39

Homogeneous Constant Matrix Systems, Part II

Let us now expand our discussions begun in the previous chapter, and consider homogeneous constant matrix systems whose matrices either have complex eigenvalues or have incomplete sets of eigenvectors.

39.1 Solutions Corresponding to Complex Eigenvalues

Eigenpairs of Real Matrices

Remember that the real and imaginary parts of a complex number z are those real numbers x and y, respectively, such that $z = x + iy$, and that the complex conjugate z^* of z is

$$z^* = x - iy \quad .$$

All these notions extend to matrices having complex components in the obvious manner: Given a matrix $\mathbf{M}$ whose entries are complex, we define the complex conjugate $\mathbf{M}^*$ of $\mathbf{M}$ to be the matrix formed by replacing each entry of $\mathbf{M}$ with that entry's complex conjugate, and the real and imaginary parts of $\mathbf{M}$ are simply the corresponding matrices of the real and imaginary parts of the components of that matrix.

It is easy to show that the standard identities for the conjugates of complex numbers such as

$$\left(z^*\right)^* = z \quad , \quad (az)^* = (a^*)(z^*) \quad \text{and} \quad z = w \iff z^* = w^*$$

also hold for analogous expression involving matrices and column vectors. In particular, for any complex value r,

$$(\mathbf{Au})^* = (\mathbf{A}^*)(\mathbf{u}^*) \quad \text{and} \quad (r\mathbf{u})^* = r^*\mathbf{u}^* \quad .$$

But if (as we've been assuming) the components of $\mathbf{A}$ are real numbers, then

$$\mathbf{A}^* = \begin{bmatrix} {a_{11}}^* & {a_{12}}^* & \cdots & {a_{1N}}^* \\ {a_{21}}^* & {a_{22}}^* & \cdots & {a_{2N}}^* \\ \vdots & \vdots & \cdots & \vdots \\ {a_{N1}}^* & {a_{N2}}^* & \cdots & {a_{NN}}^* \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \cdots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix} = \mathbf{A} \quad ,$$

and thus,

$$\mathbf{Au} = r\mathbf{u} \iff (\mathbf{Au})^* = (r\mathbf{u})^* \iff (\mathbf{A}^*)(\mathbf{u}^*) = r^*\mathbf{u}^* \iff \mathbf{A}(\mathbf{u}^*) = r^*\mathbf{u}^* \quad ,$$

giving us the first part of the following lemma:

Lemma 39.1

Let $\mathbf{A}$ be a real $N \times N$ constant matrix. Then $(r, \mathbf{u})$ is an eigenpair for $\mathbf{A}$ if and only if $(r^*, \mathbf{u}^*)$ is an eigenpair for $\mathbf{A}$. Moreover, if r is not real, then the real and imaginary parts of $\mathbf{u}$ form a linearly independent set.

The proof of the second part — the claimed linear independence of the real and imaginary parts of $\mathbf{u}$ — is not difficult, but may distract us from our core narrative. So we'll place it in an appendix for later verification (see lemma 39.10 in section 39.6).

The Corresponding Real-Valued Solutions

Let's now look at the solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ when the $N \times N$ matrix $\mathbf{A}$ has complex eigenvalues, still assuming that $\mathbf{A}$ is a real constant matrix. From lemma 39.1, we know that the complex eigenpairs occur in complex conjugate pairs. So let $(r, \mathbf{u})$ and $(r^*, \mathbf{u}^*)$ be such a pair, with

$$r = \lambda + i\omega \quad , \quad r^* = \lambda - i\omega \quad ,$$

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix} = \begin{bmatrix} a_1 + ib_1 \\ a_2 + ib_2 \\ \vdots \\ a_N + ib_N \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} + i \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix} = \mathbf{a} + i\mathbf{b}$$

and

$$\mathbf{u}^* = \cdots = \mathbf{a} - i\mathbf{b} \quad .$$

Even though the eigen-pairs are complex, the computations from the start of the last chapter still apply, and tell us that one "fundamental" pair of solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ corresponding to the complex conjugate pair of eigenvalues $\lambda + i\omega$ and $\lambda - i\omega$ is

$$\mathbf{x}(t) = \mathbf{u}e^{rt} = [\mathbf{a} + i\mathbf{b}]e^{(\lambda+i\omega)t} \qquad \text{and} \qquad \mathbf{x}^*(t) = \mathbf{u}^* e^{r^* t} = [\mathbf{a} - i\mathbf{b}]e^{(\lambda-i\omega)t} \quad ,$$

which, by the way, we can rewrite in terms of sines and cosines since

$$e^{(\lambda \pm i\omega)t} = e^{\lambda t} e^{\pm i\omega t} = e^{\lambda t} [\cos(\omega t) \pm i \sin(\omega t)] \quad .$$

Recall that we had a similar situation arise when solving single homogeneous linear differential equations with constant coefficients (see chapters 15 and 17). And here, just as there, we prefer to have solutions that do not involve complex values. So let's try adapting what we did there to get a corresponding "fundamental pair" of solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ in terms of real-valued functions only.

We start by splitting our two solutions into their real and imaginary parts:

$$\begin{aligned} e^{(\lambda \pm i\omega)t} &= [\mathbf{a} \pm i\mathbf{b}]e^{\lambda t} [\cos(\omega t) \pm i \sin(\omega t)] \\ &= \cdots \\ &= e^{\lambda t}[\mathbf{a}\cos(\omega t) - \mathbf{b}\sin(\omega t)] \pm ie^{\lambda t}[\mathbf{b}\cos(\omega t) + \mathbf{a}\sin(\omega t)] \quad . \end{aligned}$$

Letting

$$\mathbf{x}^R(t) = \text{the real part of } \mathbf{x}(t) = e^{\lambda t}[\mathbf{a}\cos(\omega t) - \mathbf{b}\sin(\omega t)] \qquad (39.1a)$$

and

$$\mathbf{x}^I(t) \;=\; \text{the imaginary part of } \mathbf{x}(t) \;=\; e^{\lambda t}[\mathbf{b}\cos(\omega t) + \mathbf{a}\sin(\omega t)] \quad , \tag{39.1b}$$

we can write our two solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ as

$$\mathbf{x}(t) \;=\; \mathbf{x}^R(t) \;+\; i\mathbf{x}^I(t) \qquad \text{and} \qquad \mathbf{x}^*(t) \;=\; \mathbf{x}^R(t) \;-\; i\mathbf{x}^I(t) \quad .$$

In turn, we can easily solve this pair of equations for $\mathbf{x}^R$ and $\mathbf{x}^I$, obtaining

$$\mathbf{x}^R(t) \;=\; \frac{1}{2}\mathbf{x}(t) \;+\; \frac{1}{2}\mathbf{x}^*(t) \qquad \text{and} \qquad \mathbf{x}^I(t) \;=\; \frac{1}{2i}\mathbf{x}(t) \;-\; \frac{1}{2i}\mathbf{x}^*(t) \quad .$$

Being linear combinations of solutions to the homogeneous linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, $\mathbf{x}^R$ and $\mathbf{x}^I$ must also be solutions to this system of differential equations. Note that both are in terms of real-valued functions only. And it is a fairly easy exercise (using the second part of lemma 39.1) to show that $\{\mathbf{x}^R, \mathbf{x}^I\}$ is a linearly independent pair. It is this pair that we will use as the "pair of real-valued solutions corresponding to eigenvalues $\lambda \pm i\omega$".

Theorem 39.2

Let $\mathbf{A}$ be a real constant $N \times N$ matrix. Then the complex eigenvalues of $\mathbf{A}$ occur as complex conjugate pairs. Moreover, if $\mathbf{u}$ is an eigenvector corresponding to eigenvalue $\lambda + i\omega$ (with $\omega \neq 0$), then a corresponding linearly independent pair of real-valued solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is given by $\{\mathbf{x}^R(t), \mathbf{x}^I(t)\}$ where

$$\mathbf{x}^R(t) \;=\; \textit{the real part of } \mathbf{x}(t) \qquad , \qquad \mathbf{x}^I(t) \;=\; \textit{the imaginary part of } \mathbf{x}(t)$$

and

$$\mathbf{x}(t) \;=\; \mathbf{u}e^{(\lambda + i\omega)t} \;=\; \mathbf{u}e^{\lambda t}\,[\cos(\omega t) \;+\; i\,\sin(\omega t)] \quad .$$

By the way, memorizing formulas (39.1a) and (39.1b) for $\mathbf{x}^R(t)$ and $\mathbf{x}^I(t)$ is not necessary, or even advisable for most people. Instead, just expand $\mathbf{u}e^{(\lambda+i\omega)t}$ into its real and imaginary parts, and use those parts for $\mathbf{x}^R(t)$ and $\mathbf{x}^I(t)$.

!►*Example 39.1:* *Let's solve the system*

$$\begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} \;=\; \begin{bmatrix} 2 & 0 & 3 \\ 0 & -4 & 0 \\ -3 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad .$$

Setting up the characteristic equation:

$$\det[\mathbf{A} - r\mathbf{I}] \;=\; 0$$

$$\hookrightarrow \quad \det \begin{bmatrix} 2-r & 0 & 3 \\ 0 & -4-r & 0 \\ -3 & 0 & 2-r \end{bmatrix} \;=\; 0$$

$$\hookrightarrow \quad (-4-r)\left[(2-r)^2 + 9\right] \;=\; 0$$

$$\hookrightarrow \quad -(r+4)\left[r^2 + 4r + 13\right] \;=\; 0 \quad .$$

Hence, either

$$r \;+\; 4 \;=\; 0 \qquad \textit{or} \qquad r^2 \;+\; 4r \;+\; 13 \;=\; 0 \quad ,$$

which you can easily solve, obtaining

$$r_1 = -4 \quad , \quad r_2 = 2 + i3 \quad \text{and} \quad r_3 = 2 - i3$$

as the eigenvalues for our matrix.

By now, you should have no problem is showing that

$$\mathbf{u}^1 = [0\,,\,1\,,\,0]^{\mathsf{T}}$$

is an eigenvector corresponding to eigenvalue $r_1 = -4$ *, giving us*

$$\mathbf{x}^1(t) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{-4t}$$

as one solution to our system.

For the pair of solutions corresponding to the conjugate pair of eigenvalues $r_2 = 2 + i3$ *and* $r_3 = 2 - i3$ *, we first need to find an eigenvector* $\mathbf{u}$ *corresponding to* $r_2 = 2 + i3$ *. Letting*

$$\mathbf{u} = \left[\alpha\,,\beta\,,\gamma\right]^{\mathsf{T}} \quad ,$$

we see that

$$\hookrightarrow \quad \left(\begin{bmatrix} 2 & 0 & 3 \\ 0 & -4 & 0 \\ -3 & 0 & 2 \end{bmatrix} - (2+i3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right.$$

$$\hookrightarrow \quad \begin{bmatrix} 2-(2+i3) & 0 & 3 \\ 0 & -4-(2+i3) & 0 \\ -3 & 0 & 2-(2+i3) \end{bmatrix}$$

$$\hookrightarrow \quad \begin{bmatrix} -3i & 0 & 3 \\ 0 & -6-3i & 0 \\ -3 & 0 & -3i \end{bmatrix}$$

So α *,* β *and* γ *must satisfy the algebraic system*

$$\begin{aligned} -3i\alpha + 0\beta + 3\gamma &= 0 \\ 0\alpha - (6+3i)\beta + 0\gamma &= 0 \\ -3\alpha + 0\beta - 3i\gamma &= 0 \end{aligned} \quad .$$

The second equation in this system clearly tells us that

$$\beta = 0 \quad .$$

The first and third equations are equivalent (if this is not obvious, multiply the third by i *). So* α *and* β *must satisfy*

$$-3i\alpha + 0\beta + 3\gamma = 0 \quad ,$$

which can be rewritten as

$$\gamma = i\alpha \quad .$$

Thus, all the eigenvectors corresponding to eigenvalue $2 + 3i$ *are given by*

$$\mathbf{u} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \\ i\alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix}$$

where α *is arbitrary. Picking* $\alpha = 1$ *, we have the single eigenvector*

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix} \quad .$$

So, a single solution corresponding to to the conjugate pair of eigenvalues $r_2 = 2 + i3$ *and* $r_3 = 2 - i3$ *is*

$$\mathbf{x}(t) = \mathbf{u}e^{r_2 t} = \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix} e^{(2+i3)t} = \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix} e^{2t} \left[\cos(3t) + i \sin(3t)\right] \quad .$$

To find the corresponding pair of real-valued solutions, we first expand out the last formula for $\mathbf{x}(t)$ *,*

$$\begin{aligned} \mathbf{x}(t) &= \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix} e^{2t} \left[\cos(3t) + i \sin(3t)\right] \\ &= e^{2t} \begin{bmatrix} \cos(3t) + i \sin(3t) \\ 0 \\ i \cos(3t) - \sin(3t) \end{bmatrix} \\ &= e^{2t} \left(\begin{bmatrix} \cos(3t) \\ 0 \\ -\sin(3t) \end{bmatrix} + i \begin{bmatrix} \sin(3t) \\ 0 \\ \cos(3t) \end{bmatrix} \right) = \begin{bmatrix} \cos(3t) \\ 0 \\ -\sin(3t) \end{bmatrix} e^{2t} + i \begin{bmatrix} \sin(3t) \\ 0 \\ \cos(3t) \end{bmatrix} e^{2t} \quad . \end{aligned}$$

Taking the real and imaginary parts, we get

$$\mathbf{x}^R(t) = \begin{bmatrix} \cos(3t) \\ 0 \\ -\sin(3t) \end{bmatrix} e^{2t} \qquad \text{and} \qquad \mathbf{x}^I(t) = \begin{bmatrix} \sin(3t) \\ 0 \\ \cos(3t) \end{bmatrix} e^{2t} \quad .$$

We now have a set

$$\left\{ \mathbf{x}^1(t),\ \mathbf{x}^R(t)\ ,\ \mathbf{x}^I(t) \right\} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{-4t}\ ,\ \begin{bmatrix} \cos(3t) \\ 0 \\ -\sin(3t) \end{bmatrix} e^{2t}\ ,\ \begin{bmatrix} \sin(3t) \\ 0 \\ \cos(3t) \end{bmatrix} e^{2t} \right\}$$

of solutions to our 3×3 *homogeneous linear system of differential equations. It should be clear that this set is linearly independent (if its not clear, compute the Wronskian at* $t = 0$ *). Hence, it is a fundamental set of solutions to our system, and a general solution is given by*

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{-4t} + c_2 \begin{bmatrix} \cos(3t) \\ 0 \\ -\sin(3t) \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} \sin(3t) \\ 0 \\ \cos(3t) \end{bmatrix} e^{2t} \quad .$$

39.2 Two-Dimensional Phase Portraits with Complex Eigenvalues

Let us now see what patterns are formed by the trajectories of a 2×2 system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ when $\mathbf{A}$ has complex eigenvalues

$$\lambda + i\omega \qquad \text{and} \qquad \lambda - i\omega \qquad \text{with} \quad \omega \neq 0 \quad .$$

As before, we'll let

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \quad .$$

The General Solutions

In the previous section, we saw that, if $\mathbf{u} = \mathbf{a} + i\mathbf{b}$ is any eigenvector corresponding to eigenvalue $\lambda + i\omega$, then $\{\mathbf{x}^R, \mathbf{x}^I\}$ is a linearly independent pair of solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where

$$\mathbf{x}^R(t) = e^{\lambda t}[\mathbf{a}\cos(\omega t) - \mathbf{b}\sin(\omega t)] \qquad \text{and} \qquad \mathbf{x}^I(t) = e^{\lambda t}[\mathbf{b}\cos(\omega t) + \mathbf{a}\sin(\omega t)] \quad .$$

Hence,

$$\mathbf{x}(t) = c_1\mathbf{x}^R(t) + c_2\mathbf{x}^I(t) \tag{39.2}$$

is a general solution for our 2×2 system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

However, for no apparent reason, let's also consider $C\mathbf{x}^r(t - t_0)$ where C and t_0 are two real constants. Using some basic trigonometric identities, you can easily confirm that

$$\begin{aligned} C\mathbf{x}^R(t - t_0) &= Ce^{\lambda[t-t_0]}\big[\mathbf{a}\cos(\omega[t - t_0]) - \mathbf{b}\sin(\omega[t - t_0])\big] \\ &= \cdots \\ &= Ce^{-\lambda t_0}\cos(\omega t_0)e^{\lambda t}[\mathbf{a}\cos(\omega t) - \mathbf{b}\sin(\omega t)] \\ &\qquad + Ce^{-\lambda t_0}\sin(\omega t_0)e^{\lambda t}[\mathbf{b}\cos(\omega t) + \mathbf{a}\sin(\omega t)] \\ &= c_1\mathbf{x}^R(t) + c_2\mathbf{x}^I(t) \end{aligned}$$

where

$$c_1 = \rho\cos(\theta) \qquad \text{and} \qquad c_2 = \rho\sin(\theta)$$

with

$$\rho = Ce^{-\lambda t_0} \qquad \text{and} \qquad \theta = \omega t_0 \quad .$$

So, $C\mathbf{x}^r(t - t_0)$ a solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for any pair of real constants C and t_0. Moreover, if we are first given real values c_1 and c_2, we can then find real constants C and t_0 so that

$$C\mathbf{x}^R(t - t_0) = c_1\mathbf{x}^R(t) + c_2\mathbf{x}^I(t)$$

by simply taking the polar coordinates (ρ, θ) of the point with Cartesian coordinates (c_1, c_2) and setting

$$t_0 = \frac{\theta}{\omega} \qquad \text{and then} \qquad C = \rho e^{\lambda t_0} \quad .$$

This means that we have two real-valued general solutions at our disposal.

Theorem 39.3
Let $\mathbf{A}$ *be a real* 2×2 *matrix with complex eigenvalues*

$$r = \lambda + i\omega \qquad \text{and} \qquad r^* = \lambda - i\omega \qquad \text{with} \quad \omega \neq 0 \quad ,$$

and let

$$\mathbf{x}^R(t) = e^{\lambda t}[\mathbf{a}\cos(\omega t) - \mathbf{b}\sin(\omega t)] \qquad \text{and} \qquad \mathbf{x}^I(t) = e^{\lambda t}[\mathbf{b}\cos(\omega t) + \mathbf{a}\sin(\omega t)]$$

where $\mathbf{a}$ *and* $\mathbf{b}$ *are, respectively, the real part and the imaginary part of an eigenvector corresponding to* r *. Then two general solutions for* $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *are*

$$c_1\mathbf{x}^R(t) + c_2\mathbf{x}^I(t) \qquad \text{and} \qquad C\mathbf{x}^R(t - t_0)$$

where c_1 *,* c_2 *,* C *and* t_0 *are arbitrary (real) constants.*

It immediately follows that, for the $\mathbf{A}$'s begin considered now, all the trajectories of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ are simply scaled versions of the trajectory of $\mathbf{x}^R(t)$.[1] Consequently, if we can get a reasonable sketch of one nonzero trajectory, all the others in our phase portrait can be obtained by just scaling the one already sketched. We will use this in constructing the phase portraits that follow.

There are three cases to consider: $\lambda = 0$, $\lambda > 0$ and $\lambda < 0$.

Trajectories when $\lambda = 0$

If $\lambda = 0$, then $e^{\lambda t} = 1$ and

$$\mathbf{x}^R(t) = \mathbf{a}\cos(\omega t) - \mathbf{b}\sin(\omega t) \quad .$$

From a simple observation,

$$\begin{aligned}\mathbf{x}^R\left(t + \frac{2\pi}{\omega}\right) &= \mathbf{a}\cos\left(\omega\left[t + \frac{2\pi}{\omega}\right]\right) - \mathbf{b}\sin\left(\omega\left[t + \frac{2\pi}{\omega}\right]\right) \\ &= \mathbf{a}\cos(\omega t + 2\pi) - \mathbf{b}\sin(\omega t + 2\pi) \\ &= \mathbf{a}\cos(\omega t) - \mathbf{b}\sin(\omega t) = \mathbf{x}^R(t) \quad ,\end{aligned}$$

we know the components of $\mathbf{x}^R(t)$ — call them $x_R(t)$ and $y_R(t)$ — are periodic with a period of $2\pi/\omega$. That is, $(x_R(t), y_R(t))$ repeatedly goes through the same points on the XY–plane as t increases by increments of $2\pi/\omega$. With a little thought, you will realize that this means these trajectories are closed loops.

Moreover, by the same sort of calculations just done, you can easily verify that

$$\mathbf{x}^R\left(t + \frac{\pi}{\omega}\right) = -\mathbf{x}^R(t) \qquad \text{for all} \quad t \quad ,$$

telling us that, for each point on the trajectory of $\mathbf{x}^R$, there is another point on the trajectory with the origin right in the middle of these two points.

So, not only is this trajectory a closed loop, this loop must be centered about the origin. In fact, it can be shown that this loop is an ellipse centered about the origin (either trust the author on this, or turn to section 39.6 where it is proven).

As a result of all this, sketching a phase portrait is fairly easy. First sketch a minimal direction field. Then sketch an elliptic trajectory about the origin using the direction field to get the general

[1] Remember: The trajectory of a solution $\mathbf{x}(t)$ is the same as that solution shifted by t_0 , $\mathbf{x}(t - t_0)$.

shape and the direction of travel.[2] Finally, scale this trajectory to get a sequence of concentric elliptic trajectories. Include the dot at the origin for the critical point, and you have a phase portrait for this system.

Note that we don't actually need the eigenvector $\mathbf{u} = \mathbf{a} + i\mathbf{b}$ to sketch a phase portrait. And the only reason you need the eigenvalue $r = i\omega$ is to verify that it is imaginary.

!►*Example 39.2:* *You can easily verify that*

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$$

has eigenvalues $r_\pm = \pm i$ *with corresponding eigenvectors*

$$\mathbf{u}^\pm = \mathbf{a} \pm i\mathbf{b} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \pm i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad .$$

So general solutions to our system are given by

$$\mathbf{x}(t) = c_1\mathbf{x}^R(t) + c_2\mathbf{x}^I(t) \qquad \text{and} \qquad \mathbf{x}(t) = C\mathbf{x}^R(t - t_0)$$

where

$$\mathbf{x}^R(t) = \text{the real part of } \mathbf{u}^+e^{it} = \cdots = \begin{bmatrix} 1 \\ -1 \end{bmatrix}\cos(t) - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin(t)$$

and

$$\mathbf{x}^I(t) = \text{the imaginary part of } \mathbf{u}^+e^{it} = \cdots = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix}\sin(t) \quad .$$

Since the eigenvalues are imaginary, we know the trajectories are ellipses centered about the origin. To get a better idea of the shape of these ellipses and the direction of travel along these ellipses, we'll compute $\mathbf{x}'$ *at a few "well-chosen' points using* $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *:*

$$\text{At } (1,0): \quad \mathbf{x}' = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad .$$

$$\text{At } (1,1): \quad \mathbf{x}' = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \quad .$$

$$\text{At } (0,1): \quad \mathbf{x}' = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad .$$

$$\text{At } (-1,1): \quad \mathbf{x}' = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad .$$

Then we sketch corresponding direction arrows at these points, and at other points using the symmetries in the direction field discussed in section 38.4. That gives the minimal direction field sketched in figure 39.1a. Sketching an ellipse that 'best matches' this direction field then gives the trajectory also sketched in figure 39.1a.

Scaling this one ellipse by various constants, adding the dot for the critical point and erasing the direction field then gives the phase portrait in figure 39.1b for our system.

[2] Alternatively, you can plot $\mathbf{x}^R(t)$ for a few well-chosen values of t, and get the direction of travel from the derivative of $\mathbf{x}^R(t)$ at some convenient value of t.

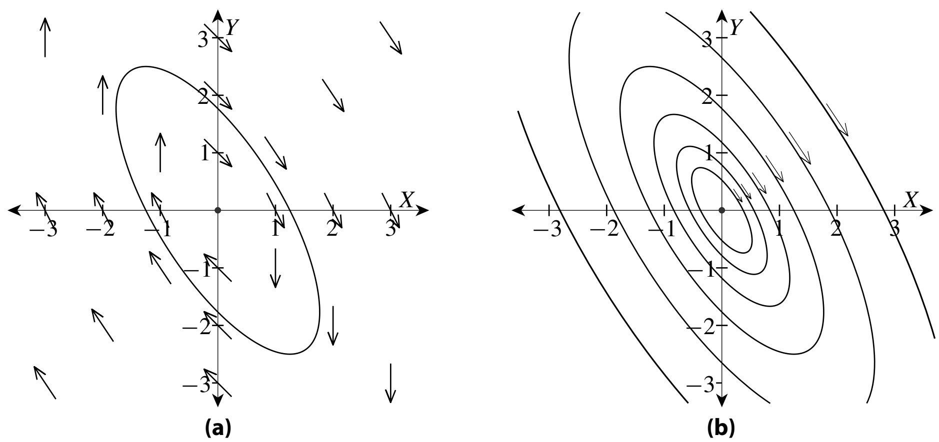


Figure 39.1: **(a)** A minimal direction field with one 'matching' elliptic trajectory, and **(b)** a phase portrait of the system in example 39.2.

In practice, we may not always need to sketch the the trajectory for $\mathbf{x}^R(t)$ as carefully as we did in the above example. Often, it is sufficient to simply note that the trajectories are all concentric ellipses centered at the origin and to determine the direction of travel on each by simply finding the direction arrow at one point.

In this case, the critical point $(0, 0)$ is not called a node or saddle point. For obvious reasons, we call it a "center". Observe that the corresponding equilibrium solution is stable, but not asymptotically stable, since the trajectory of any solution $\mathbf{x}(t)$ with $\mathbf{x}(t_0) \approx 0$ for some t_0 is an ellipse right around $(0, 0)$.

By the way, when sketching a trajectory for

$$\mathbf{a}\cos(\omega t) - \mathbf{b}\sin(\omega t) \quad ,$$

it is tempting to suspect that the vectors $\mathbf{a}$ and $\mathbf{b}$ directly correspond to the two axes of the elliptic trajectory. Well, in general, they don't. To see this, just contemplate the last example and figure 39.1a. (Still, the lines of the axes are easily determined — see exercise 39.6 on page 39–29.)

Trajectories when $\lambda > 0$

If $\lambda > 0$ then all the trajectories are given by scaling the single trajectory of

$$\mathbf{x}^R(t) = e^{\lambda t}[\mathbf{a}\cos(\omega t) - \mathbf{b}\sin(\omega t)] \quad .$$

As we just saw, the $[\mathbf{a}\cos(\omega t) - \mathbf{b}\sin(\omega t)]$ factor in this formula for $\mathbf{x}^R(t)$ repeatedly traces out an ellipse as t varies. Multiplying by $e^{\lambda t}$ — which increases to $+\infty$ as $t \to +\infty$ and decreases to 0 as $t \to -\infty$ — converts this ellipse into a spiral that spirals outward as $t \to +\infty$ and spirals inward to $(0, 0)$ as $t \to -\infty$, wrapping around the origin infinitely many times in either case. Whether the trajectories are spiralling out in the clockwise or counterclockwise direction can be determined by finding the direction of travel indicated by $\mathbf{x}'$ computed at a convenient point via $\mathbf{x}' = \mathbf{A}\mathbf{x}$. As usual, a minimal direction field may aid your sketching.

!► *Example 39.3:* *Let us just consider sketching a phase portrait for* $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *when*

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} \quad .$$

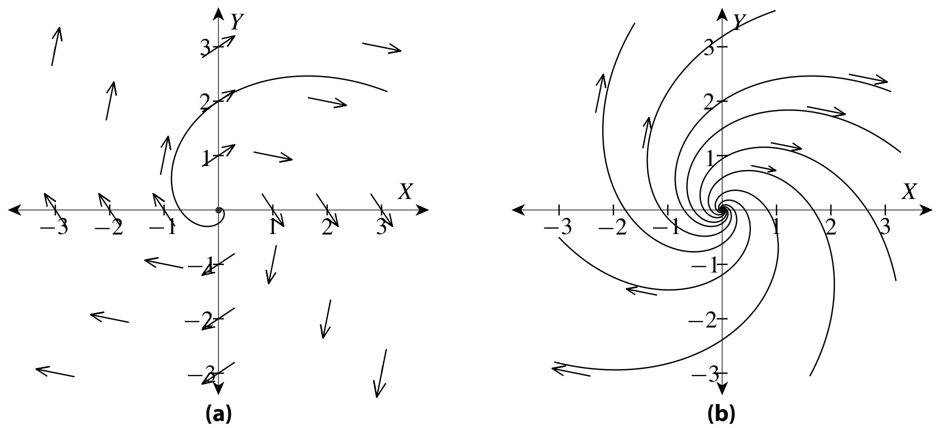


Figure 39.2: **(a)** A minimal direction field with one 'matching' spiral trajectory, and **(b)** a phase portrait of the system in example 39.3 in which $\lambda > 0$.

As you can easily verify, $\mathbf{A}$ *has eigenvalues* $r = 2 \pm i3$ *. The fact that these eigenvalues are complex with positive real parts tells us that the trajectories of* $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *are spirals with the direction of travel being away from the origin. Computing* $\mathbf{x}'$ *at a few 'well-chosen' points:*

$$At\ (1,0): \quad \mathbf{x}' = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \quad .$$

$$At\ (1,1): \quad \mathbf{x}' = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix} \quad .$$

$$At\ (0,1): \quad \mathbf{x}' = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad .$$

$$At\ (-1,1): \quad \mathbf{x}' = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \quad .$$

From these vectors and the symmetries discussed in section 38.4, we get the minimal direction field sketched in figure 39.2a, which also includes a spiral trajectory matching that direction field. Note that this trajectory is spiralling outwards in a clockwise direction.

Adding more spirals matching this minimal direction field (and erasing the direction field) then gives us the phase portrait in figure 39.2b.

For these systems, the origin classified as a "spiral point" (for obvious reasons), and is clearly unstable.

Trajectories when $\lambda < 0$

The only significant difference between this case and the previous is that $e^{\lambda t}$ decreases to 0 as $t \to +\infty$ and increases to $+\infty$ as $t \to -\infty$. All this does is to change the direction of travel. We still get spiral trajectories, but with the direction of travel on each being towards the origin. The result is a phase portrait similar to that sketched in figure 39.3.

The origin is still called a "spiral point", but you can clearly see that the equilibrium solution $\mathbf{x}(t) = \mathbf{0}$ is asymptotically stable.

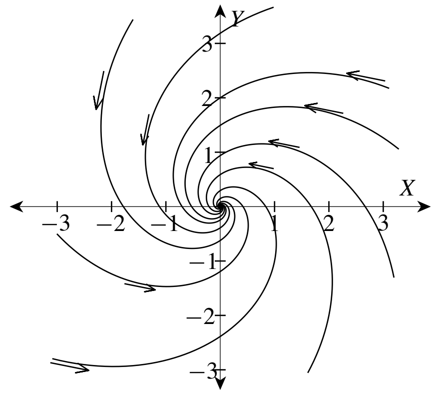


Figure 39.3: A phase portrait for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for a case where $\mathbf{A}$ has complex eigenvalues with negative real parts.

39.3 'Second Solutions' When the Set of Eigenvectors is Incomplete

Up to now, we've assumed that our matrix $\mathbf{A}$ has a complete set of eigenvectors. That is, we've assumed that, for each eigenvalue r

$$\text{the algebraic multiplicity of } r \ = \ \text{the geometric multiplicity of } r$$

where (just in case you forgot)

$$\begin{aligned}&\text{the algebraic multiplicity of } r \\ &\qquad = \text{ the multiplicity of } r \text{ as a root of the characteristic polynomial, } \det[\mathbf{A} - r\mathbf{I}] \quad ,\end{aligned}$$

while

$$\begin{aligned}&\text{the geometric multiplicity of } r \\ &\qquad = \text{ the number of eigenvectors in any basis for the eigenspace of } r \quad .\end{aligned}$$

Now, let's suppose that an eigenvalue r has algebraic multiplicity of two or more, but geometric multiplicity one, with $\mathbf{u}$ being a corresponding eigenvector. That gives us one solution

$$\mathbf{x}^1(t) \ = \ \mathbf{u}e^{rt}$$

to $\mathbf{x}' = \mathbf{A}\mathbf{x}$. It should seem reasonable to expect that there is another solution $\mathbf{x}^2$ corresponding to this eigenvalue other than a constant multiple of $\mathbf{x}^1$. Let's look for this "second solution".

Based on what we know about "second solutions" to homogeneous linear differential equations with constant coefficients (from chapter 15), we might suspect that a second solution can be generated by just multiplying the first by t,

$$\mathbf{x}^2(t) \ = \ t\mathbf{x}^1(t) \ = \ t\mathbf{u}e^{rt} \quad .$$

Alas, it doesn't work. Try it yourself to see what happens. You get a leftover term that does not cancel out. But the attempt may lead you to try adding something to deal with that leftover term. So let us try

$$\mathbf{x}^2(t) \ = \ [t\mathbf{u} + \mathbf{w}]e^{rt}$$

where $\mathbf{w}$ is a yet unknown vector:

$$\frac{d}{dt}\mathbf{x}^2(t) = \mathbf{A}\mathbf{x}^2$$

$$\hookrightarrow \qquad \frac{d}{dt}\left([t\mathbf{u}+\mathbf{w}]e^{rt}\right) = \mathbf{A}[t\mathbf{u}+\mathbf{w}]e^{rt}$$

$$\hookrightarrow \qquad [\mathbf{u}+tr\mathbf{u}+r\mathbf{w}]e^{rt} = [tr\mathbf{u}+\mathbf{A}\mathbf{w}]e^{rt}$$

$$\hookrightarrow \qquad \mathbf{u} + tr\mathbf{u} + r\mathbf{w} = tr\mathbf{u} + \mathbf{A}\mathbf{w}$$

$$\hookrightarrow \qquad \mathbf{u} + r\mathbf{w} = \mathbf{A}\mathbf{w}$$

$$\hookrightarrow \qquad \mathbf{A}\mathbf{w} - r\mathbf{w} = \mathbf{u} \quad .$$

And since

$$\mathbf{A}\mathbf{w} - r\mathbf{w} = \mathbf{A}\mathbf{w} - r\mathbf{I}\mathbf{w} = [\mathbf{A}-r\mathbf{I}]\mathbf{w} \quad ,$$

we can rewrite our last equation as

$$[\mathbf{A}-r\mathbf{I}]\mathbf{w} = \mathbf{u} \quad . \tag{39.3}$$

This is what $\mathbf{w}$ must satisfy.

!►*Example 39.4:* *Let us consider the* 2×2 *system* $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *with*

$$\mathbf{A} = \begin{bmatrix} 3 & 10 \\ 0 & 3 \end{bmatrix} \quad .$$

Solving for the eigenvalues,

$$0 = \det[\mathbf{A}-r\mathbf{I}] = \det\begin{bmatrix} 3-r & 10 \\ 0 & 3-r \end{bmatrix} = (r-3)^2 \quad ,$$

we get a single eigenvalue $r = 3$ *with algebraic multiplicity* 2 *.*

Any eigenvector $\mathbf{u} = [\alpha, \beta]^{\mathsf{T}}$ *then must satisfy*

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = [\mathbf{A}-3\mathbf{I}]\mathbf{u} = \begin{bmatrix} 0 & 10 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 10\beta \\ 0 \end{bmatrix} \quad ,$$

which tells us that $\beta = 0$ *and* α *is arbitrary. So any eigenvector of this matrix can be written as*

$$\mathbf{u} = \begin{bmatrix} \alpha \\ 0 \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad .$$

Taking $\alpha = 1$ *gives us the single eigenvector*

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad .$$

Thus, $r = 3$ *is an eigenvalue with algebraic multiplicity* 2 *but geometric multiplicity* 1 *. One solution to* $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *is then given by*

$$\mathbf{x}^1(t) = \mathbf{u}e^{rt} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t} \quad .$$

According to our derivation above, another solution is given by

$$\mathbf{x}^2(t) \;=\; [t\mathbf{u} + \mathbf{w}]e^{3t}$$

where $\mathbf{w}$ *satisfies*

$$[\mathbf{A} - 3\mathbf{I}]\mathbf{w} \;=\; \mathbf{u} \quad .$$

To find such a $\mathbf{w}$ *, plug* $\mathbf{w} = [\alpha, \beta]^{\mathsf{T}}$ *into the last equation, and solve for* α *and* β *:*

$$[\mathbf{A} - 3\mathbf{I}]\,\mathbf{w} \;=\; \mathbf{u}$$

$$\hookrightarrow \quad \left(\begin{bmatrix} 3 & 10 \\ 0 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 \\ \; \end{bmatrix}$$

$$\hookrightarrow \quad \begin{bmatrix} 10\beta \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \; \end{bmatrix}$$

Hence, $\beta = {}^1\!/_{10}$ *,* α *is arbitrary, and*

$$\mathbf{w} \;=\; \begin{bmatrix} \alpha \\ \frac{1}{10} \end{bmatrix} \;=\; \begin{bmatrix} 0 \\ \frac{1}{10} \end{bmatrix} \;+\; \alpha\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

That is, for $\mathbf{w}$ *we can use* $[0, {}^1\!/_{10}]^{\mathsf{T}}$ *plus any constant multiple of the one eigenvector* $\mathbf{u}$ *,* $\alpha\mathbf{u}$ *. For simplicity, let us take* $\alpha = 0$ *. Then, for our second solution, we have*

$$\mathbf{x}^2(t) \;=\; [t\mathbf{u} + \mathbf{w}]e^{rt} \;=\; \left(t\begin{bmatrix} 1 \\ 0 \end{bmatrix} \;+\; \begin{bmatrix} 0 \\ \frac{1}{10} \end{bmatrix}\right)e^{3t} \;=\; \begin{bmatrix} t \\ \frac{1}{10} \end{bmatrix}e^{3t} \quad .$$

Clearly, $\mathbf{x}^2$ *is not a constant multiple of* $\mathbf{x}^1$ *. Hence, our system* $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *has*

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}e^{3t} \,,\; \begin{bmatrix} t \\ \frac{1}{10} \end{bmatrix}e^{3t} \right\}$$

as a fundamental pair of solutions, and

$$\mathbf{x}(t) \;=\; c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix}e^{3t} \;+\; c_2\begin{bmatrix} t \\ \frac{1}{10} \end{bmatrix}e^{3t} \;=\; \left(c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} t \\ \frac{1}{10} \end{bmatrix}\right)e^{3t}$$

as a general solution. And if, for esthetic reasons, we don't like fractions explicitly appearing in our general solutions, we can replace c_2 *with* $10C_2$ *, rewriting the above as*

$$\mathbf{x}(t) \;=\; c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix}e^{3t} \;+\; c_2\begin{bmatrix} t \\ \frac{1}{10} \end{bmatrix}e^{3t} \;=\; \left(c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2\begin{bmatrix} 10t \\ 1 \end{bmatrix}\right)e^{3t} \quad .$$

The key to our finding a second solution is in finding a $\mathbf{w}$ satisfying

$$[\mathbf{A} - r\mathbf{I}]\mathbf{w} \;=\; \mathbf{u}$$

when $(r, \mathbf{u})$ is an eigenpair for $\mathbf{A}$. You may have some concern about this being possible. After all, since r is an eigenvalue,

$$\det[\mathbf{A} - r\mathbf{I}] \;=\; 0 \quad ,$$

telling us that the matrix $\mathbf{A} - r\mathbf{I}$ is not invertible. So what guarantee is there that the above key equation has a solution $\mathbf{w}$? Well, for a 2×2 matrix, there is the following lemma, which you'll get to verify yourself in exercise 39.10 on page 39–30.[3]

Lemma 39.4

Let $\mathbf{A}$ be a 2×2 matrix with eigenpair $(r, \mathbf{u})$. If r has algebraic multiplicity two, but geometric multiplicity of only one, then there is a $\mathbf{w}$, which is not a constant multiple of $\mathbf{u}$, such that

$$[\mathbf{A} - r\mathbf{I}]\mathbf{w} = \mathbf{u} \quad .$$

Do note that the $\mathbf{w}$ in the above lemma is not unique. After all, if

$$[\mathbf{A} - r\mathbf{I}]\mathbf{w}^1 = \mathbf{u} \qquad \text{for some} \qquad \mathbf{w}^1 \quad ,$$

then, for any constant c,

$$[\mathbf{A} - r\mathbf{I}]\mathbf{w} = \mathbf{u} \qquad \text{with} \quad \mathbf{w} = \mathbf{w}^1 + c\mathbf{u} \quad ,$$

simply because, since $(r, \mathbf{u})$ is an eigen-pair,

$$[\mathbf{A} - r\mathbf{I}](\mathbf{w}^1 + c\mathbf{u}) = [\mathbf{A} - r\mathbf{I}]\mathbf{w}^1 + c[\mathbf{A} - r\mathbf{I}]\mathbf{u} = \mathbf{u} + c\mathbf{0} \quad .$$

Back to differential equations: As an immediate corollary of the above lemma and the work done before the previous example, we have

Theorem 39.5

Let $\mathbf{A}$ be an 2×2 matrix with eigenpair $(r, \mathbf{u})$. If r has algebraic multiplicity two, but geometric multiplicity of only one, then the general solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is given by

$$\mathbf{x}(t) = c_1\mathbf{u}e^{rt} + c_2[t\mathbf{u} + \mathbf{w}]e^{rt}$$

where $\mathbf{w}$ is any vector satisfying

$$[\mathbf{A} - r\mathbf{I}]\mathbf{w} = \mathbf{u} \quad .$$

You may wonder if we can expand this discussion to find third and fourth solutions when the algebraic multiplicity is greater than two. The answer is yes, but let's look at the trajectories of the solutions we've found when $\mathbf{A}$ is a simple 2×2 matrix before discussing this expansion.

39.4 Two-Dimensional Phase Portraits with Incomplete Sets of Eigenvectors

If $\mathbf{A}$ is a 2×2 matrix with a single eigenvalue r having algebraic multiplicity two but geometric multiplicity one, then, as we just saw, a general solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is given by

$$\mathbf{x}(t) = c_1\mathbf{u}e^{rt} + c_2[t\mathbf{u} + \mathbf{w}]e^{rt}$$

where $\mathbf{u}$ is an eigenvector corresponding to eigenvalue r, and $\mathbf{w}$ satisfies $[\mathbf{A} - r\mathbf{I}]\mathbf{w} = \mathbf{u}$. The trajectories of this when $c_2 = 0$ are simply the straight-line trajectories corresponding to the eigenpair $(r, \mathbf{u})$, and, along with the critical point, are the first trajectories to sketch in our phase portrait.

[3] Much more general versions of this lemma will be discussed in section 39.5.

To simplify our sketching of the trajectories when $c_2 \neq 0$, let's rewrite the above formula for $\mathbf{x}$ as

$$\mathbf{x}(t) = c_2\left[(\alpha\mathbf{u}+\mathbf{w})e^{rt} + t\mathbf{u}e^{rt}\right]$$

where $\alpha = {}^{c_1}/_{c_2}$. Note that

$$\mathbf{x}(0) = c_2(\alpha\mathbf{u}+\mathbf{w}) \quad .$$

This may help us make rough sketches of the trajectories.

To help find the direction of travel and any tangents, we can use either the derivative,

$$\frac{d\mathbf{x}}{dt} = \cdots = c_2\left[[(r\alpha+1)\mathbf{u}+r\mathbf{w})]e^{rt} + t\mathbf{u}e^{rt}\right] \quad ,$$

or, to remove any distracting scaling effects, we can use any positive multiple of this. Let us use

$$\mathbf{T}(t) = \frac{e^{-rt}}{|t|}\frac{d\mathbf{x}}{dt} = c_2\left[\frac{(r\alpha+1)\mathbf{u}+r\mathbf{w}}{|t|} + \frac{t}{|t|}r\mathbf{u}\right] \quad .$$

Note that

$$\lim_{t\to-\infty}\mathbf{T}(t) = c_2\left[\mathbf{0}+(-1)r\mathbf{u}\right] = -c_2r\mathbf{u} \quad ,$$

while

$$\lim_{t\to+\infty}\mathbf{T}(t) = c_2\left[\mathbf{0}+(+1)r\mathbf{u}\right] = +c_2r\mathbf{u} \quad .$$

This tells us that, if t_+ is a very large positive value and t_- is very large negative value, then the tangents to the trajectories at these two values of t

1. are nearly parallel to the eigenvector $\mathbf{u}$,

but

2. are in directly opposite directions.

Trajectories when $r > 0$

If $r > 0$ and $c_2 \neq 0$, then

$$\lim_{t\to-\infty}\mathbf{x}(t) = \lim_{t\to-\infty}c_2\left[(\alpha\mathbf{u}+\mathbf{w})e^{rt} + t\mathbf{u}e^{rt}\right] = \mathbf{0}$$

and

$$\lim_{t\to\infty}\|\mathbf{x}(t)\| = \lim_{t\to\infty}\left\|c_2\left[(\alpha\mathbf{u}+\mathbf{w})+t\mathbf{u}\right]e^{rt}\right\| = \infty \quad .$$

So each trajectory appears to start at $(0,0)$ and continues infinitely beyond the origin as t goes from $-\infty$ to $+\infty$. This, alone, tells us that the direction of travel on each nonequilibrium trajectory is away from the origin.

Now, in our general comments, we saw that, if $\mathbf{x}(t)$ is a solution with $c_2 \neq 0$, then

$$\mathbf{x}(0) = c_2(\alpha\mathbf{u}+\mathbf{w}) \quad ,$$

$$\lim_{t\to\infty}\mathbf{T}(t) = c_2r\mathbf{u} \qquad \text{and} \qquad \lim_{t\to-\infty}\mathbf{T}(t) = -c_2r\mathbf{u} \quad .$$

In particular, if $c_2 > 0$, then the trajectory of $\mathbf{x}(t)$ appears to start at the origin, tangent to $\mathbf{u}$ but directed in the direction opposite to $\mathbf{u}$. As t increases to $t = 0$, $\mathbf{x}(t)$ traces a path that bends in the direction of $\mathbf{w}$ to pass through the point corresponding to $c_2(\alpha\mathbf{u}+\mathbf{w})$. As t continues to get ever

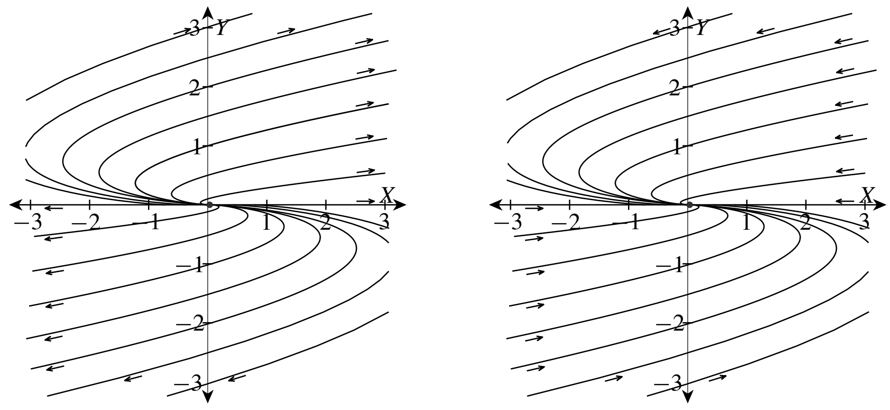


Figure 39.4: Trajectories for a 2×2 system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ when all the eigenvectors of $\mathbf{A}$ are parallel to the X–axis assuming **(a)** the eigenvalue is positive, and **(b)** the eigenvalue is negative.

larger, this path continues to bend and slowly "flattens out" until the direction of travel is in nearly the same direction as $\mathbf{u}$.

If $c_2 < 0$, then the trajectory of $\mathbf{x}(t)$ is just a mirror reflection of the trajectory of a corresponding solution having positive c_2.

The end result is a phase portrait similar to that sketched in figure 39.4a. This is a phase portrait for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with

$$\mathbf{A} = \begin{bmatrix} 3 & 10 \\ 0 & 3 \end{bmatrix} \quad .$$

from example 39.4. In that example, we found that $\mathbf{A}$ only had $r = 3$ as an eigenvalue, and that all eigenvectors were constant multiples of $\mathbf{u} = [1, 0]^{\mathsf{T}}$. We also obtained $\mathbf{w} = [0, {}^{1}/_{10}]^{\mathsf{T}}$.

Do note, however, that you don't really need to know $\mathbf{w}$ to sketch this phase portrait. You can (as we've done several times before) start by sketching the straight-line trajectories for the solutions $\pm\mathbf{u}e^{rt}$, then add a minimal direction field and use this direction field to sketch a number of other nonequilibrium trajectories. Just keep in mind that these trajectories "start" at the origin tangent to $\mathbf{u}$, and then bend around to ultimately head out in the opposite direction. And don't forget the dot at the origin for the equilibrium solution.

In these cases, the equilibrium solution $\mathbf{x}(t) = \mathbf{0}$ is clearly an unstable equilibrium. The critical point $(0, 0)$ is classified as a "node". Those who wish to be more precise refer to it as an "improper node".

By the way, we've not sketched enough of the trajectories in figure 39.4a to really see that the tangents become parallel to the eigenvector as t gets large. That is because the magnitude of $\mathbf{x}(t)$ increases like e^{rt}, much faster than $\mathbf{T}(t)$ approaches $\pm c_2 r\mathbf{u}$. Consequently, in practice, if you use a scale large enough to see "the tangents being almost parallel to the eigenvector at large values of t", then the behavior of the trajectories near the origin becomes almost invisible.

Trajectories when $r < 0$

The analysis just done assuming $r > 0$ can be redone with $r < 0$. The only real difference is that the directions of travel will be reversed. With $r < 0$, the direction of travel will be towards the origin. The result will be a phase portrait similar to that in figure 39.4b. The origin is a node (again, "improper"), and the equilibrium solution $\mathbf{x}(t) = \mathbf{0}$ is now an asymptotically stable equilibrium.

Trajectories when $r = 0$

This is left as an exercise (exercise 39.11 on page 39–31).

39.5 Complete Sets of Solutions Corresponding to Incomplete Sets of Eigenvectors

It is relatively easy to extend the ideas presented in section 39.3 for finding a 'second solution' corresponding to an eigenvalue with geometric multiplicity one but algebraic multiplicity of two. Well, it's 'relatively easy' least as long as the geometric multiplicity remains one and we know enough linear algebra. So let us start with that case.

When the Geometric Multiplicity Is One

To begin, here are two lemmas that, together, appropriately generalize lemma 39.4:

Lemma 39.6
Let $(r, \mathbf{u})$ be an eigenpair for some $N \times N$ matrix $\mathbf{A}$, and let m be the algebraic multiplicity of r. If the geometric multiplicity of r is only one, then there is a linearly independent set of m vectors

$$\left\{ \mathbf{w}^1 ,\ \mathbf{w}^2 ,\ \dots ,\ \mathbf{w}^m \right\}$$

such that

(a) $\mathbf{w}^1 = \mathbf{u}$, *and*

(b) *if* $m > 1$ *then*

$$[\mathbf{A} - r\mathbf{I}]\mathbf{w}^k \;=\; \mathbf{w}^{k-1} \qquad \text{for} \quad k \;=\; 2,\ 3,\ \dots,\ m \quad .$$

However, there is no $\mathbf{v}$ such that

$$[\mathbf{A} - r\mathbf{I}]\mathbf{v} \;=\; \mathbf{w}^m \quad .$$

Lemma 39.7
Let $\{r_1, r_2, \dots, r_J\}$ be the complete set of distinct eigenvalues for an $N \times N$ matrix $\mathbf{A}$, and suppose that the geometric multiplicity of each eigenvalue is one. For each r_j, let m_j be the algebraic multiplicity of r_j, and let

$$\mathcal{W}_j \;=\; \left\{ \mathbf{w}^{j,1} ,\ \mathbf{w}^{j,2} ,\ \dots ,\ \mathbf{w}^{j,m_j} \right\}$$

be any set of vectors such that

1. $\mathbf{w}^{j,1}$ *is an eigenvector for* $\mathbf{A}$ *corresponding to eigenvalue* r_j, *and*

2. *if* $m_j > 1$ *then*

$$[\mathbf{A} - r\mathbf{I}]\mathbf{w}^{j,k} \;=\; \mathbf{w}^{j,k-1} \qquad \text{for} \quad k \;=\; 2,\ 3,\ \dots,\ m_j \quad .$$

Then the union of the $\mathcal{W}_j$'s is a linearly independent set of N vectors.

The $\mathbf{w}^k$'s and $\mathbf{w}^{j,k}$'s in the above lemmas are sometimes referred to as *generalized eigenvectors* for the matrix. Unfortunately, the arguments outlined in exercise 39.10 for proving lemma 39.4 do not readily extend to the more general cases considered by the above lemmas. To verify these lemmas, we really should delve deeper into the general theory of linear algebra and further develop such concepts as generalized eigenvectors and "the Jordan form of a matrix" — concepts rarely mentioned in elementary linear algebra courses. But that would take us far outside the appropriate bounds of this text. So either "trust the author" or (better yet) take a more advanced course in linear algebra and learn how to prove the above lemmas yourself.

Back to differential equations: Let $\mathbf{A}$ be an $N \times N$ matrix with eigenpair $(r, \mathbf{u})$, and assume r has algebraic multiplicity m greater than one, but geometric multiplicity of only one. By the arguments already given in section 39.3, it immediately follows that the 'first two' solutions corresponding to eigenvalue r are

$$\mathbf{x}^1(t) = \mathbf{w}^1 e^{rt} \qquad \text{and} \qquad \mathbf{x}^2(t) = \left[t\mathbf{w}^1 + \mathbf{w}^2\right] e^{rt} \tag{39.4a}$$

where $\mathbf{w}^1 = \mathbf{u}$, and $\mathbf{w}^2$ is any column vector statisfying

$$[\mathbf{A} - r\mathbf{I}]\,\mathbf{w}^2 = \mathbf{w}^1 \quad . \tag{39.4b}$$

And if the algebraic multiplicity of r is greater than two, then you can easily extend the computations done in section 39.3 to show that a third solution is given by

$$\mathbf{x}^3(t) = \left[t^2\mathbf{w}^1 + 2t\mathbf{w}^2 + \mathbf{w}^3\right] e^{rt} \tag{39.4c}$$

where $\mathbf{w}^3$ is any column vector satisfying

$$[\mathbf{A} - r\mathbf{I}]\,\mathbf{w}^3 = \mathbf{w}^2 \tag{39.4d}$$

(see exercise 39.12). And once you've done that (and noted that $\mathbf{x}^k(0) = \mathbf{w}^k$ ensures the linear independence of the set of $\mathbf{x}^k$'s), the road to generating a linearly independent set of m solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ should be clear, at least when r has algebraic multiplicity m and geometric multiplicity one. Traveling that road is left to the interested reader.

But we can at least do a simple example using the above formulas.

!►Example 39.5: *Consider the system* $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *where*

$$\mathbf{A} = \begin{bmatrix} 4 & 3 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

It is easily verified that

$$\det(\mathbf{A} - r\mathbf{I}) = (4 - r)^3 \quad .$$

So this matrix has one eigenvalue $r = 4$, *and this eigenvalue has algebraic multiplicity three. It is also easily seen that all eigenvectors are constant multiples of*

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad .$$

So the geometric multiplicity of $r = 4$ *is one. From this and our last lemma it follows that we have a linearly independent set* $\left\{\mathbf{w}^1, \mathbf{w}^2, \mathbf{w}^3\right\}$ *with*

$$\mathbf{w}^1 = \mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad ,$$

and with $\mathbf{w}^2$ *and* $\mathbf{w}^3$ *being any vectors satisfying*

$$[\mathbf{A} - 4\mathbf{I}]\mathbf{w}^2 = \mathbf{w}^1 \qquad \text{and} \qquad [\mathbf{A} - 4\mathbf{I}]\mathbf{w}^3 = \mathbf{w}^2 \quad .$$

Letting $\mathbf{w}^2 = [\alpha, \beta, \gamma]^\mathsf{T}$, *we see that* $[\mathbf{A} - 4\mathbf{I}]\mathbf{w}^2 = \mathbf{w}^1$ *is simply*

$$\begin{bmatrix} 0 & 3 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad ,$$

yielding

$$\alpha \text{ is arbitrary} \quad , \quad \beta = \frac{1}{3} \quad \text{and} \quad \gamma = 0 \quad .$$

Taking $\alpha = 0$ *for convenience, we have*

$$\mathbf{w}^2 = \begin{bmatrix} 0 \\ \frac{1}{3} \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad .$$

Now consider $[\mathbf{A} - 4\mathbf{I}]\mathbf{w}^3 = \mathbf{w}^2$. *Letting* $\mathbf{w}^3 = [\alpha, \beta, \gamma]^\mathsf{T}$, *this becomes*

$$\begin{bmatrix} 0 & 3 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad ,$$

which reduces to the easily solved system

$$3\beta + 2\gamma = 0 \qquad \text{and} \qquad \gamma = \frac{1}{3} \quad .$$

From this, it immediately follows that

$$\alpha \text{ is arbitrary} \quad , \quad \beta = -\frac{2}{9} \quad \text{and} \quad \gamma = \frac{1}{3} \quad .$$

Again, we might as well take $\alpha = 0$, *giving us*

$$\mathbf{w}^3 = \begin{bmatrix} 0 \\ -\frac{2}{9} \\ \frac{1}{3} \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$$

Applying the formulas given in set (39.4) for the corresponding solutions to the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, *we have*

$$\mathbf{x}^1(t) = \mathbf{w}^1 e^{rt} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{4t} \quad ,$$

$$\mathbf{x}^2(t) = \left[t\mathbf{w}^1 + \mathbf{w}^2\right] e^{rt} = \left(t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) e^{4t} = \frac{1}{3}\begin{bmatrix} 3t \\ 1 \\ 0 \end{bmatrix} e^{4t}$$

and

$$\mathbf{x}^3(t) = \left[t^2\mathbf{w}^1 + 2t\mathbf{w}^2 + \mathbf{w}^3\right] e^{rt}$$

$$= \left(t^2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2t \cdot \frac{1}{3} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{9} \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix} \right) e^{4t} = \frac{1}{9} \begin{bmatrix} t^2 \\ 6t - 2 \\ 3 \end{bmatrix} e^{4t} \quad .$$

So, a general solution to our system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *is*

$$\begin{aligned} \mathbf{x}(t) &= c_1 \mathbf{x}^1(t) + c_2 \mathbf{x}^2(t) + c_3 \mathbf{x}^3(t) \\ &= c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{4t} + \frac{c_2}{3} \begin{bmatrix} 3t \\ 1 \\ 0 \end{bmatrix} e^{4t} + \frac{c_3}{9} \begin{bmatrix} t^2 \\ 6t - 2 \\ 3 \end{bmatrix} e^{4t} \\ &= \left(C_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 3t \\ 1 \\ 0 \end{bmatrix} + C_3 \begin{bmatrix} t^2 \\ 6t - 2 \\ 3 \end{bmatrix} \right) e^{4t} \quad . \end{aligned}$$

When the Geometric Multiplicity Is Greater Than One

If the geometric multiplicity of an eigenvalue is two or more, then we must account for the fact that the corresponding "basic set" of eigenvectors contains two or more eigenvectors. This complicates things a bit, especially since the eigenvectors first found might not be quite the ones generating the other generalized eigenvectors. You can see that in the following generalization of lemma 39.6:

Theorem 39.8
Assume r *is an eigenvalue for some* $N \times N$ *matrix* $\mathbf{A}$, *and let* m *and* μ *be, respectively, the algebraic and geometric multiplicities of* r. *Then* $N \geq m \geq \mu \geq 1$, *and there is a unique collection of* μ *integers* $\{\gamma_1, \gamma_2, \ldots \gamma_\mu\}$, *along with corresponding linearly independent sets of vectors*

$$\mathcal{W}_{r,k} = \left\{ \mathbf{w}^{k,1},\ \mathbf{w}^{k,2},\ \ldots,\ \mathbf{w}^{k,\gamma_k} \right\} \qquad \text{for} \quad k = 1, 2, \ldots, \mu$$

such that:

1. $\gamma_1 + \gamma_2 + \cdots + \gamma_\mu = m$.
2. *The set* $\{\mathbf{w}^{1,1}, \mathbf{w}^{2,1}, \ldots, \mathbf{w}^{\mu,1}\}$ *is a linearly independent set of eigenvectors for* $\mathbf{A}$ *corresponding to eigenvalue* r.
3. *The set of all* $\mathbf{w}^{k,j}$*'s is linearly independent.*
4. *For* $k = 1, 2, \ldots, \mu$*:*
   (a) *If* $\gamma_k > 1$, *then*
   $$[\mathbf{A} - r\mathbf{I}]\mathbf{w}^{k,j} = \mathbf{w}^{k,j-1} \qquad \text{for} \quad j = 2, \ldots, \gamma_k \quad .$$
   (b) *There is no* $\mathbf{v}$ *such that* $[\mathbf{A} - r\mathbf{I}]\mathbf{v} = \mathbf{w}^{k,\gamma_k}$.

While the above theorem assures us that the collection of γ_k's is unique for each eigenvalue r of $\mathbf{A}$, it does not assure us that the corresponding sets of $\mathbf{w}^{k,j}$'s are unique. In fact, they are not. For convenience, let us refer to any set of vectors $\mathcal{W}_r$ as a *complete set of generalized eigenvectors* for an $N \times N$ matrix $\mathbf{A}$ corresponding to eigenvalue r if and only if $\mathcal{W}_r$ can be given as

$$\mathcal{W}_r = \left\{ \mathbf{w}^{k,1},\ \mathbf{w}^{k,2},\ \ldots,\ \mathbf{w}^{k,\gamma_k} : k = 1, 2, \ldots, \mu \right\}$$

where:

1. The value μ is the geometric multiplicity of eigenvalue r .
2. The set $\{\mathbf{w}^{1,1}, \mathbf{w}^{2,1}, \dots, \mathbf{w}^{\mu,1}\}$ is a linearly independent set of eigenvectors for $\mathbf{A}$ corresponding to eigenvalue r .
3. For $k = 1, 2, \dots, \mu$:
   (a) If $[\mathbf{A} - r\mathbf{I}]\mathbf{v} = \mathbf{w}^{k,1}$ has no solution, then $\gamma_k = 1$. Otherwise $\gamma_k > 1$.
   (b) If $\gamma_k > 1$, then
   $$[\mathbf{A} - r\mathbf{I}]\mathbf{w}^{k,j} = \mathbf{w}^{k,j-1} \qquad \text{for} \quad j = 2, \dots, \gamma_k \quad .$$
   (c) There is no $\mathbf{v}$ such that $[\mathbf{A} - r\mathbf{I}]\mathbf{v} = \mathbf{w}^{k,\gamma_k}$.

In practice, a complete set of generalized eigenvectors corresponding to an eigenvalue r can be found starting with the eigenvectors originally found, and requiring that the $\mathbf{w}^{k,j}$'s satisfy the three conditions above (with the last being used to find the γ_k's). The previous theorem assures us that such sets exist. The one below, generalizing lemma 39.7, assures us that any process of generating complete sets of generalized functions will generate all the generalized eigenvectors needed for our purposes.

Theorem 39.9
Let $\{r_1, r_2, \dots, r_J\}$ be the set of all distinct eigenvalues for an $N \times N$ matrix $\mathbf{A}$, and, for $j = 1, 2, \dots, J$, let

$$\mathcal{W}_j = \text{a complete set of generalized eigenvectors corresponding to } r_j \quad .$$

Then:

1. *Each $\mathcal{W}_j$ is a linearly independent set of m_j vectors with m_j being the algebraic multiplicity of r_j .*
2. *The set of all the vectors in all these $\mathcal{W}_j$'s is a linearly independent set of N vectors.*

These two theorems can either be developed using lemmas 39.6 and 39.7, or developed independently, and the lemmas viewed as corollaries of the theorems. Alternatively, all the lemmas and theorems in this section can be viewed as corollaries of a theorem we will discuss (but not prove) later (theorem 40.8 on page 40–21). In any case, you either must trust the author on these theorems, or take a suitable course on linear algebra.

When the situation is as described in the two theorems, then each set of $\mathbf{w}^{k,j}$'s described in theorem 39.8,

$$\left\{\mathbf{w}^{k,1},\ \mathbf{w}^{k,2},\ \dots,\ \mathbf{w}^{k,\gamma_k}\right\} \quad ,$$

generates a corresponding linearly independent set of solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$,

$$\left\{\mathbf{x}^{k,1},\ \mathbf{x}^{k,2},\ \dots,\ \mathbf{x}^{k,\gamma_k}\right\} \quad ,$$

following the scheme just discussed for the case where the geometric multiplicity is one,

$$\mathbf{x}^{k,1}(t) = \mathbf{w}^{k,1}e^{rt} \quad ,$$

$$\mathbf{x}^{k,2}(t) = \left[t\mathbf{w}^{k,1} + \mathbf{w}^{k,2}\right]e^{rt} \quad ,$$

$$\mathbf{x}^{k,3}(t) = \left[t^2\mathbf{w}^{k,1} + 2t\mathbf{w}^{k,2} + \mathbf{w}^{k,3}\right]e^{rt} \quad ,$$

$$\vdots$$

One major difference, however, is that you do not know the initial eigenvector $\mathbf{w}^{k,1}$. All you know is that it must be some linear combination of the μ eigenvectors $\mathbf{u}^1$, ... and $\mathbf{u}^\mu$ originally found corresponding to the eigenvalue being used. So finding a suitable linear combination

$$\mathbf{w}^{k,1} = a_1\mathbf{u}^1 + a_2\mathbf{u}^2 + \cdots + a_\mu\mathbf{u}^\mu$$

is part of solving

$$[\mathbf{A} - r\mathbf{I}]\mathbf{w}^{k,2} = \mathbf{w}^{k,1} \qquad \text{for each} \quad k \quad .$$

In addition, you do not initially know the γ_k's. So you just have to iteratively solve for the $\mathbf{w}^{k,j}$'s until you reach the point where

$$[\mathbf{A} - r\mathbf{I}]\mathbf{w}^{k,j} = \mathbf{w}^{k,j-1}$$

has no solution $\mathbf{w}^{k,j}$. You then know $\gamma_k = j - 1$ for that k and j.

!►*Example 39.6:* *Consider the system* $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *where*

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 1 & 4 \end{bmatrix} \quad .$$

It is easily verified that

$$\det(\mathbf{A} - r\mathbf{I}) = (4 - r)^3 \quad .$$

So this matrix has one eigenvalue $r = 4$, *and this eigenvalue has algebraic multiplicity three. It is also easily seen that all eigenvectors are linear combinations of the two eigenvectors*

$$\mathbf{u}^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{u}^2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad .$$

So the geometric multiplicity of $r = 4$ *is two. From this and our last lemma it follows that we have a linearly independent set*

$$\left\{\mathbf{w}^{1,1}, \mathbf{w}^{2,1}, \mathbf{w}^{2,2}\right\}$$

with $\mathbf{w}^{1,1}$ *and* $\mathbf{w}^{2,1}$ *being eigenvectors of* $\mathbf{A}$, *and with*

$$[\mathbf{A} - 4\mathbf{I}]\,\mathbf{w}^{2,2} = \mathbf{w}^{2,1} \quad .$$

We also know the corresponding fundamental set of solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *is* $\{\mathbf{x}^{1,1}, \mathbf{x}^{2,1}, \mathbf{x}^{2,2}\}$ *where*

$$\mathbf{x}^{1,1}(t) = \mathbf{w}^{1,1}e^{4t} \quad , \quad \mathbf{x}^{2,1}(t) = \mathbf{w}^{2,1}e^{4t} \qquad \text{and} \qquad \mathbf{x}^{2,2}(t) = \left[t\mathbf{w}^{2,1} + \mathbf{w}^{2,2}\right]e^{4t} \quad .$$

Now, consider the equation

$$[\mathbf{A} - 4\mathbf{I}]\,\mathbf{w}^{2,2} = \mathbf{w}^{2,1} \quad .$$

Since $\mathbf{w}^{2,1}$ *must be an eigenvector, it must be a linear combination of the two eigenvectors already obtained; that is,*

$$\mathbf{w}^{2,1} = \sigma\mathbf{u}^1 + \tau\mathbf{u}^2 = \sigma\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \tau\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sigma \\ 0 \\ \tau \end{bmatrix}$$

for some, yet unknown, pair of constants σ *and* τ *. Letting* $\mathbf{w}^{2,2} = [\alpha, \beta, \gamma]^{\mathsf{T}}$ *, we now have*

$$[\mathbf{A} - 4\mathbf{I}]\,\mathbf{w}^{2,2} \;=\; \mathbf{w}^{2,1}$$

$$\hookrightarrow \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} \sigma \\ 0 \\ \tau \end{bmatrix}$$

$$\hookrightarrow \quad \begin{bmatrix} \beta \\ 0 \\ \beta \end{bmatrix} = \begin{bmatrix} \sigma \\ 0 \\ \tau \end{bmatrix}$$

Consequently, α *,* β *,* γ *,* σ *and* τ *can be any values satisfying* $\beta = \sigma = \tau \neq 0$ *. In particular, taking* $\sigma = 1$ *and* $\alpha = \gamma = 0$ *gives us*

$$\mathbf{w}^{2,1} \;=\; \begin{bmatrix} \sigma \\ 0 \\ \tau \end{bmatrix} \;=\; \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \;=\; \mathbf{u}^1 \;+\; \mathbf{u}^2 \qquad \text{and} \qquad \mathbf{w}^{2,2} \;=\; \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \;=\; \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad .$$

For $\mathbf{w}^{1,1}$ *we can take any linear combination of* $\mathbf{u}^1$ *and* $\mathbf{u}^2$ *such that* $\{\mathbf{w}^{1,1}, \mathbf{w}^{2,1}, \mathbf{w}^{2,2}\}$ *is linearly independent. An obvious choice is*

$$\mathbf{w}^{1,1} \;=\; \mathbf{u}^1 \;-\; \mathbf{u}^2 \;=\; \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad .$$

Finally, then, we can write out the general solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *as*

$$\mathbf{x}(t) \;=\; c_1\mathbf{x}^{1,1}(t) \;+\; c_2\mathbf{x}^{2,1}(t) \;+\; c_2\mathbf{x}^{2,2}(t)$$

where

$$\mathbf{x}^{1,1}(t) \;=\; \mathbf{w}^{1,1}e^{4t} \;=\; \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{4t} \qquad , \qquad \mathbf{x}^{2,1}(t) \;=\; \mathbf{w}^{2,1}e^{4t} \;=\; \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{4t}$$

and

$$\mathbf{x}^{2,2}(t) \;=\; \left[t\mathbf{w}^{2,1} + \mathbf{w}^{2,2}\right] e^{4t} \;=\; \left(t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \;+\; \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) e^{4t} \;=\; \begin{bmatrix} t \\ 1 \\ t \end{bmatrix} e^{4t} \quad .$$

That is,

$$\mathbf{x}(t) \;=\; \left(c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \;+\; c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \;+\; c_3 \begin{bmatrix} t \\ 1 \\ t \end{bmatrix} \right) e^{4t} \quad .$$

Finally, we should note that the eigenvalue in question may be complex, $r = \lambda + i\omega$. If $\mathbf{A}$ is real (as we've been assuming), then all of the above still holds, and, following the discussion at the start of this chapter, we can find a 'complete' set of real-valued solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ corresponding to r and r^* by taking the real and imaginary parts of the complex-valued solutions generated by the procedures just discussed.

39.6 Appendix for Sections 39.1 and 39.2

In this section, we will verify two claims made in sections 39.1 and 39.2; namely,

1. The claim of the linear independence of of the real and imaginary parts of a complex eigenvector made in lemma 39.1 on page 39–2.
2. The claim made on page 39–7 that the trajectory of $\mathbf{x}^R$ is an ellipse about the origin.

Linear Independence of Two Vectors

Here is the claim and the proof:

Lemma 39.10
Let $(r, \mathbf{u})$ be an eigenpair for a real $N \times N$ matrix $\mathbf{A}$, and assume r is not real. Then the real and imaginary parts of $\mathbf{u}$ form a linearly independent set.

PROOF: Let λ and ω be, respectively, the real and imaginary parts of r, and let $\mathbf{a}$ and $\mathbf{b}$ be the real and imaginary parts of $\mathbf{u}$, so that,

$$\mathbf{u} = \mathbf{a} + i\mathbf{b} \quad \text{and} \quad \mathbf{u}^* = \mathbf{a} - i\mathbf{b} \quad ,$$

and

$$r - r^* = (\lambda + i\omega) - (\lambda - i\omega) = 2i\omega \quad .$$

Since we are assuming r is not real, we must have

$$r - r^* \neq 0 \quad . \tag{39.5}$$

For the moment, suppose $\{\mathbf{a}, \mathbf{b}\}$ is not linearly independent. This would mean that either $\mathbf{b} = \mathbf{0}$ or that $\mathbf{a}$ is a constant multiple of $\mathbf{b}$, $\mathbf{a} = \gamma\mathbf{b}$. If $\mathbf{b} = 0$, then

$$\mathbf{u} = \mathbf{a} = \mathbf{u}^*$$

On the other hand, if $\mathbf{a} = \gamma\mathbf{b}$, then

$$\mathbf{u} = \mathbf{a} + i\mathbf{b} = (\gamma + i)\mathbf{b} \quad \text{and} \quad \mathbf{u}^* = \mathbf{a} - i\mathbf{b} = (\gamma - i)\mathbf{b}$$

Since eigenvectors are automatically nonzero, we know that $\mathbf{u}$, $\mathbf{u}^*$ and $\gamma \pm i$ are nonzero. Hence, we would have

$$\mathbf{u} = \frac{\gamma + i}{\gamma - i}\mathbf{u}^* \quad .$$

Either way, if $\{\mathbf{a}, \mathbf{b}\}$ is not linearly independent, then $\mathbf{u}$ must be a constant nonzero multiple of $\mathbf{u}^*$, and, thus, must be an eigenvector corresponding to r^* as well as r. But then,

$$(r - r^*)\mathbf{u} = r\mathbf{u} - r^*\mathbf{u} = \mathbf{A}\mathbf{u} - \mathbf{A}\mathbf{u} = \mathbf{0} \quad ,$$

which, since $\mathbf{u} \neq \mathbf{0}$, tells us that

$$r - r^* = 0 \quad ,$$

contrary to inequality (39.5), which followed immediately from the basic assumption that r is not real. So it is not possible to have both that

"r is not real." and "$\{\mathbf{a}, \mathbf{b}\}$ is not linearly independent." .

If r is not real, then it cannot be true that $\{\mathbf{a}, \mathbf{b}\}$ is not linearly independent; that is, if r is not real, $\{\mathbf{a}, \mathbf{b}\}$ must be linearly independent . ∎

Ellipticity of Trajectories

On page 39–7, it was claimed that the trajectory of

$$\mathbf{x}^R(t) \;=\; \mathbf{a}\cos(\omega t) - \mathbf{b}\sin(\omega t)$$

is an ellipse when $(i\omega, \mathbf{u})$ is an eigenpair and $\mathbf{a}$ and $\mathbf{b}$ are the real and imaginary parts of $\mathbf{u}$. Since, as we just saw, $\{\mathbf{a}, \mathbf{b}\}$ is linearly independent (and, hence, so is $\{\mathbf{a}, -\mathbf{b}\}$), this claim will follow immediately from the following lemma.

Lemma 39.11
Let $\mathbf{x}(t) = [x(t), y(t)]^{\mathsf{T}}$ be given by

$$\mathbf{x}(t) \;=\; \mathbf{v}\cos(\omega t) \;+\; \mathbf{w}\sin(\omega t)$$

where ω is a nonzero real number and $\{\mathbf{v}, \mathbf{w}\}$ is a linearly independent pair of vectors whose components are real numbers. Then the path traced out by $(x(t), y(t))$ on the Cartesian plane as t varies is an ellipse centered at the origin.

Our approach to verifying this lemma will be to verify that the x and y components of $\mathbf{x}(t)$ satisfy an equation for such an ellipse. So let us start with little review (and possibly an extension) of what you know about equations for ellipses.

Coordinate Equations for Ellipses

Recall the canonical equation for an ellipse centered at the origin and with its axes parallel to the coordinate axes is

$$\frac{x^2}{\alpha^2} \;+\; \frac{y^2}{\beta^2} \;=\; 1$$

where α and β are nonzero real numbers. With the same assumptions on α and β,

$$\frac{x^2}{\alpha^2} \;-\; \frac{y^2}{\beta^2} \;=\; 1 \qquad \text{and} \qquad -\,\frac{x^2}{\alpha^2} \;+\; \frac{y^2}{\beta^2} \;=\; 1$$

are equations for symmetric pairs of hyperbolas about the origin, and the graph of

$$-\frac{x^2}{\alpha^2} \;-\; \frac{y^2}{\beta^2} \;=\; 1$$

is empty (i.e., contains no points) since no (x, y) can satisfy this equation. Since these are the only possibilities, we have

Lemma 39.12
Let a and c be real numbers, and assume the polynomial equation

$$ax^2 \;+\; cy^2 \;=\; 1$$

has a nonempty graph. Then:

1. *If $ac > 0$, the graph is an ellipse centered at the origin.*
2. *If $ac < 0$, the graph is a pair of hyperbolas.*

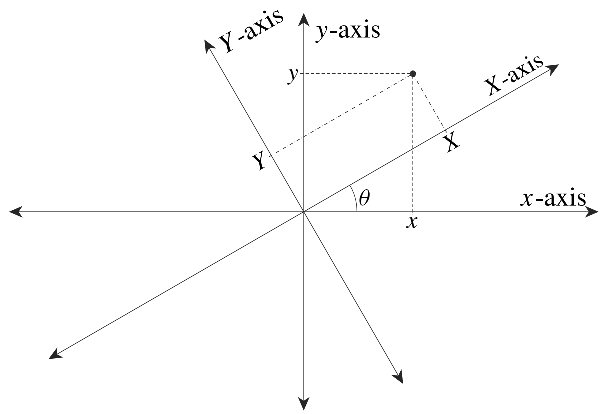


Figure 39.5: An XY-coordinate system obtained by rotating the xy-coordinate system by an angle θ .

Now assume we have a nonempty graph of a polynomial equation of the form

$$Ax^2 + Bxy + Cy^2 = 1 \tag{39.6}$$

where A , B and C are real numbers. To deal with this, we invoke a new XY-coordinate system obtained by rotating the original xy-coordinate system by an angle θ as indicated in figure 39.5. If you check, you will find that the (x, y) and (X, Y) coordinates of any single point are related by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} \quad . \tag{39.7}$$

Using this, we can rewrite our polynomial equation (equation (39.6)) in terms of X and Y , obtaining

$$aX^2 + bXY + cY^2 = 1 \tag{39.8}$$

where a , b and c are formulas involving $\cos(\theta)$ and $\sin(\theta)$. Now, the clever choice for θ is the one that makes $b = 0$, thus giving

$$aX^2 + cY^2 = 1 \tag{39.9}$$

as the polynomial equation of our graph in terms of the XY-coordinate system. As an exercise (exercise 39.1, following the next theorem), you can show that this choice is possible, and that by this choice

$$ac = AC - \frac{1}{4}B^2$$

This and lemma 39.12 then yields:

Lemma 39.13
Let A , B and C be real numbers, and assume the polynomial equation

$$Ax^2 + Bxy + Cy^2 = 1$$

has a nonempty graph. Then:

1. *If $AC - \frac{1}{4}B^2 > 0$, the graph is an ellipse centered at the origin.*

2. *If $AC - \frac{1}{4}B^2 < 0$, the graph is a pair of hyperbolas.*

?►*Exercise 39.1* ***a:*** *Find the formulas for* A *,* B *and* C *when using the change of variables given by equation (39.7) to convert equation (39.6) to equation (39.8).*

b: *Show that equation (39.6) reduces to equation (39.9) under the change of variables given by equation (39.7) if*

$$\theta \;=\; \begin{cases} \frac{1}{2}\operatorname{Arctan}\left(\frac{B}{A-C}\right) & \text{if} \quad A \neq C \\ \frac{\pi}{4} & \text{if} \quad A = C \end{cases} \quad .$$

c: *Assume that* θ *is as above, and verify that* $AC - \frac{1}{4}B^2 \;=\; ac$ *.*

Verifying Lemma 39.11

Now consider

$$\mathbf{x}(t) \;=\; \mathbf{v}\cos(\omega t) \;+\; \mathbf{w}\sin(\omega t)$$

assuming ω is some nonzero real number, and

$$\mathbf{v} \;=\; \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{w} \;=\; \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad .$$

is a linearly independent pair of vectors whose components are real numbers. Rewriting the above formula for $\mathbf{x}$ in component form, we see that

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \;=\; \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\cos(\omega t) \;+\; \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\sin(\omega t) \;=\; \begin{bmatrix} v_1\cos(\omega t) + w_1\sin(\omega t) \\ v_2\cos(\omega t) + w_2\sin(\omega t) \end{bmatrix} \quad .$$

That is, (x, y) is the point on the trajectory given by $\mathbf{x}(t)$ if and only if

$$\begin{bmatrix} x \\ y \end{bmatrix} \;=\; \mathbf{M}\begin{bmatrix} \cos(\omega t) \\ \sin(\omega t) \end{bmatrix} \qquad \text{where} \quad \mathbf{M} \;=\; \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \quad . \tag{39.10}$$

Since the two columns of $\mathbf{M}$ are the column vectors $\mathbf{v}$ and $\mathbf{w}$, and $\{\mathbf{v}, \mathbf{w}\}$ is linearly independent, we know (from linear algebra) that

$$\delta \;=\; \det(\mathbf{M}) \;=\; v_1 w_2 \;-\; w_1 v_2 \;\neq\; 0 \quad .$$

This tells us $\mathbf{M}$ is invertible. The inverse of $\mathbf{M}$ is easily found, and using it, we can then invert relation (39.10), obtaining

$$\begin{bmatrix} \cos(\omega t) \\ \sin(\omega t) \end{bmatrix} \;=\; \frac{1}{\delta}\begin{bmatrix} w_2 & -w_1 \\ -v_2 & v_1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \;=\; \frac{1}{\delta}\begin{bmatrix} w_2 x - w_1 y \\ -v_2 x + v_1 y \end{bmatrix} \quad .$$

From this and our favorite trigonometric identity, we have

$$1 \;=\; \cos^2(\omega t) \;+\; \sin^2(\omega t) \;=\; \frac{1}{\delta^2}(w_2 x - w_1 y)^2 \;+\; \frac{1}{\delta^2}(-v_2 x + v_1 y)^2 \quad ,$$

which, after a little algebra, simplifies to

$$Ax^2 \;+\; Bxy \;+\; Cy^2 \;=\; 1$$

with

$$A = \frac{(w_2)^2 + (v_2)^2}{\delta^2} \quad , \quad B = -2\frac{w_1 w_2 + v_1 v_2}{\delta^2} \quad \text{and} \quad C = \frac{(w_1)^2 + (v_1)^2}{\delta^2} \quad .$$

From theorem 39.13 we know this trajectory is an ellipse centered at the origin if $AC - {}^{B^2}/_4 > 0$. Remarkably, when you carry out the details of computation, you get

$$AC - \frac{1}{4}B^2 = \frac{1}{\delta^4}\left[\left(w_2^2 + v_2^2\right)\left(w_1^2 + v_1^2\right) - (w_1 w_2 + v_1 v_2)^2\right] = \cdots = 1 > 0 \quad .$$

So, yes, the trajectory is an ellipse centered at the origin, and lemma 39.11 is confirmed. ■

?►*Exercise 39.2:* *Fill in the details of the computations in the above proof.*

Additional Exercises

39.3. *Find a general solution, in terms of real-valued functions only, for each of the following systems:*

a. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

b. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 8 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

c. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

d. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

e. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

f. $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -2 & 3 & 0 \\ -3 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

g. $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 8 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

h. $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 2 \\ 0 & -4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

i. $\begin{bmatrix} {x_1}' \\ {x_2}' \\ {x_3}' \\ {x_4}' \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 15 & 0 \\ 0 & -8 & 0 & 16 \\ 0 & 0 & 4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

j. $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

39.4. *For each of the following systems:*

i *Determine whether the critical point is a center or a spiral point, and describe its stability.*

ii *Sketch (by hand) a phase portrait.*

a. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

b. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

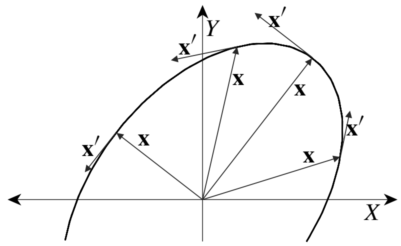


Figure 39.6: The $\mathbf{x}$ and $\mathbf{x}'$ vectors for various points on an elliptic trajectory.

c. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

d. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -5 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

e. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

f. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

g. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & -10 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

h. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 5 & -6 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

39.5. *Let*

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where a *,* b *,* c *and* d *are fixed real numbers. Show that the eigenvalues of* $\mathbf{A}$ *are purely imaginary (and, hence, the system* $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *has elliptic trajectories) if and only if*

$$d = -a \qquad \text{and} \qquad a^2 + bc < 0 \quad .$$

Further show that, if the above holds, then the eigenvalues of $\mathbf{A}$ *are* $\pm i\sqrt{bc + a^2}$ *.*

39.6. *Let* $\mathbf{x} = \mathbf{x}(t)$ *be a nonequilibrium solution to a* 2×2 *constant matrix system* $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *having elliptic trajectories, and consider the dot product* $\mathbf{x}' \cdot \mathbf{x}$ *. Observe that the vectors* $\mathbf{x}$ *and* $\mathbf{x}'$ *are perpendicular (and, hence,* $\mathbf{x}' \cdot \mathbf{x} = 0$ *) if and only if* $\mathbf{x}$ *is pointing along one the axes of the elliptic trajectory of* $\mathbf{x}(t)$ *(see figure 39.6). Using this observation with the fact that* $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *:*

a. *Show that the two axes for the system's elliptic trajectories are the coordinate axes if the diagonal elements of* $\mathbf{A}$ *are zero.*

b. *Find the slopes of the straight lines* $y = mx$ *containing the axes of the elliptic trajectories for each of the following systems:*

i. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -5 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

ii. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -10 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

iii. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

iv. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

39.7. *Find a general solution for each of the following systems:*

a. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

b. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -12 & 4 \\ -1 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

c. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -10 & 1 \\ -1 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

d. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 6 & -4 \\ 9 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

e. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & -6 \\ 6 & -10 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ **f.** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

g. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 12 & -5 \\ 20 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ **h.** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

39.8. *For each of the following systems:*

i *Describe the stability of the equilibrium solution.*

ii *Sketch (by hand) a phase portrait.*

a. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -2 & -10 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ **b.** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 10 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

c. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & -10 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ **d.** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

e. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -\frac{3}{2} \\ \frac{3}{2} & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ **f.** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -8 \\ 2 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

g. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -5 & 9 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ **h.** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

39.9. *Find a general solution for each of the following systems:*

a. $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 8 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ **b.** $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -3 & 1 & -1 \\ 0 & -3 & 3 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

c. $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & -3 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ **d.** $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 5 & 0 & -5 \\ 0 & 0 & -3 \\ 5 & 0 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

e. $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 4 & 5 & 0 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ **f.** $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

g. $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & 3 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ **h.** $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -4 & 5 & 0 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

39.10. *The main goal of this exercise is to verify lemma 39.4 on page 39–14.*

a. *Assume* $\mathbf{A}$ *is any* $N \times N$ *matrix, and let* $\mathbf{B} = \mathbf{A} - \gamma\mathbf{I}$ *for some scalar* γ *. Show that* $(r, \mathbf{u})$ *is an eigenpair for* $\mathbf{A}$ *if and only if* $(\rho, \mathbf{u})$ *is an eigenpair for* $\mathbf{B}$ *with* $r = \rho + \gamma$ *.*

b. *For the rest of this exercise, assume* $\mathbf{A}$ *is a* 2×2 *constant matrix, and set*

$$\mathbf{B} = \mathbf{A} - r\mathbf{I}$$

where r *is an eigenvalue for* $\mathbf{A}$ *with corresponding eigenvector* $\mathbf{u} = [u_1, u_2]^\mathsf{T}$ *.*

i. *Using the first part of this exercise set, verify that* $\mathbf{B}\mathbf{u} = \mathbf{0}$ *.*

ii. *Then, using the fact that* $\mathbf{Bu} = \mathbf{0}$ *, find formulas for the components of* $\mathbf{B}$ *.*

iii. *Using the formulas just found for the components of* $\mathbf{B}$ *, show that there is a vector* $\mathbf{a} = [a_1, a_2]^\mathsf{T}$ *such that*

$$\mathbf{Bv} = (u_1 v_2 - u_2 v_1)\mathbf{a} \qquad \text{for each} \quad \mathbf{v} = [v_1, v_2]^\mathsf{T} \quad .$$

iv. *Assume that the* $\mathbf{a}$ *just found is nonzero, and, using the above, verify that it is an eigenvector for both* $\mathbf{B}$ *and* $\mathbf{A}$ *, and determine the eigenvalue* r_2 *for* $\mathbf{A}$ *corresponding to eigenvector* $\mathbf{a}$ *.*

v. *Let* $(r_2, \mathbf{a})$ *be as just found, and suppose* $\{\mathbf{u}, \mathbf{a}\}$ *is linearly independent. Show that* $r_2 \neq r$ *.*

vi. *Let* $(r_2, \mathbf{a})$ *be as just found, and suppose* $\mathbf{a} = \mathbf{0}$ *. Show that there is a linearly independent pair* $\{\mathbf{u}, \mathbf{v}\}$ *where* $(r, \mathbf{v})$ *is an eigenpair for* $\mathbf{A}$ *.*

c. *Let* $\mathbf{A}$ *,* $(r, \mathbf{u})$ *and* $\mathbf{a}$ *be as above, and assume, in addition, that eigenvalue* r *has algebraic multiplicity two but geometric multiplicity one. Using the above:*

i. *Show that* $\mathbf{a}$ *must be a nonzero constant multiple of* $\mathbf{u}$ *.*

ii. *Verify that there is a nonzero* $\mathbf{w}$ *satisfying*

$$(\mathbf{A} - r\mathbf{I})\mathbf{w} = \mathbf{u}$$

with $\mathbf{w}$ *not being a constant multiple of* $\mathbf{u}$ *.*

39.11. *Assume* $r = 0$ *is the only eigenvalue for some* 2×2 *real constant matrix* $\mathbf{A}$ *. Further assume that every eigenvector for* $\mathbf{A}$ *is a multiple of* $\mathbf{u}$ *, and recall from exercise 38.13 on page 38–28 that every point on the straight line parallel to* $\mathbf{u}$ *through the origin is a critical point for* $\mathbf{x}' = \mathbf{Ax}$ *.*

a. *Describe the general solution to* $\mathbf{x}' = \mathbf{Ax}$ *.*

b. *Describe the trajectories of the nonequilibrium solutions.*

c. *Sketch a phase portrait for each of the following systems.*

i. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ **ii.** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

iii. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ **iv.** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

39.12. *Let* $(r, \mathbf{u})$ *be an eigenpair for an* $N \times N$ *matrix* $\mathbf{A}$ *. Assume* r *has geometric multiplicity one but algebraic multiplicity three or greater. We know that two solutions to* $\mathbf{x}' = \mathbf{Ax}$ *are given by*

$$\mathbf{x}^1(t) = \mathbf{u}e^{rt} \qquad \text{and} \qquad \mathbf{x}^2(t) = \left[t\mathbf{u} + \mathbf{w}^1\right]e^{rt}$$

where $\mathbf{w}^1$ *is any vector satisfying*

$$[\mathbf{A} - r\mathbf{I}]\mathbf{w}^1 = \mathbf{u} \quad .$$

a. *Set*

$$\mathbf{x}^3(t) = \left[t^2\mathbf{u} + \alpha t\mathbf{w}^1 + \mathbf{w}^2\right]e^{rt}$$

and derive the fact that $\mathbf{x}^3$ *is a solution to* $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *if*

$$\alpha = 2 \qquad \text{and} \qquad [\mathbf{A} - r\mathbf{I}]\,\mathbf{w}^2 = \mathbf{w}^1 \quad .$$

b. *Assume the algebraic multiplicity of* r *is at least four, and derive the conditions for* α , β *and* $\mathbf{w}^3$ *so that*

$$\mathbf{x}^4(t) = \left[t^3\mathbf{u} + \alpha t^2\mathbf{w}^1 + \beta t\mathbf{w}^2 + \mathbf{w}^3\right]e^{rt}$$

is a solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Some Answers to Some of the Exercises

WARNING! Most of the following answers were prepared hastily and late at night. They have not been properly proofread! Errors are likely!

3a. $c_1 \begin{bmatrix} -\sin(4t) \\ \cos(4t) \end{bmatrix} + c_2 \begin{bmatrix} \cos(4t) \\ \sin(4t) \end{bmatrix}$

3b. $c_1 \begin{bmatrix} \cos(4t) \\ 2\sin(4t) \end{bmatrix} + c_2 \begin{bmatrix} \sin(4t) \\ -2\cos(4t) \end{bmatrix}$

3c. $c_1 \begin{bmatrix} 2\cos(4t) \\ \cos(4t)+2\sin(4t) \end{bmatrix} + c_2 \begin{bmatrix} 2\sin(4t) \\ -2cos(4t)+sin(4t) \end{bmatrix}$

3d. $c_1 \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -\cos(2t) \\ \sin(2t) \end{bmatrix} e^{3t}$

3e. $c_1 \begin{bmatrix} 2\cos(3t) \\ 3\sin(3t)-\cos(3t) \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 2\sin(3t) \\ -3\cos(3t)-\sin(3t) \end{bmatrix} e^{2t}$

3f. $c_1 \begin{bmatrix} \cos(3t) \\ -\sin(3t) \\ 0 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} \sin(3t) \\ \cos(3t) \\ 0 \end{bmatrix} e^{-2t} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-t}$

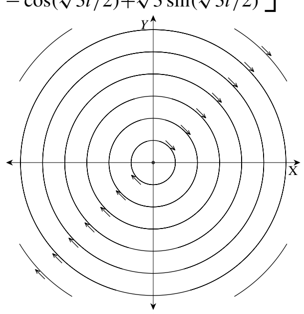
3g. $c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -\cos(4t) \\ 2\cos(4t)-2\sin(4t) \\ 4\cos(4t)+4\sin(4t) \end{bmatrix} + c_3 \begin{bmatrix} -\sin(4t) \\ 2\cos(4t)+2\sin(4t) \\ -4\cos(4t)+4\sin(4t) \end{bmatrix}$

3h. $c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -\cos(2t) \\ \sin(2t) \\ 2\cos(2t) \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} \sin(2t) \\ \cos(2t) \\ -2\sin(2t) \end{bmatrix} e^{3t}$

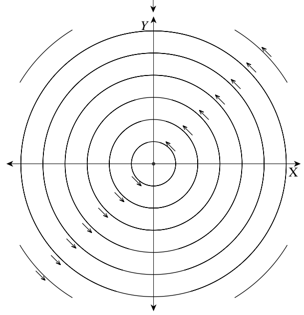
3i. $c_1 \begin{bmatrix} -15 \\ 15 \\ -4 \\ 8 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 15 \\ 15 \\ 4 \\ 8 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} \sin(8t) \\ 4\cos(8t) \\ -2\sin(8t) \\ \cos(8t) \end{bmatrix} + c_4 \begin{bmatrix} -\cos(8t) \\ 4\sin(8t) \\ 2\cos(8t) \\ \sin(8t) \end{bmatrix}$

3j. $c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 2\cos(\sqrt{3}t/2) \\ -\cos(\sqrt{3}t/2)-\sqrt{3}\sin(\sqrt{3}t/2) \\ -\cos(\sqrt{3}t/2)+\sqrt{3}\sin(\sqrt{3}t/2) \end{bmatrix} e^{-t/2} + c_3 \begin{bmatrix} 2\sin(\sqrt{3}t/2) \\ -\sin(\sqrt{3}t/2)+\sqrt{3}\cos(\sqrt{3}t/2) \\ -\sin(\sqrt{3}t/2)-\sqrt{3}\cos(\sqrt{3}t/2) \end{bmatrix} e^{-t/2}$

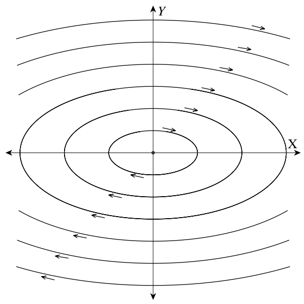
4a. A stable center.



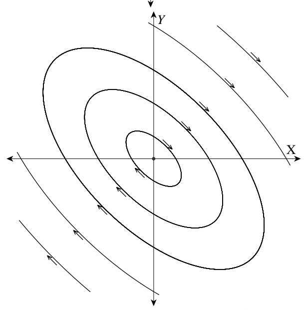
4b. A stable center.



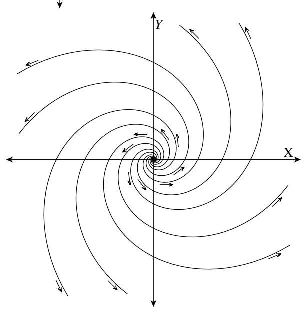
4c. A stable center.



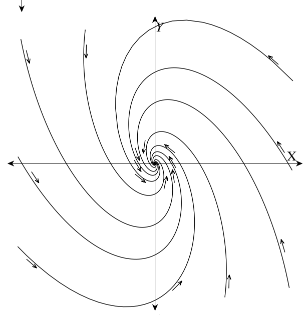
4d. A stable center.



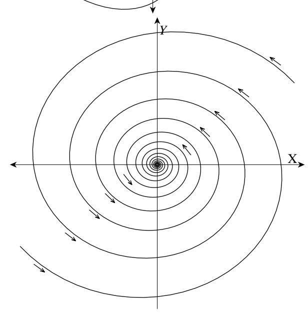
4e. An unstable spiral point.



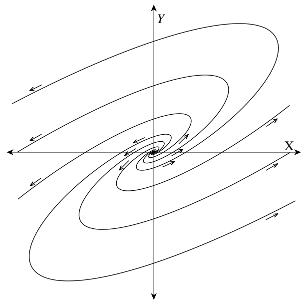
4f. An asymptotically stable spiral point.



4g. An asymptotically stable spiral point.



4h. An unstable spiral point.



6b i. $m = \pm 1$

6b ii. $m = 3,\ -\frac{1}{3}$

6b iii. $m = -2,\ \frac{1}{2}$

6b iv. $m = (-1 \pm \sqrt{5})/2$

7a. $\left(c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} t \\ 1 \end{bmatrix}\right) e^{t}$

7b. $\left(c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2t-1 \\ t \end{bmatrix}\right) e^{-10t}$

7c. $\left(c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} t \\ t+1 \end{bmatrix}\right) e^{-9t}$

7d. $c_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 2t+1 \\ 3t+1 \end{bmatrix}$

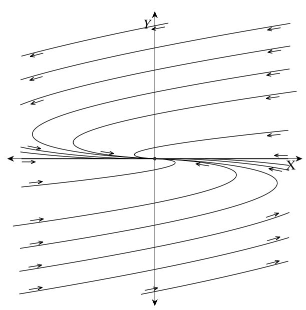
7e. $\left(c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 6t+1 \\ 6t \end{bmatrix}\right) e^{-4t}$

7f. $\left(c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 3t+1 \\ -3t \end{bmatrix}\right) e^{t}$

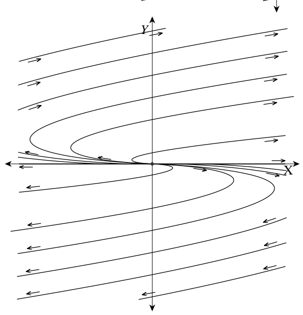
7g. $\left(c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 5t+2 \\ 10t \end{bmatrix}\right) e^{t/2}$

7h. $\left(c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2t+1 \\ t \end{bmatrix}\right) e^{2t}$

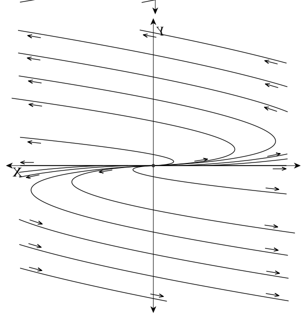
8a. Asymtotically stable.



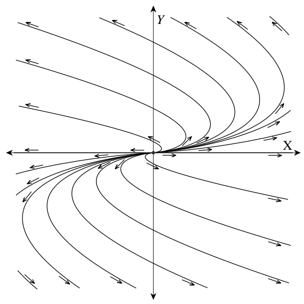
8b. Unstable.



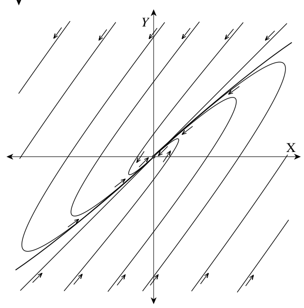
8c. Unstable.



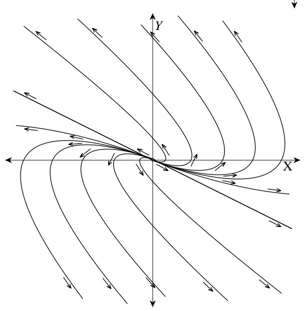
8d. Unstable.



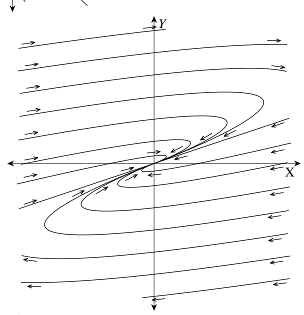
8e. Asymptotically stable.



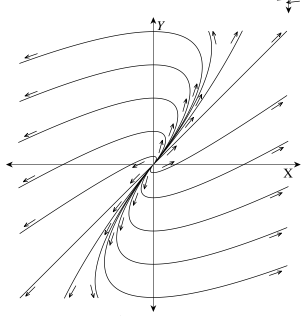
8f. Unstable.



8g. Asymptotically stable.



8h. Unstable.



9a. $c_1 \begin{bmatrix} 1 \\ 4 \\ -8 \end{bmatrix} e^{-4t} + \left(c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} t \\ 1 \\ 2 \end{bmatrix} \right) e^{4t}$

9b. $\left(c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 3t^2 \\ 6t+1 \\ 1 \end{bmatrix} \right) e^{-3t}$

9c. $\left(c_1\begin{bmatrix}0\\1\\0\end{bmatrix}+c_2\begin{bmatrix}1\\6t\\-1\end{bmatrix}+c_3\begin{bmatrix}4t-1\\12t^2\\-4t-1\end{bmatrix}\right)e^{2t}$

9d. $c_1\begin{bmatrix}0\\1\\0\end{bmatrix}+c_2\begin{bmatrix}1\\-3t\\1\end{bmatrix}+c_3\begin{bmatrix}10t+1\\-15t^2\\10t\end{bmatrix}$

9e. $\left(c_1\begin{bmatrix}0\\0\\1\end{bmatrix}+c_2\begin{bmatrix}-1\\2\\2t\end{bmatrix}+c_3\begin{bmatrix}1-4t\\8t\\4t^2\end{bmatrix}\right)e^{3t}$

9f. $\left(c_1\begin{bmatrix}1\\0\\0\end{bmatrix}+c_2\begin{bmatrix}2t\\0\\1\end{bmatrix}+c_3\begin{bmatrix}0\\1\\0\end{bmatrix}\right)e^{-t}$

9g. $\left(c_1\begin{bmatrix}-1\\3\\1\end{bmatrix}+c_2\begin{bmatrix}1-t\\3t\\t\end{bmatrix}+c_3\begin{bmatrix}-1\\0\\1\end{bmatrix}\right)e^{2t}$

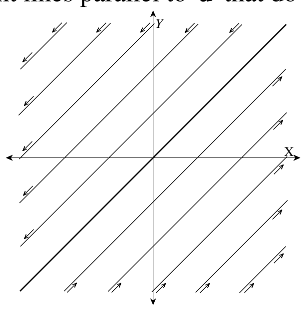
9h. $\left(c_1\begin{bmatrix}1\\2\\-1\end{bmatrix}+c_2\begin{bmatrix}t\\2t+1\\-t\end{bmatrix}+c_3\begin{bmatrix}0\\0\\1\end{bmatrix}\right)e^{3t}$

10b iv. $r_2 = r + u_1a_2 - u_2a_1$

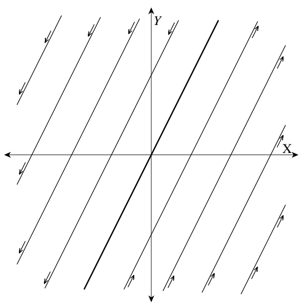
11a. $\mathbf{x}(t) = c_1\mathbf{u} + c_2[t\mathbf{u}+\mathbf{w}]$ where $\mathbf{w}$ is any solution to $\mathbf{Aw}=\mathbf{u}$.

11b. They are the straight lines parallel to $\mathbf{u}$ that do not pass through the origin.

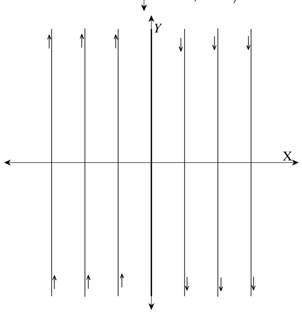
11c i. $c_1\begin{bmatrix}1\\1\end{bmatrix}+c_2\begin{bmatrix}t+1\\t\end{bmatrix}$



11c ii.



11c iii.



11c iv.