\mathbf{X}$, is simply the matrix obtained by differentiating each component of $\mathbf{X}$. Conversely, if $\mathbf{X}$ is a matrix-valued solution to the above, then $\mathbf{X}$ is easily seen to be a fundamental matrix for $\mathbf{x}' = \mathbf{P}\mathbf{x}$. Moreover, it should be clear that the theory discussed for the matrix/vector system $\mathbf{x}' = \mathbf{P}\mathbf{x}$ extends naturally to the matrix/matrix system $\mathbf{X}' = \mathbf{P}\mathbf{X}$. This includes the facts regarding the existence and uniqueness of solutions, as summarized in the following 'matrix/matrix system' version of lemma 37.2 on page 37–5:

Lemma 40.1 (existence and uniqueness of solutions)
Assume $\mathbf{P}$ *is a continuous* $N \times N$ *matrix-valued function over the interval* (α, β) *, and let* t_0 *and* $\mathbf{A}$ *be, respectively, a point in the interval* (α, β) *and a constant* $N \times N$ *matrix. Then the initial-value problem*

$$\mathbf{X}' = \mathbf{P}\mathbf{X} \qquad \textit{with} \quad \mathbf{X}(t_0) = \mathbf{A} \quad ,$$

has exactly one matrix-valued solution $\mathbf{X}$ *over the interval* (α, β) *. Moreover, this solution and its derivative are continuous over that interval.*

Now consider solving the initial-value problem

$$\mathbf{x}' = \mathbf{P}\mathbf{x} \qquad \text{with} \quad \mathbf{x}(t_0) = \mathbf{a}$$

where t_0 is some point in (α, β) and $\mathbf{a}$ is some constant vector. Again, let $\{\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^N\}$ be a fundamental set of solutions for $\mathbf{x}' = \mathbf{P}\mathbf{x}$, and let $\mathbf{X}$ be the corresponding fundamental matrix. In section 37.4 (see page 37–13), we saw that the general solution

$$\mathbf{x}(t) = c_1\mathbf{x}^1(t) + c_2\mathbf{x}^2(t) + \cdots + c_N\mathbf{x}^N$$

can be written more concisely as

$$\mathbf{x}(t) = [\mathbf{X}(t)]\mathbf{c}$$

where $\mathbf{c} = [c_1, c_2, \ldots, c_N]^{\mathsf{T}}$. So the solution to our initial-value problem can be given by

$$\mathbf{x}(t) = [\mathbf{X}(t)]\mathbf{c} \tag{40.2}$$

with $\mathbf{c}$ chosen so that

$$[\mathbf{X}(t_0)]\mathbf{c} = \mathbf{a} \quad .$$

But, as noted in section 37.5, fundamental matrices are invertible. So

$$\mathbf{c} = [\mathbf{X}(t_0)]^{-1}\mathbf{a} \quad .$$

Combining this with formula (40.2) for $\mathbf{x}$ gives us:

Theorem 40.2
Let $\mathbf{X}(t)$ *be a fundamental matrix for an* $N \times N$ *system* $\mathbf{x}' = \mathbf{P}\mathbf{x}$ *where* $\mathbf{P}$ *is a continuous matrix-valued function over an interval* (α, β) *. Let* t_0 *be in* (α, β) *, and let* $\mathbf{a}$ *be any constant vector. Then the solution to*

$$\mathbf{x}' = \mathbf{P}\mathbf{x} \qquad \textit{with} \quad \mathbf{x}(t_0) = \mathbf{a}$$

is given by

$$\mathbf{x}(t) = [\mathbf{X}^0(t)]\mathbf{a} \qquad \textit{where} \quad \mathbf{X}^0(t) = [\mathbf{X}(t)][\mathbf{X}(t_0)]^{-1} \quad .$$

Before going further, it should be noted that you can easily verify that the standard product rule

$$(\mathbf{F}\mathbf{G})' = \mathbf{F}'\mathbf{G} + \mathbf{F}\mathbf{G}' \tag{40.3}$$

holds whenever $\mathbf{F}$ and $\mathbf{G}$ are any matrix-valued functions whose product $\mathbf{F}\mathbf{G}$ exists. This holds whether or not both are $N \times N$.[1] Moreover, if $\mathbf{G}$ is a constant matrix, then

$$\mathbf{G}' = \mathbf{O}$$

where $\mathbf{O}$ is the corresponding constant matrix whose every component is 0, and

$$(\mathbf{F}\mathbf{G})' = \mathbf{F}'\mathbf{G} + \mathbf{F}\mathbf{G}' = \mathbf{F}'\mathbf{G} + \mathbf{F}\mathbf{O} = \mathbf{F}'\mathbf{G} \quad .$$

[1] Warning: In extending calculus to matrix-valued functions, it is important to remember that matrix multiplication is not commutative; that is, $\mathbf{F}\mathbf{G} \neq \mathbf{G}\mathbf{F}$ in general. Because of this, the matrix versions of some of the standard identities are not generally valid. For example, in exercise 40.8 you will show that, in general,

$$\left(\mathbf{F}^2\right)' \neq 2\mathbf{F}\mathbf{F}' \quad .$$

Applying the above and basic linear algebra, we see that the $\mathbf{X}^0$ described in the last theorem satisfies both

$$\frac{d\mathbf{X}^0}{dt} = \frac{d}{dt}\Big([\mathbf{X}(t)][\mathbf{X}(t_0)]^{-1}\Big) = [\mathbf{X}'][\mathbf{X}(t_0)]^{-1}$$
$$= (\mathbf{PX})[\mathbf{X}(t_0)]^{-1} = \mathbf{P}\Big([\mathbf{X}][\mathbf{X}(t_0)]^{-1}\Big) = \mathbf{PX}^0$$

and

$$\mathbf{X}^0(t_0) = [\mathbf{X}(t_0)][\mathbf{X}(t_0)]^{-1} = \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \quad .$$

So $\mathbf{X}^0$ is the one solution to the matrix/matrix initial-value problem

$$\mathbf{X}' = \mathbf{PX} \qquad \text{with} \quad \mathbf{X}(t_0) = \mathbf{I} \quad .$$

From this, our discussions in chapter 37, and the fact $\det(\mathbf{X}(t_0)) = \det(\mathbf{I}) = 1$, it quickly follows that $\mathbf{X}^0$ is a fundamental matrix for $\mathbf{x}' = \mathbf{Px}$. Clearly, it is the one we would want if we had to solve

$$\mathbf{x}' = \mathbf{Px} \qquad \text{with} \quad \mathbf{x}(t_0) = \mathbf{a}$$

for several different choices of $\mathbf{a}$.

If you think about it, there are at least two ways of finding this $\mathbf{X}^0$:

1. Take any fundamental matrix $\mathbf{X}(t)$ already found for $\mathbf{x}' = \mathbf{Px}$, compute $\mathbf{X}(t_0)$ and $[\mathbf{X}(t_0)]^{-1}$, and then set $\mathbf{X}^0(t) = [\mathbf{X}(t)][\mathbf{X}(t_0)]^{-1}$.

2. First solve the N initial-value problems

$$\frac{d\mathbf{x}^k}{dt} = \mathbf{Px}^k \qquad \text{with} \quad \mathbf{x}^k(t_0) = \mathbf{e}^k$$

where $\mathbf{e}^k$ is the $N \times 1$ column matrix whose components are all 0 except for the k^{th} component which is 1. Then use each $\mathbf{x}^k(t)$ just found as the k^{th} column of $\mathbf{X}^0(t)$.

We'll usually use the first method above in what follows. But before doing so, let's further limit our choices for the matrix $\mathbf{P}$.

The Exponential Matrix

The Exponential Matrix as a Fundamental Matrix

Consider, now, just the cases we've been considering in the last few chapters; namely, where $\mathbf{P}(t) = \mathbf{A}$ with $\mathbf{A}$ being a constant real $N \times N$ matrix. If $t_0 = 0$, then we would be particularly interested in the fundamental matrix $\mathbf{X} = \mathbf{X}^0$ satisfying

$$\mathbf{X}' = \mathbf{AX} \qquad \text{with} \quad \mathbf{X}(0) = \mathbf{I} \quad . \tag{40.4}$$

Observe the similarity between this initial-value problem and the first-order initial-value problem

$$x' = ax \qquad \text{with} \quad x(0) = 1$$

where a is some constant. This is a simple problem with a simple solution,

$$x(t) = e^{at} \qquad \text{for all} \quad t \quad .$$

In analogy to this, we will refer to the solution $\mathbf{X}^0$ of initial-value problem (40.4) as the *exponential fundamental matrix (for* $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *)*, writing

$$\mathbf{X}^0(t) = e^{\mathbf{A}t} = \exp(\mathbf{A}t) \quad .$$

Thus, by definition, $e^{\mathbf{A}t}$ is the matrix-valued function on $(-\infty, \infty)$ satisfying

$$\frac{de^{\mathbf{A}t}}{dt} = \mathbf{A}e^{\mathbf{A}t} \qquad \text{with} \quad e^{\mathbf{A}0} = \mathbf{I} \quad .$$

We can then write the solution to

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} \qquad \text{with} \quad \mathbf{x}(0) = \mathbf{a}$$

as

$$x(t) = e^{\mathbf{A}t}\mathbf{a} \quad .$$

From the discussion in the previous subsection, we know of two ways of finding $e^{\mathbf{A}t}$:

1. Set $e^{\mathbf{A}t} = [\mathbf{X}(t)][\mathbf{X}(0)]^{-1}$ where $\mathbf{X}$ is any known fundamental matrix for $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

2. Set $e^{\mathbf{A}t}$ equal to the $N \times N$ matrix whose k^{th} column is the solution to

$$\frac{d\mathbf{x}^k}{dt} = \mathbf{P}\mathbf{x}^k \qquad \text{with} \quad \mathbf{x}^k(0) = \mathbf{e}^k$$

where $\mathbf{e}^k$ is the $N \times 1$ column matrix whose components are all 0 except for the k^{th} component which is 1 .

!►Example 40.4: *Let us find* $e^{\mathbf{A}t}$ *when*

$$\mathbf{A} = \begin{bmatrix} 2 & -\frac{12}{5} & \frac{4}{5} \\ 2 & -3 & 1 \\ 0 & \frac{6}{5} & \frac{8}{5} \end{bmatrix}$$

A few pages ago, in example 40.3 on page 40–8, we found one fundamental matrix for the corresponding system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *to be*

$$\mathbf{X}(t) = \begin{bmatrix} e^{2t} & 2e^{-2t} & 3e^{2t} \\ e^{2t} & 3e^{-2t} & e^{2t} \\ 3e^{2t} & -e^{-2t} & 3e^{2t} \end{bmatrix} \quad .$$

So

$$[\mathbf{X}(0)] = \begin{bmatrix} e^{2\cdot 0} & 2e^{-2\cdot 0} & 3e^{2\cdot 0} \\ e^{2\cdot 0} & 3e^{-2\cdot 0} & e^{2\cdot 0} \\ 3e^{2\cdot 0} & -e^{-2\cdot 0} & 3e^{2\cdot 0} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 3 & -1 & 3 \end{bmatrix} \quad .$$

Computing the inverse of this (using whichever method you prefer), you get

$${}^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 3 & -1 & 3 \end{bmatrix}^{-1} = \cdots = \begin{bmatrix} -\frac{1}{2} & \frac{9}{20} & \frac{7}{20} \\ 0 & \frac{3}{10} & -\frac{1}{10} \\ \frac{1}{2} & -\frac{7}{20} & -\frac{1}{20} \end{bmatrix} \quad .$$

Thus,

$$
\begin{aligned}
e^{\mathbf{A}t} &= \mathbf{X}^0(t) = [\mathbf{X}(t)][\mathbf{X}(t_0)]^{-1} \\
&= \begin{bmatrix} e^{2t} & 2e^{-2t} & 3e^{2t} \\ e^{2t} & 3e^{-2t} & e^{2t} \\ 3e^{2t} & -e^{-2t} & 3e^{2t} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{9}{20} & \frac{7}{20} \\ 0 & \frac{3}{10} & -\frac{1}{10} \\ \frac{1}{2} & -\frac{7}{20} & -\frac{1}{20} \end{bmatrix} \\
&= \begin{bmatrix} e^{2t} & -\frac{3}{5}e^{2t} + \frac{3}{5}e^{-2t} & \frac{1}{5}e^{2t} - \frac{1}{5}e^{-2t} \\ 0 & \frac{1}{10}e^{2t} + \frac{9}{10}e^{-2t} & \frac{3}{10}e^{2t} - \frac{3}{10}e^{-2t} \\ 0 & \frac{3}{10}e^{2t} - \frac{3}{10}e^{-2t} & \frac{9}{10}e^{2t} + \frac{1}{10}e^{-2t} \end{bmatrix} \quad .
\end{aligned}
$$

While the formula for the fundamental matrix $e^{\mathbf{A}t}$ *is not as simple as what we had originally obtained for the first fundamental matrix* $\mathbf{X}$ *, this more complicated formula will simplify solving*

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} \qquad \text{with} \quad \mathbf{x}(0) = \mathbf{a} \quad .$$

For example, if $\mathbf{a} = [1, 2, 3]^{\mathsf{T}}$ *, then*

$$
\begin{aligned}
\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{a} &= \begin{bmatrix} e^{2t} & -\frac{3}{5}e^{2t} + \frac{3}{5}e^{-2t} & \frac{1}{5}e^{2t} - \frac{1}{5}e^{-2t} \\ 0 & \frac{1}{10}e^{2t} + \frac{9}{10}e^{-2t} & \frac{3}{10}e^{2t} - \frac{e}{10}e^{-2t} \\ 0 & \frac{3}{10}e^{2t} - \frac{3}{10}e^{-2t} & \frac{9}{10}e^{2t} + \frac{1}{10}e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
&= \cdots = \begin{bmatrix} \frac{2}{5}e^{2t} + \frac{3}{5}e^{-2t} \\ \frac{11}{10}e^{2t} + \frac{9}{10}e^{-2t} \\ \frac{33}{10}e^{2t} - \frac{3}{10}e^{-2t} \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 4 \\ 11 \\ 33 \end{bmatrix} e^{2t} + \frac{1}{10}\begin{bmatrix} 6 \\ 9 \\ 3 \end{bmatrix} e^{-2t} \quad .
\end{aligned}
$$

And if $\mathbf{a} = [20, 0, 30]^{\mathsf{T}}$ *, then*

$$
\begin{aligned}
\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{a} &= \begin{bmatrix} e^{2t} & -\frac{3}{5}e^{2t} + \frac{3}{5}e^{-2t} & \frac{1}{5}e^{2t} - \frac{1}{5}e^{-2t} \\ 0 & \frac{1}{10}e^{2t} + \frac{9}{10}e^{-2t} & \frac{3}{10}e^{2t} - \frac{e}{10}e^{-2t} \\ 0 & \frac{3}{10}e^{2t} - \frac{3}{10}e^{-2t} & \frac{9}{10}e^{2t} + \frac{1}{10}e^{-2t} \end{bmatrix} \begin{bmatrix} 20 \\ 0 \\ 30 \end{bmatrix} \\
&= \cdots = \begin{bmatrix} 26e^{2t} - 6e^{-2t} \\ 9e^{2t} - 9e^{-2t} \\ 27e^{2t} + 3e^{-2t} \end{bmatrix} = \begin{bmatrix} 26 \\ 9 \\ 27 \end{bmatrix} e^{2t} + \begin{bmatrix} -6 \\ -9 \\ 3 \end{bmatrix} e^{-2t} \quad .
\end{aligned}
$$

The Exponential Matrix as a Power Series

More generally, the *exponential of any* $N \times N$ *matrix* $\mathbf{M}$ is defined by the exponential Taylor series

$$e^{\mathbf{M}} = \exp(\mathbf{M}) = \sum_{k=0}^{\infty} \frac{\mathbf{M}^k}{k!} \tag{40.5}$$

where

$$\mathbf{M}^0 = \mathbf{I} \quad , \quad \mathbf{M}^1 = \mathbf{M} \quad , \quad \mathbf{M}^2 = \mathbf{MM} \quad , \quad \mathbf{M}^3 = \mathbf{MMM} \quad , \quad \ldots \quad .$$

In particular,

$$\begin{aligned} e^{\mathbf{A}t} = \exp(\mathbf{A}t) &= \sum_{k=0}^{\infty} \frac{(\mathbf{A}t)^k}{k!} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!} t^k \\ &= \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2 t^2 + \frac{1}{3!}\mathbf{A}^3 t^3 + \cdots \quad . \end{aligned} \tag{40.6}$$

The theory for power series of square matrices is a straightforward extension of the theory for power series discussed in chapter 29, and we can safely use the "matrix" versions of the results discussed in 29 (provided we take into account the fact that matrix multiplication is not commutative). From that, we know the series for $e^{\mathbf{M}}$ converges for every square matrix $\mathbf{M}$. Moreover, for any constant $N \times N$ matrix $\mathbf{A}$,

$$\begin{aligned} \frac{de^{\mathbf{A}t}}{dt} = \frac{d}{dt}\left[\sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!} t^k\right] &= \sum_{k=1}^{\infty} \frac{\mathbf{A}^k}{k!} k t^{k-1} \\ &= \mathbf{A} \sum_{k=1}^{\infty} \frac{\mathbf{A}^{k-1}}{(k-1)!} t^{k-1} = \mathbf{A} \sum_{n=0}^{\infty} \frac{\mathbf{A}^n}{(n)!} t^n = \mathbf{A} e^{\mathbf{A}t} \end{aligned}$$

and

$$e^{\mathbf{A}0} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!} 0^k = \mathbf{I} + \mathbf{A}0 + \frac{1}{2!}\mathbf{A}^2 0^2 + \frac{1}{3!}\mathbf{A}^3 0^3 + \cdots = \mathbf{I} \quad ,$$

confirming that the solution to initial-value problem (40.4) is, indeed, given by $\mathbf{X}^0(t) = e^{\mathbf{A}t}$ using the above power series definition of the exponential.

This means we have two definitions for $e^{\mathbf{A}t}$: The above power series definition, and the definition of $e^{\mathbf{A}t}$ as the fundamental matrix for $\mathbf{X}' = \mathbf{AX}$ with $\mathbf{X}(0) = \mathbf{I}$. By the uniqueness of solutions to this system, we know that the two definitions yield the very same matrix-valued function for $e^{\mathbf{A}t}$. Do observe, however, that the power series definition makes it clear that $e^{\mathbf{A}t}$ depends just on 'the matrix $\mathbf{M} = \mathbf{A}t$'. And since

$$\mathbf{A}t = t\mathbf{A} \qquad \text{and} \qquad \mathbf{A}(t - t_0) = \mathbf{A}t - \mathbf{A}t_0 \quad ,$$

we immediately have

$$e^{\mathbf{A}t} = e^{t\mathbf{A}} \qquad \text{and} \qquad e^{\mathbf{A}(t-t_0)} = e^{\mathbf{A}t - \mathbf{A}t_0} \quad .$$

(We'll expand on the last equality in a little bit.)

Formula (40.6) provides another way for computing the fundamental matrix satisfying initial-value problem (40.4). However, unless $\mathbf{A}$ is particularly simple, it may be easier to compute formula (40.6) for a given $\mathbf{A}$ by solving initial-value problem (40.4) as discussed earlier in this section.

!►Example 40.5: *Let α be any constant or function of t, and set*

$$\mathbf{P} = \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} \quad \textit{and} \quad \mathbf{Q} = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix} \quad .$$

By basic matrix multiplication,

$$\mathbf{P}^2 = \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \alpha^2 & 0 \\ 0 & 0 \end{bmatrix} \quad \textit{and} \quad \mathbf{Q}^2 = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad .$$

Continuing these calculations, we see that

$$\mathbf{P}^k = \begin{bmatrix} \alpha^k & 0 \\ 0 & 0 \end{bmatrix} \quad \textit{for} \quad k = 1, 2, 3, \ldots \quad ,$$

but that

$$\mathbf{Q}^k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{O} \quad \textit{for} \quad k = 2, 3, \ldots \quad .$$

So

$$\begin{aligned} e^{\mathbf{P}} &= \sum_{k=0}^{\infty} \frac{\mathbf{P}^k}{k!} = \mathbf{P}^0 + \sum_{k=1}^{\infty} \frac{\mathbf{P}^k}{k!} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sum_{k=1}^{\infty} \frac{1}{k!}\begin{bmatrix} \alpha^k & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{\alpha} & 0 \\ 0 & 1 \end{bmatrix} \quad , \end{aligned}$$

while

$$\begin{aligned} e^{\mathbf{Q}} &= \sum_{k=0}^{\infty} \frac{\mathbf{Q}^k}{k!} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{1!}\begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \cdots = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \quad . \end{aligned}$$

In exercise 40.12, you will extend the computations done above for $e^{\mathbf{P}}$ to show that

$$\text{if} \quad \mathbf{P} = \begin{bmatrix} r_1 & 0 & 0 & \cdots & 0 \\ 0 & r_2 & 0 & \cdots & 0 \\ 0 & 0 & r_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ldots \\ 0 & 0 & 0 & \cdots & r_N \end{bmatrix} \quad \text{then} \quad e^{\mathbf{P}} = \begin{bmatrix} e^{r_1} & 0 & 0 & \cdots & 0 \\ 0 & e^{r_2} & 0 & \cdots & 0 \\ 0 & 0 & e^{r_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ldots \\ 0 & 0 & 0 & \cdots & e^{r_N} \end{bmatrix} \quad .$$

However, the computation of $e^{\mathbf{Q}}$ in the last example shows that, in general, the entries in the exponential of a given matrix are not simply the exponentials of the corresponding entries of the given matrix.

Shifted Exponentials

Keep in mind that $\mathbf{X}^0(t) = e^{\mathbf{A}t}$ is the matrix-valued function of t satisfying

$$\frac{d\mathbf{X}^0}{dt} = \mathbf{A}\mathbf{X}^0 \quad \text{with} \quad \mathbf{X}^0(0) = \mathbf{I} \quad .$$

Remember, also, that if $\mathbf{x}^0(t)$ is the vector-valued function of t satisfying

$$\frac{d\mathbf{x}^0}{dt} = \mathbf{A}\mathbf{x}^0 \qquad \text{with} \quad \mathbf{x}^0(0) = \mathbf{a}$$

for some vector $\mathbf{a}$, then, for any real value t_0, $\mathbf{x}(t) = \mathbf{x}^0(t - t_0)$ is the vector-valued function of t satisfying[2]

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \qquad \text{with} \quad \mathbf{x}(t_0) = \mathbf{a} \quad .$$

Using either this or the power series formula for the exponential, you can easily verify that, for any real value t_0, the solution to

$$\mathbf{X}' = \mathbf{A}\mathbf{X} \qquad \text{with} \quad \mathbf{X}(t_0) = \mathbf{I}$$

is given by the corresponding *shifted exponential fundamental matrix*, $\mathbf{X}(t) = \mathbf{X}^0(t - t_0) = e^{\mathbf{A}(t-t_0)}$. Accordingly, the solution to

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \qquad \text{with} \quad \mathbf{x}(t_0) = \mathbf{a}$$

can be given by

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{a} \quad .$$

Let us also observe that

$$\mathbf{X}(0) = \mathbf{X}(0 - t_0) = e^{\mathbf{A}(0-t_0)} = e^{-\mathbf{A}t_0} \quad .$$

So this shifted exponential, $\mathbf{X}(t) = e^{\mathbf{A}(t-t_0)}$, is also the solution to the matrix/matrix system

$$\mathbf{X}' = \mathbf{A}\mathbf{X} \qquad \text{with} \quad \mathbf{X}(0) = e^{-\mathbf{A}t_0} \quad ,$$

a fact that will soon be useful.

Identities and Limitations of Matrix Exponentials

Just as in calculus of the matrices, the matrix versions of the classic exponential identities are sometimes NOT valid. For example, while we know $e^{a+b} = e^a e^b$ for any two numbers a and b, we can also find two $N \times N$ matrices $\mathbf{A}$ and $\mathbf{B}$ such that

$$e^{\mathbf{A}+\mathbf{B}} \neq e^{\mathbf{A}} e^{\mathbf{B}}$$

(see exercise 40.13). Fortunately, we do have the identity in the next lemma. We will use it later when solving nonhomogeneous systems of differential equations.

Lemma 40.3
Let $\mathbf{A}$ be any $N \times N$ matrix, and let t and s be any two real numbers. Then

$$e^{\mathbf{A}(t-s)} = e^{\mathbf{A}t} e^{-\mathbf{A}s} \quad . \tag{40.7}$$

PROOF: As we noted earlier, $e^{\mathbf{A}(t-s)}$ is the solution to

$$\frac{d\mathbf{X}}{dt} = \mathbf{A}\mathbf{X} \qquad \text{with} \quad \mathbf{X}(0) = e^{-\mathbf{A}s} \quad .$$

Now let

$$\mathbf{X}(t) = e^{\mathbf{A}t} e^{-\mathbf{A}s} \quad ,$$

[2] If you don't remember, see the discussion of *Trajectories and Solutions* starting on page 36–12.

and observe that, for each value of s ,

$$\frac{d\mathbf{X}}{dt} = \frac{d}{dt}\left[e^{\mathbf{A}t}e^{-\mathbf{A}s}\right] = \frac{de^{\mathbf{A}t}}{dt}e^{-\mathbf{A}s} = \left[\mathbf{A}e^{\mathbf{A}t}\right]e^{-\mathbf{A}s} = \mathbf{A}\left[e^{\mathbf{A}t}e^{-\mathbf{A}s}\right] = \mathbf{AX}$$

and

$$\mathbf{X}(0) = e^{\mathbf{A}0}e^{-\mathbf{A}s} = e^{\mathbf{O}}e^{-\mathbf{A}s} = \mathbf{I}e^{-\mathbf{A}s} = e^{-\mathbf{A}s} \quad .$$

So both

$$e^{\mathbf{A}(t-s)} \quad \text{and} \quad e^{\mathbf{A}t}e^{-\mathbf{A}s}$$

are solutions to

$$\frac{d\mathbf{X}}{dt} = \mathbf{AX} \quad \text{with} \quad \mathbf{X}(0) = e^{-\mathbf{A}s} \quad .$$

Since such a system can only have one solution, these two solutions must be the same; that is,

$$e^{\mathbf{A}(t-s)} = e^{\mathbf{A}t}e^{-\mathbf{A}s} \quad \text{for all} \quad t \quad .$$

∎

Since identity (40.7) holds for every pair of real numbers t and s , we can use it with $s = t$,

$$e^{\mathbf{A}t}e^{-\mathbf{A}t} = e^{\mathbf{A}(t-t)} = e^{\mathbf{A}0} = \mathbf{I} \quad .$$

Thus,

Lemma 40.4
Let $\mathbf{A}$ be any $N \times N$ matrix. Then $e^{\mathbf{A}t}$ is invertible, and its inverse is $e^{-\mathbf{A}t}$.

As an immediate consequence, the inverse of $e^{\mathbf{A}t}$ can be obtained by simply replacing the t with $-t$ in whatever formula you have for $e^{\mathbf{A}t}$.

40.6 Using Laplace Transforms

There are those who like to use the Laplace transform to solve constant matrix systems of differential equations. Since this approach works equally well for nonhomogeneous systems, we will delay any further discussion of this use of Laplace transforms until we are discussing nonhomogeneous systems. (If you cannot wait, go ahead and turn to section 41.4 on page 41–16.)

40.7 Euler Systems

What Is An Euler System?

We will refer to any $N \times N$ system of differential equations as an *Euler* system if it can be given by

$$\mathbf{x}' = \frac{1}{t}\mathbf{Ax} \quad \text{for} \quad 0 < t < \infty$$

or, equivalently, by

$$t\mathbf{x}' = \mathbf{Ax} \quad \text{for} \quad 0 < t < \infty$$

where $\mathbf{A}$ is constant $N \times N$ matrix with real components. Such a system is a natural first-order system extension of the Euler equations discussed in chapter 18 (see exercises 35.17 and 35.18 in the *Addendum to Chapter 35*. As with the Euler equation solutions, the domains of the solutions to Euler systems will not include $t = 0$. For convenience, we will insist that $0 < t < \infty$.

Our interest in Euler systems is mainly in having systems other than constant matrix systems that we can 'easily' solve. And Euler systems can be solved just as easily as constant matrix systems. In fact, with only a few hints, you should be able to figure out how to solve Euler systems by building on what you know about Euler equations and what we've learned about solving constant matrix systems.

Direction Fields and Trajectories

Observe that an Euler system is a homogeneous linear system, but, because of the t^{-1} factor, it is not an autonomous system. So the "velocity" $\mathbf{x}'(t)$ of a solution as it goes through a given position will depend both on that position and on when (i.e., t) it goes through that point. Still, as illustrated in the next example, changes in t only affects the magnitude of $\mathbf{x}'(t)$, not the direction. Consequently, as we'll see, we can still construct a direction field for an Euler system that does not depend on t.

!►Example 40.6: *Let us consider the direction arrows at various points on the XY–plane for both*

$$\mathbf{x}' = \frac{1}{t}\mathbf{A}\mathbf{x} \qquad \text{and} \qquad \mathbf{x}' = \mathbf{A}\mathbf{x} \qquad \text{with} \quad \mathbf{A} = \frac{1}{2}\begin{bmatrix} 5 & -3 \\ 3 & -5 \end{bmatrix} \quad .$$

The first is an Euler system, while the second is a constant matrix system.

Clearly, $\mathbf{x}(t) = \mathbf{0}$ is a constant solution for each.

Now, let $\mathbf{x}^E(t)$ and $\mathbf{x}^{CM}(t)$ be solutions, respectively, to the Euler system and the constant matrix system that satisfy

$$\mathbf{x}^E(t_0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}^{CM}(t_0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

for some $t_0 > 0$.

For the constant matrix system, the direction arrow that we would sketch at $(1, 2)$ would be a short arrow in the same direction as

$$\left.\frac{d\mathbf{x}^{CM}}{dt}\right|_{t=t_0} = \frac{1}{2}\begin{bmatrix} 5 & -3 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 \\ -7 \end{bmatrix} \quad ,$$

which, as should be expected, does not depend on t_0.

However, for the Euler system, the direction arrow that we would sketch at $(1, 2)$ would be a short arrow in the same direction as

$$\left.\frac{d\mathbf{x}^{E}}{dt}\right|_{t=t_0} = \frac{1}{2t_0}\begin{bmatrix} 5 & -3 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{2t_0}\begin{bmatrix} -1 \\ -7 \end{bmatrix} \quad ,$$

which does depend on t_0, but only very simple manner: Its dependence is in a factor that scales the vector by some positive quantity. But that does not affect the direction. In fact,

$$\left.\frac{d\mathbf{x}^{E}}{dt}\right|_{t=t_0} = \frac{1}{t_0}\left.\frac{d\mathbf{x}^{CM}}{dt}\right|_{t=t_0} \quad .$$

So, the direction arrow of the Euler system at $(x, y) = (1, 2)$

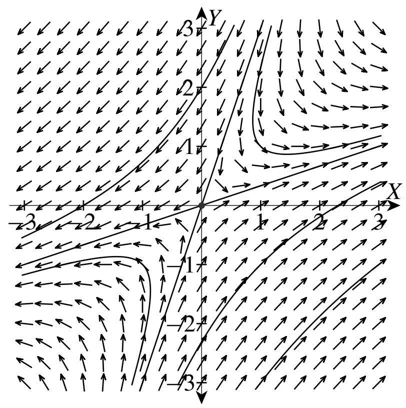


Figure 40.3: A direction field and some trajectories for the Euler system in example 40.6.

1. *does not change with* t *, and*

2. *is the same as the direction arrow at* $(x, y) = (1, 2)$ *for the constant matrix system.*

Of course, there was nothing special about the point $(1, 2)$ *. If* (x_0, y_0) *is any point other than the origin, and* $\mathbf{x}^E(t)$ *and* $\mathbf{x}^{CM}(t)$ *are solutions, respectively, to the Euler system and the constant matrix system that satisfy*

$$\mathbf{x}^E(t_0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}^{CM}(t_0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

for some $t_0 > 0$ *, then*

$$\left. \frac{d\mathbf{x}^E}{dt} \right|_{t=t_0} = \frac{1}{2t_0} \begin{bmatrix} 5 & -3 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \frac{1}{t_0} \left. \frac{d\mathbf{x}^{CM}}{dt} \right|_{t=t_0} \quad ,$$

telling us that the direction arrow of the Euler system at (x_0, y_0)

1. *does not change with* t *, and*

2. *is the same as the direction arrow at* (x_0, y_0) *for the constant matrix system.*

Thus, we can construct a well-defined direction field for our Euler system, and this direction field

1. *does not change with* t *, and*

2. *is also a direction field for the above constant matrix system.*

Moreover, since the trajectories of the solutions can be determined from the direction field, the trajectories of the solutions to the above Euler system

1. *do not vary with* t *, and*

2. *are the same as the trajectories for the above constant matrix system.*

A direction field for our Euler system, along with a few trajectories have been sketched in figure 40.3.

It should be clear that the observations made in the example hold for any Euler system. In summary:

Lemma 40.5
Let $\mathbf{A}$ *be a constant* $N \times N$ *matrix with real components. Then direction fields and trajectories for the Euler system*

$$\mathbf{x}' = \frac{1}{t}\mathbf{A}\mathbf{x} \qquad \textit{for} \quad 0 < t < \infty$$

are well-defined and do not depend on t *. Moreover these direction fields and trajectories are also direction fields and trajectories for the constant matrix system*

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \qquad \textit{for} \quad 0 < t < \infty \quad .$$

Keep in mind, however, that the trajectories of an Euler system are traced out by the solutions as t goes from 0 to ∞, unlike the trajectories of a constant matrix system that are traced out by the solutions as t goes from $-\infty$ to ∞.

Solving Euler Systems

Our solving of a constant matrix system began with the observation that $\mathbf{x}(t) = \mathbf{u}e^{rt}$ satisfies $\mathbf{x}' = \mathbf{A}\mathbf{x}$ if $(r, \mathbf{u})$ is an eigenpair for $\mathbf{A}$, with the e^{rt} factor in this solution being inspired by the basic solutions to the linear homogeneous differential equations with constant coefficients. From our study of Euler equations in chapter 18 you probably suspect that, for Euler systems, we will want to use t^r instead of e^{rt}. Well, you are correct.

Theorem 40.6
Let $\mathbf{A}$ *be a constant* $N \times N$ *matrix with real components. If* $(r, \mathbf{u})$ *is an eigenpair for* $\mathbf{A}$*, then* $\mathbf{x}(t) = \mathbf{u}t^r$ *is a solution to the Euler system*

$$\mathbf{x}' = \frac{1}{t}\mathbf{A}\mathbf{x} \qquad \textit{for} \quad 0 < t < \infty \quad .$$

?►*Exercise 40.2:* *Verify the above theorem.*

As an immediate corollary, we have:

Corollary 40.7
Let $\mathbf{A}$ *be a constant* $N \times N$ *matrix with real components. Assume* $\mathbf{A}$ *has a complete set of eigenvectors* $\{\mathbf{u}^1, \mathbf{u}^2, \ldots, \mathbf{u}^N\}$*, and, for* $k = 1, 2, \ldots, N$*, let* r_k *be the eigenvalue corresponding to* $\mathbf{u}^k$*. Then the Euler system*

$$\mathbf{x}' = \frac{1}{t}\mathbf{A}\mathbf{x} \qquad \textit{for} \quad 0 < t < \infty$$

has

$$\left\{\mathbf{u}^1 t^{r_1},\ \mathbf{u}^2 t^{r_2},\ \cdots,\ \mathbf{u}^N t^{r_N}\right\}$$

as a fundamental set of solutions, and

$$\mathbf{x}(t) = c_1\mathbf{u}^1 t^{r_1} + c_2\mathbf{u}^2 t^{r_2} + \cdots + c_N\mathbf{u}^N t^{r_N}$$

as a general solution.

!► ***Example 40.7:*** *You can easily verify that*

$$\left(-2\,,\begin{bmatrix}1\\3\end{bmatrix}\right) \qquad \text{and} \qquad \left(2\,,\begin{bmatrix}3\\1\end{bmatrix}\right)$$

are eigenpairs for the matrix

$$\mathbf{A} = \frac{1}{2}\begin{bmatrix}5 & -3\\3 & -5\end{bmatrix} \quad .$$

So, according to the above, the Euler system

$$\mathbf{x}' = \frac{1}{t}\mathbf{A}\mathbf{x}$$

has

$$\left\{\mathbf{x}^1(t),\ \mathbf{x}^2(t)\right\} = \left\{\begin{bmatrix}1\\3\end{bmatrix}t^{-2}\,,\ \begin{bmatrix}3\\1\end{bmatrix}t^2\right\}$$

as a fundamental set of solutions, and

$$\mathbf{x}(t) = c_1\begin{bmatrix}1\\3\end{bmatrix}t^{-2} + c_2\begin{bmatrix}3\\1\end{bmatrix}t^2$$

as a general solution.

In addition, from this fundamental set of solutions, we can construct the corresponding fundamental matrix for the Euler system,

$$\mathbf{X}(t) = \begin{bmatrix}t^{-2} & 3t^2\\3t^{-2} & t^2\end{bmatrix}$$

and the fundamental matrix $\mathbf{X}^0$ *that also satisfies* $\mathbf{X}^0(1) = \mathbf{I}$ *is*

$$\begin{aligned}\mathbf{X}^0(t) &= [\mathbf{X}(t)][\mathbf{X}(1)]^{-1}\\ &= \begin{bmatrix}t^{-2} & 3t^2\\3t^{-2} & t^2\end{bmatrix}\left(\begin{bmatrix}1 & 3\\3 & 1\end{bmatrix}^{-1}\right)\\ &= \begin{bmatrix}t^{-2} & 3t^2\\3t^{-2} & t^2\end{bmatrix}\left(\frac{1}{8}\begin{bmatrix}-1 & 3\\3 & -1\end{bmatrix}\right) = \frac{1}{8}\begin{bmatrix}-t^{-2}+9t^2 & 3t^{-2}-3t^2\\-3t^{-2}+3t^2 & 9t^{-2}-t^2\end{bmatrix} \quad .\end{aligned}$$

As with the constant matrix systems, there are two particular complications that may arise:

1. One is that $\mathbf{A}$ may have a complex eigenvalue r. If so, then taking the real and imaginary parts of the corresponding solution $\mathbf{u}t^r$ will yield real-valued solutions to the Euler system, just as taking the real and imaginary parts of solutions of the form $\mathbf{u}e^{rt}$ yielded real-valued solutions to the constant matrix system in section 39.1.

2. The other is that $\mathbf{A}$ might not have a complete set of eigenvectors. If so, then an adaptation of the development discussed in sections 39.3 and 39.5 is in order. The details of dealing with this complication will be left to the interested reader.

40.8 Using Similarity Transforms

Two constant $N \times N$ matrices $\mathbf{A}$ and $\mathbf{B}$ are said to be related by a *similarity transform* if and only if there is an invertible $N \times N$ matrix $\mathbf{T}$ such that

$$\mathbf{B} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} \quad . \tag{40.8}$$

Note that this is completely equivalent to saying

$$\mathbf{A} = \mathbf{T}\mathbf{B}\mathbf{T}^{-1}$$

(just multiply both sides of equation (40.8) on the left by $\mathbf{T}$ and on the right by $\mathbf{T}^{-1}$).

Similarity transforms are quite important in linear algebra. For example, from the theory of linear algebra, we have the following:

Theorem 40.8
Let $\mathbf{A}$ be a constant $N \times N$ matrix. Then there is an invertible matrix $\mathbf{T}$ such that

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{B}$$

with

$$\mathbf{B} = \begin{bmatrix} \rho_1 & \sigma_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \rho_2 & \sigma_3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \rho_3 & \sigma_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \rho_{N-2} & \sigma_{N-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \rho_{N-1} & \sigma_N \\ 0 & 0 & 0 & \cdots & 0 & 0 & \rho_N \end{bmatrix}$$

where:

1. *The ρ_k's are the eigenvalues of $\mathbf{A}$ with*

$$\rho_1 \leq \rho_2 \leq \rho_3 \leq \cdots \leq \rho_N \quad .$$

2. *The number of times a particular eigenvalue appears in the main diagonal is equal to the algebraic multiplicity of that eigenvalue (as an eigenvalue of $\mathbf{A}$).*
3. *Each σ_k is either 0 or 1 .*
4. *If $\rho_{k-1} \neq \rho_k$, then $\sigma_k = 0$.*

This theorem is proven in the more advanced courses and textbooks on linear algebra[3,4] It tells us that every $N \times N$ matrix is related by a similarity transform to a particularly simple matrix. To see the importance of this to us, let $\mathbf{A}$ and $\mathbf{B}$ be two constant $N \times N$ matrices related by

$$\mathbf{B} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} \qquad \text{(equivalently, } \mathbf{T}\mathbf{B}\mathbf{T}^{-1} = \mathbf{A}\text{)}$$

[3] Such as *Linear Algebra* by Friedberg, Insel and Spence

[4] Actually, what is usually proven is that each $N \times N$ matrix of real or complex numbers is related by a similarity transform to a matrix in "Jordan canonical form". The description of matrix $\mathbf{B}$ in theorem 40.8 is a partial description of that form. A more complete description would say more about the "zero or one" pattern of the σ_k's . It should also be noted that the ordering of the eigenvalues given in statement 1 of the theorem is not necessary, but is always possible and allowed for the partial description of the "zero or one" pattern of the σ_k's in statement 4.

for some invertible matrix $\mathbf{T}$. Also let $\mathbf{x}$ and $\mathbf{y}$ be two vector-valued functions on the real line related to each other via

$$\mathbf{y} = \mathbf{T}^{-1}\mathbf{x} \qquad (\text{equivalently, } \mathbf{Ty} = \mathbf{x}) \quad .$$

Now suppose $\mathbf{x}$ satisfies $\mathbf{x}' = \mathbf{Ax}$. Using the above, the product rule for matrices, and the fact that $\mathbf{x} = \mathbf{Ix} = \mathbf{TT}^{-1}\mathbf{x}$, we see that

$$\begin{aligned} \mathbf{y}' &= \left(\mathbf{T}^{-1}\mathbf{x}\right)' = \mathbf{T}^{-1}\mathbf{x}' = \mathbf{T}^{-1}\mathbf{Ax} \\ &= \mathbf{T}^{-1}\mathbf{A}\left(\mathbf{TT}^{-1}\mathbf{x}\right) = \left(\mathbf{T}^{-1}\mathbf{AT}\right)\left(\mathbf{T}^{-1}\mathbf{x}\right) = \mathbf{By} \quad , \end{aligned} \tag{40.9a}$$

showing that $\mathbf{y}$ satisfies the constant matrix system $\mathbf{y}' = \mathbf{By}$. And if we had instead assumed $\mathbf{y}' = \mathbf{By}$, then we would have

$$\mathbf{x}' = (\mathbf{Ty})' = \mathbf{Ty}' = \mathbf{TBy} = \mathbf{TBT}^{-1}\mathbf{Ty} = \mathbf{Ax} \quad , \tag{40.9b}$$

showing that $\mathbf{x}$ satisfies the constant matrix system $\mathbf{x}' = \mathbf{Ax}$. And if you then take into account the fact that, because matrix multiplication is linear,

$$\mathbf{T}\left[c_1\mathbf{y}^1(t) + c_2\mathbf{y}^2(t) + \cdots + c_N\mathbf{y}^N(t)\right] = c_1\mathbf{Ty}^1(t) + c_2\mathbf{Ty}^2(t) + \cdots + c_N\mathbf{Ty}^N(t) \quad ,$$

you can easily finish proving the next theorem.

Theorem 40.9

Let $\mathbf{A}$ and $\mathbf{B}$ be two constant $N \times N$ matrices related by a similarity transform

$$\mathbf{B} = \mathbf{T}^{-1}\mathbf{AT} \quad ,$$

and let

$$\left\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N\right\} \qquad \text{and} \qquad \left\{\mathbf{y}^1, \mathbf{y}^2, \ldots, \mathbf{y}^N\right\}$$

be two sets of N vector-valued functions on the real line with

$$\mathbf{x}^k = \mathbf{Ty}^k \qquad \text{for} \quad k = 1, 2, \ldots, N \quad .$$

Then

$$\left\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N\right\}$$

is a fundamental set of solutions for $\mathbf{x}' = \mathbf{Ax}$ if and only if

$$\left\{\mathbf{y}^1, \mathbf{y}^2, \ldots, \mathbf{y}^N\right\}$$

is a fundamental set of solutions for $\mathbf{y}' = \mathbf{By}$.

So, if we can find a $\mathbf{T}$ so that $\mathbf{B} = \mathbf{T}^{-1}\mathbf{AT}$ is of the form described in theorem 40.8, then we can find a fundamental set of solutions $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N\}$ for $\mathbf{x}' = \mathbf{Ax}$ by first finding a fundamental set of solutions $\{\mathbf{y}^1, \mathbf{y}^2, \ldots, \mathbf{y}^N\}$ for the much simpler system $\mathbf{y}' = \mathbf{By}$, and then setting

$$\mathbf{y}^k = \mathbf{T}^{-1}\mathbf{x}^k \qquad \text{for} \quad k = 1, 2, \ldots, N \quad .$$

Do observe how simple this system $\mathbf{y}' = \mathbf{By}$ is: If $\mathbf{A}$ has a complete set of eigenvectors, then $\mathbf{B}$ is diagonal, and a fundamental set can be found almost by inspection. Otherwise, it is still a very simple, weakly coupled system which is still easily solved, possibly using the material discussed in sections 39.3 and 39.5.

As an exercise, you should take a little closer look at the eigenvectors and 'generalized eigenvectors' of $\mathbf{B}$:

?►*Exercise 40.3:* *Let* $\mathbf{B}$ *be as in theorem 40.8, and, for each positive integer* k *less than or equal to* N *, let* $\mathbf{b}^k$ *be the column vector whose entries are all* 0 *except for the* k^{th} *entry which is* 1 *,*

$$\mathbf{b}^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad , \quad \mathbf{b}^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad , \quad \cdots \quad , \quad \mathbf{b}^N = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad .$$

Verify the following:

a: $(\rho_1, \mathbf{b}^1)$ *is an eigenpair for* $\mathbf{B}$ *.*

b: *For* $k = 2, 3, \ldots, N$ *:*

i: *If* $\sigma_k = 0$ *, then* $(\rho_k, \mathbf{b}^k)$ *is an eigenpair for* $\mathbf{B}$ *.*

ii: *If* $\sigma_k = 1$ *, then* $[\mathbf{B} - \rho_k \mathbf{I}]\mathbf{b}^k = \mathbf{b}^{k-1}$ *.*

c: *The matrix* $\mathbf{B}$ *has a complete set of eigenvectors if and only if the* σ_k*'s are all zero.*

Doubtlessly, you are now asking how we find that matrix $\mathbf{T}$. Here is part of the answer:

Corollary 40.10

Let $\mathbf{A}$ *be a constant* $N \times N$ *matrix with a complete set of eigenvectors. Then:*

1. *If matrices* $\mathbf{T}$ *and* $\mathbf{B}$ *are as in theorem 40.8 (with* $\mathbf{B} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ *), and* $\mathbf{u}^k$ *is the column vector given by the* k^{th} *column of* $\mathbf{T}$ *for* $k = 1, 2, \ldots, N$ *, then* $\{\mathbf{u}^1, \mathbf{u}^2, \ldots, \mathbf{u}^N\}$ *is a complete set of eigenvectors for* $\mathbf{A}$ *, with* ρ_k *, the* k^{th} *element in the main diagonal of* $\mathbf{B}$ *, being the eigenvalue corresponding to eigenvector* $\mathbf{u}^k$ *.*

2. *If* $\{\mathbf{u}^1, \mathbf{u}^2, \ldots, \mathbf{u}^N\}$ *is any complete set of eigenvectors for* $\mathbf{A}$ *, labeled so that the corresponding eigenvalues form an increasing sequence, and if* $\mathbf{T}$ *is the matrix whose* k^{th} *column is given by* $\mathbf{u}^k$ *, and* $\mathbf{B} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ *, then the matrices* $\mathbf{T}$ *and* $\mathbf{B}$ *are as described in theorem 40.8.*

This corollary follows from theorem 40.8 using the results from the previous exercise and basic linear algebra. It tells us that finding a suitable matrix $\mathbf{T}$ for our similarity transform is equivalent to finding a complete set of eigenvectors for $\mathbf{A}$, provided $\mathbf{A}$ has a complete set of eigenvectors. A more general version of this corollary ("straightforward" in concept but awkward to concisely describe) further tells us that, in general, the construction a suitable matrix $\mathbf{T}$ is equivalent to finding a complete set of generalized eigenvectors for $\mathbf{A}$.

This relation between the $\mathbf{T}$ in theorem 40.8 and the eigenvectors and generalized eigenvectors of $\mathbf{A}$ rather lowers the value of similarity transforms as practical tools for solving homogeneous constant matrix systems of differential equations. After all, if we have already found those vectors, then it is easier to finish solving $\mathbf{x}' = \mathbf{A}\mathbf{x}$ via the methods discussed in the last two chapters. Still, similarity transforms have theoretical value. In fact, the lemmas and theorems in section 39.6 can all be viewed as corollaries of theorem 40.8. In addition, we may find some use for similarity transforms when dealing with nonhomogeneous constant matrix systems.

Let's just do one example, and move on to other topics.

!►Example 40.8: *Let us reconsider (again) the system in example 40.4,*

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \qquad \text{with} \quad \mathbf{A} = \begin{bmatrix} 2 & -\frac{12}{5} & \frac{4}{5} \\ 2 & -3 & 1 \\ 0 & \frac{6}{5} & \frac{8}{5} \end{bmatrix} \quad .$$

We already know that $\mathbf{A}$ *has eigenpairs*

$$\left(-2, \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}\right) \quad , \quad \left(2, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}\right) \quad \text{and} \quad \left(2, \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}\right)$$

with the three above eigenvectors forming a complete set of eigenvectors for $\mathbf{A}$ *. So it immediately follows that*

$$\left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} e^{-2t} , \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} e^{2t} , \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} e^{2t} \right\}$$

is a fundamental set of solutions for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *, and*

$$\mathbf{X}(t) = \mathbf{X}(t) = \begin{bmatrix} 2e^{-2t} & e^{2t} & 3e^{2t} \\ 3e^{-2t} & e^{2t} & e^{2t} \\ -e^{-2t} & 3e^{2t} & 3e^{2t} \end{bmatrix}$$

is the corresponding fundamental matrix.

Ignoring the fact that we can so readily find the solutions from the eigenpairs of $\mathbf{A}$ *, let's observe that the matrix* $\mathbf{T}$ *whose columns are given by the above eigenvectors is*

$$\mathbf{T} = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 1 \\ -1 & 3 & 3 \end{bmatrix} \quad .$$

(In fact, it's worth observing that $\mathbf{T} = \mathbf{X}(0)$ *!) Computing the inverse of* $\mathbf{T}$ *however you wish, you will find that*

$$\mathbf{T}^{-1} = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 1 \\ -1 & 3 & 3 \end{bmatrix}^{-1} = \cdots = \frac{1}{20} \begin{bmatrix} 0 & 6 & -2 \\ -10 & 9 & 7 \\ 10 & -7 & -1 \end{bmatrix} \quad .$$

Thus,

$$\mathbf{B} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$$

$$= \frac{1}{20} \begin{bmatrix} 0 & 6 & -2 \\ -10 & 9 & 7 \\ 10 & -7 & -1 \end{bmatrix} \begin{bmatrix} 2 & -\frac{12}{5} & \frac{4}{5} \\ 2 & -3 & 1 \\ 0 & \frac{6}{5} & \frac{8}{5} \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 1 \\ -1 & 3 & 3 \end{bmatrix} = \cdots = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad .$$

Clearly $\mathbf{B}$ *has eigenpairs*

$$\left(-2, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) \quad , \quad \left(2, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) \quad \text{and} \quad \left(2, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

with the three above eigenvectors forming a complete set of eigenvectors for $\mathbf{B}$. *Hence, a fundamental set of solutions for* $\mathbf{y}' = \mathbf{B}\mathbf{y}$ *is* $\{\mathbf{y}^1, \mathbf{y}^2, \mathbf{y}^3\}$ *with*

$$\mathbf{y}^1(t) = \begin{bmatrix}1\\0\\0\end{bmatrix} e^{-2t} \quad , \quad \mathbf{y}^2(t) = \begin{bmatrix}0\\1\\0\end{bmatrix} e^{2t} \quad \text{and} \quad \mathbf{y}^3(t) = \begin{bmatrix}0\\0\\1\end{bmatrix} e^{2t} \quad .$$

Theorem 40.9 now assures us that a corresponding fundamental set of solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *is* $\{\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3\}$ *with*

$$\mathbf{x}^1(t) = \mathbf{T}\mathbf{y}^1(t) = \begin{bmatrix}2 & 1 & 3\\3 & 1 & 1\\-1 & 3 & 3\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} e^{-2t} = \begin{bmatrix}2\\3\\-1\end{bmatrix} e^{-2t} \quad ,$$

$$\mathbf{x}^2(t) = \mathbf{T}\mathbf{y}^1(t) = \begin{bmatrix}2 & 1 & 3\\3 & 1 & 1\\-1 & 3 & 3\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} e^{2t} = \begin{bmatrix}1\\1\\3\end{bmatrix} e^{2t}$$

and

$$\mathbf{x}^3(t) = \mathbf{T}\mathbf{y}^3(t) = \begin{bmatrix}2 & 1 & 3\\3 & 1 & 1\\-1 & 3 & 3\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} e^{2t} = \begin{bmatrix}3\\1\\3\end{bmatrix} e^{2t} \quad ,$$

just as we had noted near the start of this example.

Additional Exercises

40.4. *For each shifted constant matrix systems given below:*

i. *Find a general solution.*

ii. *Describe the type and stability of the critical point.*

iii. *Sketch a phase portrait.*

a. $\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}\left(\begin{bmatrix}x\\y\end{bmatrix} - \begin{bmatrix}2\\1\end{bmatrix}\right)$

b. $\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}2 & 0\\0 & 4\end{bmatrix}\begin{bmatrix}x\\y-3\end{bmatrix}$

c. $\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}-2 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}x-2\\y+3\end{bmatrix}$

d. $\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}0 & 2\\-2 & 0\end{bmatrix}\begin{bmatrix}x+3\\y+1\end{bmatrix}$

e. $\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}-1 & -9\\9 & -1\end{bmatrix}\begin{bmatrix}x-4\\y-4\end{bmatrix}$

f. $\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}5 & -3\\3 & -5\end{bmatrix}\begin{bmatrix}x-3\\y+3\end{bmatrix}$

40.5. *Assume* $\mathbf{A}$ *is* 2×2 *constant matrix with real components, and let*

$$r_1 = \lambda_1 + i\omega \quad \text{and} \quad r_2 = \lambda_2 - i\omega$$

be approximations of the true eigenvalues, with $\lambda_1 = \lambda_2$ *if* $\omega \neq 0$ *and* $r_1 \leq r_2$ *if* $\omega = 0$. *Assume each computed* λ *and* ω *is known to be "within* ϵ*" of the true value, where* ϵ *is some (hopefully small) positive value. To simplify notation,* $r_1 \approx r_0$ *or* $\omega \approx 0$ *means these computed values are close enough that it is possible for the corresponding equalities to hold for the true values.*

Several choices for r_1 *and* r_2 *are given in each of the following. For each, state what these eigenvalues tell you about the critical point* $(0,0)$*, the stability of the equilibrium solution* $\mathbf{x}(t) \equiv \mathbf{0}$ *and the phase portraits for* $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *both assuming the computed eigenvalues are correct, and for all the other possible values of the eigenvalues.*

a. $r_1 < r_2 < -\epsilon$ *with* $r_1 \not\approx r_2$

b. $r_1 < r_2 < -\epsilon$ *with* $r_1 \approx r_2$

c. $r_1 < -\epsilon$ *and* $\epsilon < r_2$

d. $r_1 \approx 0$ *and* $r_2 \approx 0$

40.6. *Sketch phase portraits for each of the following (simple)* $3{\times}3$ *constant matrix systems:*

a. $$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

b. $$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

c. $$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

d. $$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

e. $$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

40.7. *Let* $\mathbf{F}$ *and* $\mathbf{G}$ *be any two matrix-valued functions with differentiable components, and show that the standard product rule*

$$(\mathbf{FG})' = \mathbf{F}'\mathbf{G} + \mathbf{F}\mathbf{G}'$$

holds when

a. $\mathbf{F} = \begin{bmatrix} f_{1,1} & f_{1,2} \\ f_{2,1} & f_{2,2} \end{bmatrix}$ *and* $\mathbf{G} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$

b. $\mathbf{F} = \begin{bmatrix} f_{1,1} & f_{1,2} \\ f_{2,1} & f_{2,2} \end{bmatrix}$ *and* $\mathbf{G} = \begin{bmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{bmatrix}$

c. *The matrix product* $\mathbf{FG}$ *exists.*

40.8. *Let*

$$\mathbf{F}(t) = \begin{bmatrix} 1 & t \\ 0 & 0 \end{bmatrix} \quad ,$$

and verify, by direct computation, the following:

a. $\mathbf{F}\dfrac{d\mathbf{F}}{dt} \neq \dfrac{d\mathbf{F}}{dt}\mathbf{F}$ **b.** $\dfrac{d}{dt}\left[\mathbf{F}^2\right] \neq 2\mathbf{F}\dfrac{d\mathbf{F}}{dt}$ **c.** $\dfrac{d}{dt}\left[\mathbf{F}^2\right] \neq 2\dfrac{d\mathbf{F}}{dt}\mathbf{F}$

40.9. *Now let* $\mathbf{M}(t)$ *be any* $N \times N$ *matrix of differentiable functions that commutes with its derivative,* $\mathbf{M}\mathbf{M}' = \mathbf{M}'\mathbf{M}$. *Show that, in this case,*

a. $\frac{d}{dt}\left[\mathbf{M}^2\right] = 2\mathbf{M}\frac{d\mathbf{M}}{dt}$. **b.** $\frac{d}{dt}\left[\mathbf{M}^k\right] = k\frac{d\mathbf{M}}{dt}\mathbf{M}^{k-1}$ *for* $k = 2, 3, \dots$.

40.10. *Each constant matrix system* $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *given below is accompanied by a general solution* $\mathbf{x}$ *to the system (a solution that you probably found in an earlier exercise). For each system, write out the fundamental matrix* $\mathbf{X}$ *corresponding to the given general solution, and find the exponential fundamental matrix* $e^{\mathbf{A}t}$.

a. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$, $\mathbf{x}(t) = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^t + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-t}$

b. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$, $\mathbf{x}(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}e^t + c_2\begin{bmatrix} t \\ 1 \end{bmatrix}$

c. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$, $\mathbf{x}(t) = c_1\begin{bmatrix} 1 \\ -4 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{9t}$

d. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 1 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$, $\mathbf{x}(t) = c_1\begin{bmatrix} -3 \\ 1 \end{bmatrix}e^{3t} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{7t}$

e. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 8 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$, $\mathbf{x}(t) = c_1\begin{bmatrix} \cos(4t) \\ 2\sin(4t) \end{bmatrix} + c_2\begin{bmatrix} \sin(4t) \\ -2\cos(4t) \end{bmatrix}$

f. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$, $\mathbf{x}(t) = c_1\begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}e^{3t} + c_2\begin{bmatrix} -\cos(2t) \\ \sin(2t) \end{bmatrix}e^{3t}$

g. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 6 & -4 \\ 9 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$, $\mathbf{x}(t) = c_1\begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2\begin{bmatrix} 2t+1 \\ 3t+1 \end{bmatrix}$

h. $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 2 \\ 0 & -4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$,

$$\mathbf{x}(t) = c_1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}e^{3t} + c_2\begin{bmatrix} -\cos(2t) \\ \sin(2t) \\ 2\cos(2t) \end{bmatrix}e^{3t} + c_3\begin{bmatrix} \sin(2t) \\ \cos(2t) \\ -2\sin(2t) \end{bmatrix}e^{3t}$$

40.11. *Compute* $e^{\mathbf{M}}$ *for each of the following choices of* $\mathbf{M}$ *, using either power series formula (40.5) or (40.6):*

a. $\begin{bmatrix} 2t & 0 \\ 0 & 3t \end{bmatrix}$ **b.** $\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$ **c.** $\begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}$ **d.** $\begin{bmatrix} 0 & 0 \\ \beta & 0 \end{bmatrix}$

e. $\begin{bmatrix} 1 & t \\ 0 & 0 \end{bmatrix}$ **f.** $\begin{bmatrix} t & 1 \\ 0 & 0 \end{bmatrix}$

40.12. *Let* r_1 , r_2 , ... *and* r_N *be any set of constants, and show that*

$$e^{\mathbf{A}} = \begin{bmatrix} e^{r_1} & 0 & 0 & \cdots & 0 \\ 0 & e^{r_2} & 0 & \cdots & 0 \\ 0 & 0 & e^{r_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{r_N} \end{bmatrix} \quad \text{when} \quad \mathbf{A} = \begin{bmatrix} r_1 & 0 & 0 & \cdots & 0 \\ 0 & r_2 & 0 & \cdots & 0 \\ 0 & 0 & r_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & r_N \end{bmatrix} .$$

using either the power series definition of $e^{\mathbf{A}}$ or by solving the appropriate system of differential equations.

40.13. *Let*

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad , \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad .$$

a. *Using results from exercise 40.11, compute $e^{\mathbf{A}}$, and $e^{\mathbf{B}}$.*

b. *Note that $\mathbf{C} = \mathbf{A} + \mathbf{B}$. Using this and the answer from exercise 40.10 a, compute $e^{\mathbf{A}+\mathbf{B}}$.*

c. *Now, by direct computation, show that*

$$e^{\mathbf{A}}e^{\mathbf{B}} \neq e^{\mathbf{B}}e^{\mathbf{A}} \quad , \quad e^{\mathbf{A}}e^{\mathbf{B}} \neq e^{\mathbf{A}+\mathbf{B}} \quad \text{and} \quad e^{\mathbf{B}}e^{\mathbf{A}} \neq e^{\mathbf{A}+\mathbf{B}} \quad .$$

for the above choices for $\mathbf{A}$ and $\mathbf{B}$.

40.14. *In exercise 40.11 you found $e^{\mathbf{M}(t)}$ when*

$$\mathbf{M}(t) = \begin{bmatrix} 1 & t \\ 0 & 0 \end{bmatrix} \quad .$$

Now verify, by direct computation, that

$$\frac{d}{dt}\left[e^{\mathbf{M}(t)}\right] \neq e^{\mathbf{M}(t)}\frac{d\mathbf{M}}{dt} \quad \text{and} \quad \frac{d}{dt}\left[e^{\mathbf{M}(t)}\right] \neq \frac{d\mathbf{M}}{dt}e^{\mathbf{M}(t)}$$

when $\mathbf{M}$ is as above.

40.15. *For each of the following Euler systems below:*

i. *Find the general solution $\mathbf{x}$ over $(0, \infty)$.*

ii. *Write out a corresponding fundamental matrix $\mathbf{X}$ for the system.*

iii. *Sketch a representative sampling of the system's trajectories.*

a. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \frac{1}{t}\begin{bmatrix} 5 & 1 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

b. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \frac{1}{t}\begin{bmatrix} 6 & 7 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

c. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \frac{1}{t}\begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

d. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \frac{1}{t}\begin{bmatrix} -3 & -3 \\ 3 & -13 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

e. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \frac{1}{t}\begin{bmatrix} 5 & -3 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

f. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \frac{1}{3t}\begin{bmatrix} -9 & 2 \\ 9 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

g. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \frac{1}{3t}\begin{bmatrix} 6 & 2 \\ 9 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

h. $\begin{bmatrix} x' \\ y' \end{bmatrix} = -\frac{1}{9t}\begin{bmatrix} 6 & 10 \\ 45 & 21 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

40.16. *For each of the following Euler systems:*

i *Find a general solution over $(0, \infty)$ in terms of real-valued functions only.*

ii *Sketch a representative sampling of the system's trajectories.*

(Note: The eigenvalues for these matrices are all complex.)

a. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \frac{1}{t}\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

b. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \frac{1}{t}\begin{bmatrix} 3 & 5 \\ -5 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

c. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \frac{1}{t}\begin{bmatrix} 3 & -6 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

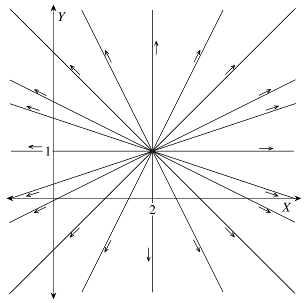
d. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \frac{1}{t}\begin{bmatrix} -3 & -2 \\ 5 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

Some Answers to Some of the Exercises

WARNING! Most of the following answers were prepared hastily and late at night. They have not been properly proofread! Errors are likely!

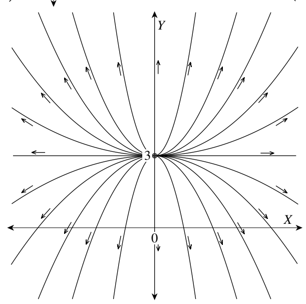
4a. $c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2x} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2x} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

An unstable star node



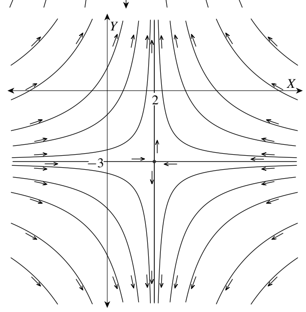
4b. $c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2x} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{4x} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

An unstable node



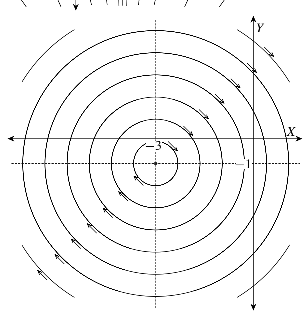
4c. $c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-2x} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2x} + \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

An unstable saddle point



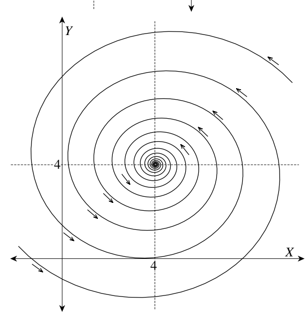
4d. $c_1 \begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix} + c_2 \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix} + \begin{bmatrix} -3 \\ -1 \end{bmatrix}$

A stable center



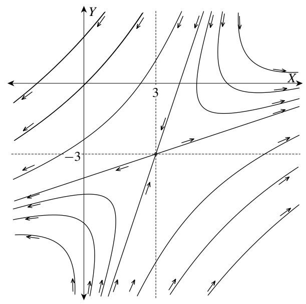
4e. $c_1 \begin{bmatrix} -\sin(9t) \\ \cos(9t) \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} \cos(9t) \\ \sin(9t) \end{bmatrix} e^{-t} + \begin{bmatrix} 4 \\ 4 \end{bmatrix}$

An asymptotically stable spiral point

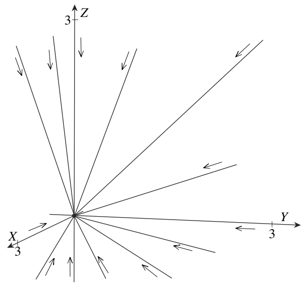


4f. $c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-4t} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{4t} + \begin{bmatrix} 3 \\ -3 \end{bmatrix}$

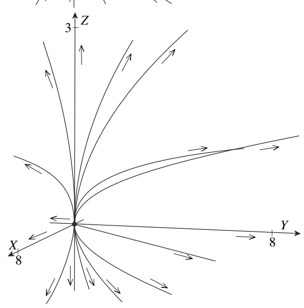
An unstable saddle point



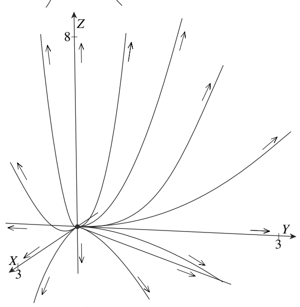
6a.



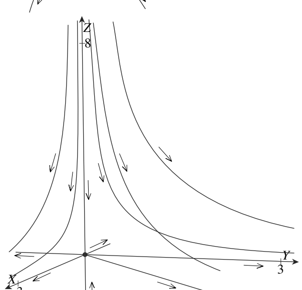
6b.



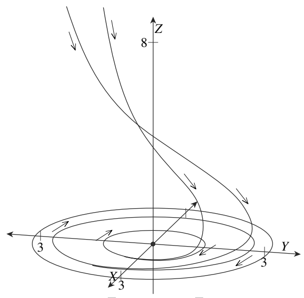
6c.



6d.



6e.



10a. $\mathbf{X}(t) = \begin{bmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{bmatrix}$, $\exp(\mathbf{A}t) = \frac{1}{2}\begin{bmatrix} e^t+e^{-t} & e^t-e^{-t} \\ e^t-e^{-t} & e^t+e^{-t} \end{bmatrix}$

10b. $\mathbf{X}(t) = e^t\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$, $\exp(\mathbf{A}t) = e^t\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

10c. $\mathbf{X}(t) = \begin{bmatrix} 1 & 2e^{9t} \\ -4 & e^{9t} \end{bmatrix}$, $\exp(\mathbf{A}t) = \frac{1}{9}\begin{bmatrix} 1+8e^{9t} & -2+2e^{9t} \\ -4+4e^{9t} & 8+e^{9t} \end{bmatrix}$

10d. $\mathbf{X}(t) = \begin{bmatrix} -3e^{3t} & e^{7t} \\ e^{3t} & e^{7t} \end{bmatrix}$, $\exp(\mathbf{A}t) = \frac{1}{4}\begin{bmatrix} 3e^{3t}+e^{7t} & -3e^{3t}+3e^{7t} \\ -e^{3t}+e^{7t} & e^{3t}+e^{7t} \end{bmatrix}$

10e. $\mathbf{X}(t) = \begin{bmatrix} \cos(4t) & \sin(4t) \\ 2\sin(4t) & -2\cos(4t) \end{bmatrix}$, $\exp(\mathbf{A}t) = \frac{1}{2}\begin{bmatrix} 2\cos(4t) & -\sin(4t) \\ 4\sin(4t) & 2\cos(4t) \end{bmatrix}$

10f. $\mathbf{X}(t) = e^{3t}\begin{bmatrix} \sin(2t) & -\cos(2t) \\ \cos(2t) & \sin(2t) \end{bmatrix}$, $\exp(\mathbf{A}t) = e^{3t}\begin{bmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{bmatrix}$

10g. $\mathbf{X}(t) = \begin{bmatrix} 2 & 2t+1 \\ 3 & 3t+1 \end{bmatrix}$, $\exp(\mathbf{A}t) = \begin{bmatrix} 1+6t & -4t \\ 9t & 1-6t \end{bmatrix}$

10h. $\mathbf{X}(t) = e^{3t}\begin{bmatrix} 1 & -\cos(2t) & \sin(2t) \\ 0 & \sin(2t) & \cos(2t) \\ -1 & 2\cos(2t) & -2\sin(2t) \end{bmatrix}$, $\exp(\mathbf{A}t) = e^{3t}\begin{bmatrix} 2-\cos(2t) & \sin(2t) & 1-\cos(2t) \\ \sin(2t) & \cos(2t) & \sin(2t) \\ -2+2\cos(2t) & -2\sin(2t) & -1+2\cos(2t) \end{bmatrix}$

11a. $\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix}$

11b. $\begin{bmatrix} e^{\alpha t} & 0 \\ 0 & e^{\beta t} \end{bmatrix}$

11c. $\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$

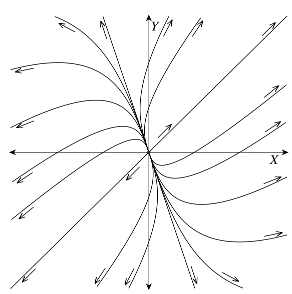
11d. $\begin{bmatrix} 1 & 0 \\ \beta & 1 \end{bmatrix}$

11e. $\begin{bmatrix} e & t(e-1) \\ 0 & 1 \end{bmatrix}$

11f. $\begin{bmatrix} e^t & t^{-1}(e^t-1) \\ 0 & 1 \end{bmatrix}$

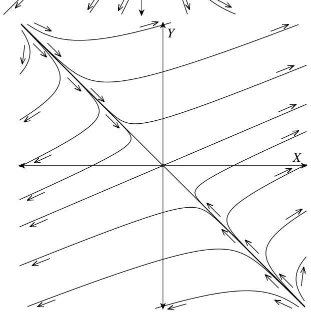
15a. $\mathbf{x}(t) = c_1\begin{bmatrix} 1 \\ -3 \end{bmatrix}t^2 + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}t^6$

$\mathbf{X}(t) = \begin{bmatrix} t^2 & t^6 \\ -3t^2 & t^6 \end{bmatrix}$



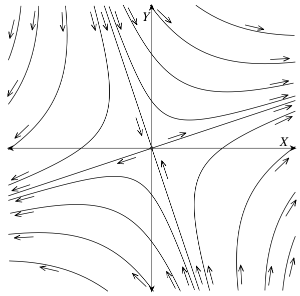
15b. $\mathbf{x}(t) = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}t^{-1} + c_2\begin{bmatrix} 7 \\ 3 \end{bmatrix}t^9$

$\mathbf{X}(t) = \begin{bmatrix} t^{-1} & 7t^9 \\ -t^{-1} & 3t^9 \end{bmatrix}$

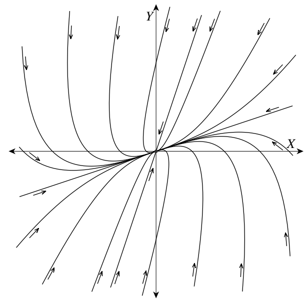


.

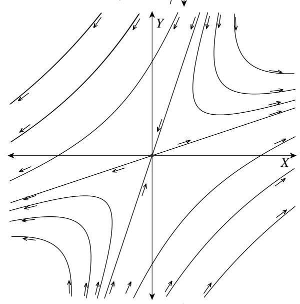
15c.
$$c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} t^{-5} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} t^{5}$$
$$\mathbf{X}(t) = \begin{bmatrix} t^{-5} & 3t^{5} \\ -3t^{-5} & t^{5} \end{bmatrix}$$



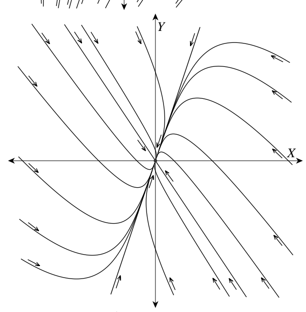
15d.
$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} t^{-12} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} t^{-4}$$
$$\mathbf{X}(t) = \begin{bmatrix} t^{-12} & 3t^{-4} \\ 3t^{-12} & t^{-4} \end{bmatrix}$$



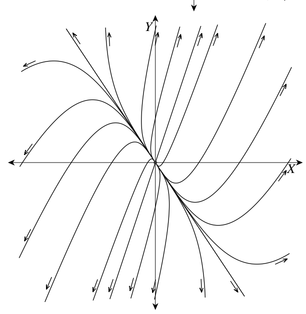
15e.
$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} t^{-4} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} t^{4}$$
$$\mathbf{X}(t) = \begin{bmatrix} t^{-4} & 3t^{4} \\ 3t^{-4} & t^{4} \end{bmatrix}$$



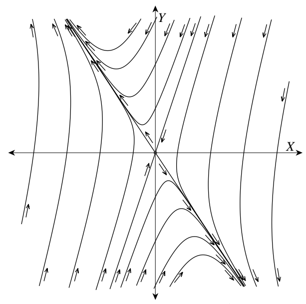
15f.
$$\mathbf{x}(t) = c_1 \begin{bmatrix} -2 \\ 3 \end{bmatrix} t^{-4} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} t^{-1}$$
$$\mathbf{X}(t) = \begin{bmatrix} -2t^{-4} & t^{-1} \\ 3t^{-4} & 3t^{-1} \end{bmatrix}$$



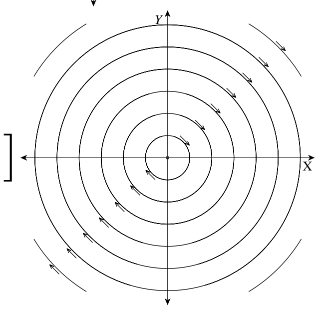
15g.
$$\mathbf{x}(t) = c_1 \begin{bmatrix} -2 \\ 3 \end{bmatrix} t + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} t^{4}$$
$$\mathbf{X}(t) = \begin{bmatrix} -2t & t^{4} \\ 3t & 3t^{4} \end{bmatrix}$$



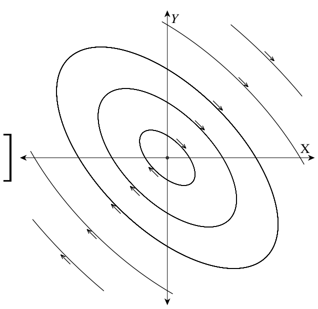
15h.
$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} t^{-4} + c_2 \begin{bmatrix} -2 \\ 3 \end{bmatrix} t$$
$$\mathbf{X}(t) = \begin{bmatrix} t^{-4} & -2t \\ 3t^{-4} & 3t \end{bmatrix}$$



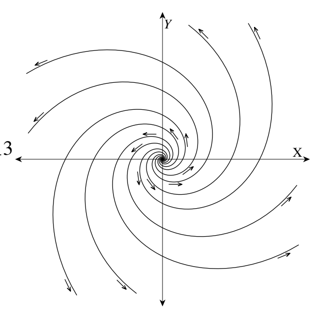
16a. $\mathbf{x}(t) = c_1 \begin{bmatrix} \cos(2\ln|t|) \\ -\sin(2\ln|t|) \end{bmatrix} + c_2 \begin{bmatrix} \sin(2\ln|t|) \\ \cos(2\ln|t|) \end{bmatrix}$



16b. $\mathbf{x}(t) = c_1 \begin{bmatrix} -5\cos(4\ln|t|) \\ 3\cos(4\ln|t|)+4\sin(4\ln|t|) \end{bmatrix} + c_2 \begin{bmatrix} 5\sin(4\ln|t|) \\ 4\cos(4\ln|t|)-3\sin(4\ln|t|) \end{bmatrix}$



16c. $\mathbf{x}(t) = c_1 \begin{bmatrix} \cos(6\ln|t|) \\ \sin(6\ln|t|) \end{bmatrix} t^3 + c_2 \begin{bmatrix} \sin(6\ln|t|) \\ -\cos(6\ln|t|) \end{bmatrix} t^3$



16d. $\mathbf{x}(t) = c_1 \begin{bmatrix} 2\cos(3\ln|t|) \\ 3\sin(3\ln|t|)-\cos(3\ln|t|) \end{bmatrix} t^{-2} + c_2 \begin{bmatrix} -2\sin(3\ln|t|) \\ 3\cos(3\ln|t|)+\sin(3\ln|t|) \end{bmatrix} t^{-2}$